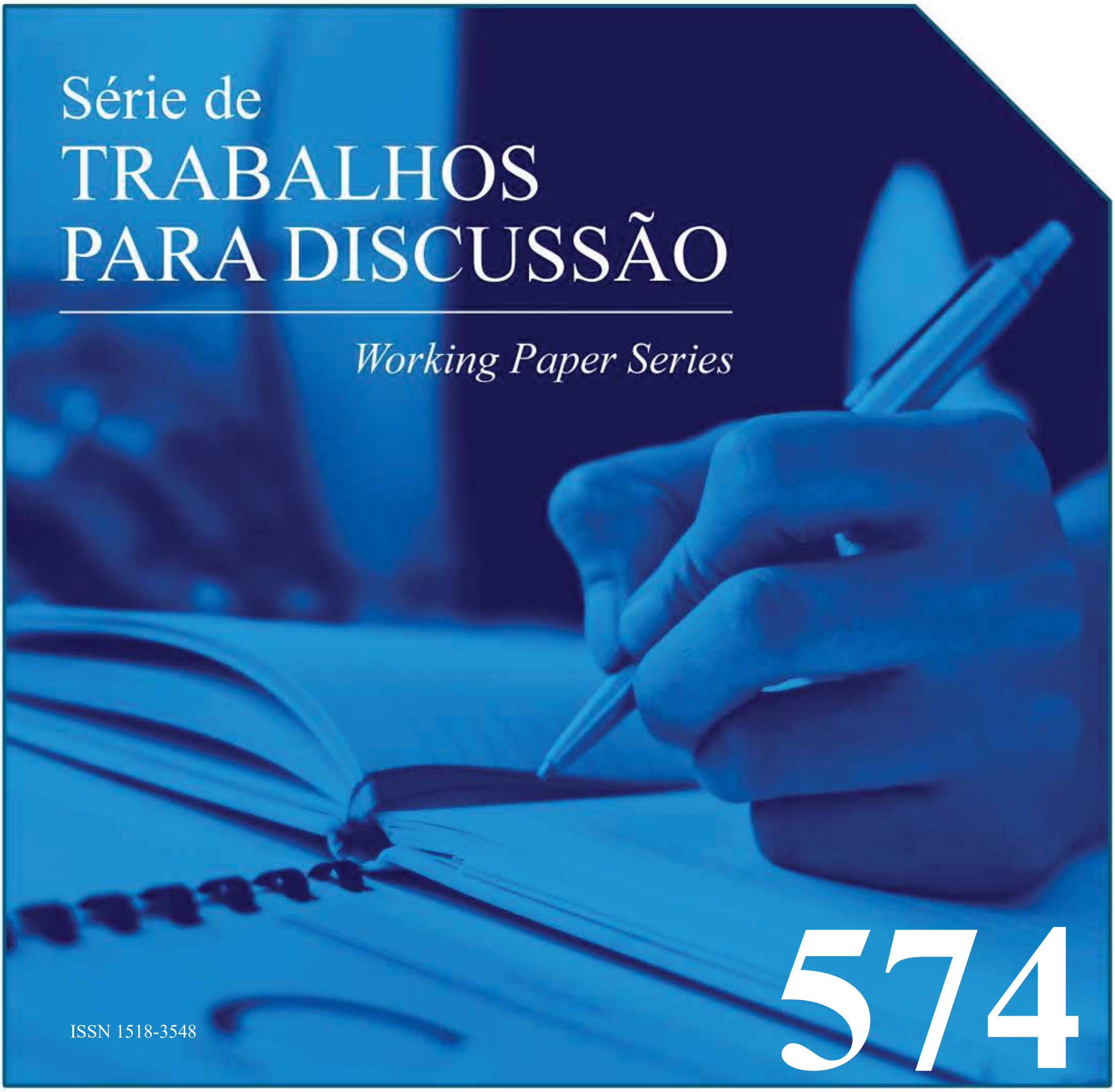

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Non-technical Summary

In this paper, we explore different ways to improve a traditional econometric method to forecast inflation. The goal is to predict the consumer price inflation using a hybrid approach that combines a standard Vector Autoregression (VAR) model with survey expectations from consumers or professional forecasters.

The basic idea is to cast a VAR model into a state-space setup, which allows for general parameter restriction. In particular, we impose restrictions that guarantee the long-run forecast produced by the model equals the long-run survey prediction. This way, the proposed approach can timely incorporate new survey information into the multi-step-ahead forecast. The method also allows for exogenous variables in the system of equations, as a way to further improve the information set.

An empirical exercise with Brazilian data shows the usefulness of the proposed method using a pre-COVID-19 sample. Indeed, forecasts from the hybrid model, thus incorporating survey expectations about future inflation, tend to prevail over traditional methods at longer horizons, confirming the benefits of using forward-looking information in the forecasting process. The main reason behind this result is that the proposed method entails relevant transformations of the Brazilian economy, occurred in recent years and present in survey expectations, such as monetary policy credibility gains and lower inflation targets, which affect the inflation dynamics.

In turn, the empirical results using a more recent sample, up to August 2022, show larger forecast errors after the pandemic, due to unexpected shocks and outliers in macroeconomic variables, observed not only in Brazil but also worldwide. Altogether, these results offer a valuable contribution to applied macroeconomics, especially with regard to forecasting inflation in Brazil.

Sumário Não Técnico

Neste artigo, exploramos diferentes formas de aperfeiçoar um método econométrico tradicional de previsão da inflação. O objetivo é prever a inflação de preços ao consumidor usando uma abordagem híbrida, que combina um modelo de Vetor Autorregressivo (VAR) com expectativas de pesquisa (*survey*) de consumidores ou analistas profissionais.

A ideia básica consiste em reescrever um modelo VAR em um arcabouço de espaço de estados (*state-space*), que permite a imposição de restrições genéricas nos parâmetros do modelo. Em particular, utilizamos restrições que garantem que a previsão de longo prazo produzida pelo modelo seja igual à previsão de longo prazo do *survey* de expectativas. Dessa forma, a abordagem proposta pode incorporar tempestivamente novas informações do *survey* na previsão multipassos à frente. O método também permite incluir variáveis exógenas no sistema de equações como forma de aperfeiçoar o conjunto de informação.

Um exercício empírico com dados brasileiros mostra a utilidade do método proposto utilizando uma amostra pré-Covid-19. De fato, as previsões do modelo híbrido, que incorporam as expectativas do *survey* sobre a inflação futura, tendem a prevalecer sobre os métodos tradicionais em horizontes mais longos, confirmando os benefícios do uso de informações prospectivas (*forward-looking*) no processo de previsão. A principal razão por trás desse resultado é que o método proposto incorpora transformações relevantes na economia brasileira, ocorridas nos últimos anos e presentes nas expectativas de inflação, tais como ganhos de credibilidade da política monetária e metas de inflação mais baixas, que afetam a dinâmica inflacionária.

Por sua vez, os resultados empíricos utilizando uma amostra mais recente, até agosto de 2022, apresentam maiores erros de previsão após a pandemia, devidos a choques inesperados e *outliers* em variáveis macroeconômicas, observados não apenas no Brasil, mas em todo o mundo. Em suma, esses resultados oferecem uma valiosa contribuição para a macroeconomia aplicada, especialmente no que diz respeito à previsão de inflação no Brasil.

Anchoring Long-term VAR Forecasts Based On Survey Data and State-space Models*

Marta Baltar Moreira Areosa †

Wagner Piazza Gaglianone ‡

Abstract

The objective of this paper is to forecast Brazilian inflation using a hybrid approach that combines a standard Vector Autoregression (VAR) model with expectations from surveys of consumers or professional forecasters. We cast a VAR model with parameter restriction into a state-space setup, where the long-run forecast from the model matches the long-run survey prediction. The proposed method also allows for exogenous variables in the system of equations as a way to enlarge the information set, and is designed to quickly adapt the multi-step-ahead forecasts in response to new survey information. An empirical exercise with Brazilian data illustrates the usefulness of the method. The results using a pre-COVID-19 sample indicate forecasts obtained from the proposed model prevail over traditional methods at longer horizons, thus confirming the benefits of using forward-looking information from survey in the forecasting process. The main reason is that the method incorporates relevant transformations observed in the Brazilian economy in recent years, such as monetary policy credibility gains and lower inflation targets. In turn, the results based on the full sample, up to August 2022, show larger forecast errors after the pandemic, which caused huge outliers in macroeconomic variables worldwide. Altogether, these findings offer a valuable contribution to applied macroeconomics, especially with regard to forecasting inflation in Brazil using VARs and survey data.

Keywords: Inflation; Expectations; Survey Data; Shifting Endpoint; Kalman Filter.

JEL Classification: C22, C53, E17, E31, E37, E52.

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1 Introduction

Reliable macroeconomic forecasts are crucial for economic agents in an ever-changing world. In particular, forecasting how the price level evolves is an essential part of economic planning and decision-making. For instance, higher inflation brings uncertainty to economic agents and shortens investment horizon, especially in emerging countries, making the construction of accurate macroeconomic forecasts an important issue in these economies.

Since the seminal paper of Sims (1980), the Vector Autoregressive (VAR) model has been a workhorse tool in applied macroeconomics, widely used by econometricians for forecasting, structural inference and policy analysis. A VAR model is a set of linear regression equations, describing the evolution of a set of endogenous variables, where each equation treats a given dependent variable as a function of lagged values of all variables considered in the model. Such method can be viewed as a *backward-looking* approach reflecting past information.

On the other hand, information obtained from surveys of consumers or professional forecasters naturally pertains to future prospects for the economy, representing a rich source of *forward-looking* information. Indeed, the usefulness of such data in forecasting has been shown in many empirical studies. For example, Ang et al. (2007) report true out-of-sample survey forecasts in the U.S. (such as Michigan or Livingston) outperform a large number of single-equation and multivariate time series competitors. Faust and Wright (2013) point out inflation subjective forecasts obtained from surveys seem to prevail over model-based forecasts in certain dimensions, often by a wide margin, possibly because forecasters have access to econometric models, but can also add expert judgment to these models.

In fact, in order to improve the forecast accuracy of traditional econometric models, a whole new set of methods emerged, along the past decades, with the objective of adding extra information to traditional VARs. The first attempt was the VARX model, in which an exogenous variable is included in the system of equations.[1] More recently, Krüger et al. (2017) combine medium-term forecasts from Bayesian VARs with short-term forecasts from surveys using entropic tilting.[2] As result, the accuracy of both point and density forecasts is improved, especially for persistent variables.

In a similar approach, Tallman and Zaman (2018) build hybrid forecasts that combine survey information with VAR forecasts. They use relative entropy to tilt one-step ahead and long-run VAR forecasts to match, respectively, the nowcast and long-term forecasts from the *Survey of Professional Forecasters*. The results indicate meaningful gains in multi-horizon forecast accuracy, particularly for inflation, relative to model forecasts that do not incorporate long-run survey conditions.[3]

[1] A VAR process can be affected by other observable variables that are determined outside the system of interest. Such variables are called exogenous (or independent) variables.

[2] Tilting is a technique for modifying a baseline distribution to match moment conditions of interest.

[3] See also Giannone et al. (2019), who propose a class of prior distributions that also discipline the long-run behavior of VAR forecasts. These priors come from economic theory and provide guidance on the joint dynamics of macroeconomic time series in the long run. According to the authors, VARs with standard macro variables and priors based on the long-run predictions of a wide class of theoretical models yield substantial improvements in forecasting performance.

Survey data can also provide immediate information about perceived structural changes in the economy.[4] These perceptions of structural shifts can be useful to improve forecast accuracy. For example, surveys can be used to guide long-run inflation forecasts of VAR models, which are mean-reverting, by design, and often too insensitive to recent inflation. In this sense, Kozicki and Tinsley (2012) propose a shifting endpoint[5] autoregressive (AR) model to approximate the implicit inflation forecasting model that underlies survey expectations.

In this paper, we generalize the shifting endpoint univariate approach of Kozicki and Tinsley (2012) to a multivariate context with exogenous variables. Our goal is to propose an econometric setup that combines VAR and VARX forecasts with survey data, where survey expectations are used to *anchor* long-run forecasts. This way, the multi-step-ahead forecast is driven in the short- and medium-run by the usual VAR and VARX dynamics, but gradually converges to the survey prediction in the long run.

To do so, we rewrite VAR and VARX models in a state-space setup,[6] that embodies not only a VAR model structure with parameter restriction, but also delivers a long-run forecast grounded on *forward-looking* survey data. This way, the approach allows for exogenous variables, and also imposes parameter restriction to ensure the shifting endpoint dynamics driven by survey data. To the best of our knowledge, this is the first paper to address this particular problem.

The advantages of the proposed method are the following: (i) to timely capture structural changes in the economy, as perceived by survey respondents, and translate that into a prompt shift in long-run predictions; and (ii) to enlarge the information set through macroeconomic and financial variables that can be included as exogenous variables in the model. We illustrate the usefulness of the proposed methodology using Brazilian data.

The outline of the paper is as follows. In Section 2, we present our methodology to build VAR and VARX models, imposing the shifting endpoint restriction, within a state-space framework. Section 3 presents an out-of-sample forecasting exercise to predict inflation in Brazil, and Section 4 concludes. The Appendix presents proofs of the propositions as well as examples of the proposed models, besides some additional results.

2 Methodology

Our objective is to improve the accuracy of inflation forecasts, generated by VAR or VARX models, using information from surveys (although other exogenous variables could be used as well to anchor the long-run inflation forecasts, such as breakeven inflation extracted from

[4] Since surveys naturally include judgmental views derived, for instance, from a diverse set of available macroeconomic and financial variables as well as from different econometric models.

[5] According to Kozicki and Tinsley (2012): "*The endpoint is the level to which inflation expectations eventually converge as the forecast horizon is increased, conditional on a given information set... Endpoints may shift according to information and beliefs at the time the forecast is made. The potential for endpoint shifts is an essential feature of the model of expectations as endpoint shifts can accommodate the possibility of rapid reaction to structural change in survey expectations independent of recent movements in actual inflation.*"

[6] Building a VAR model within a state-space setup is not a novelty in the literature. For instance, Mittnik (1989) designs an autoregressive process with exogenous variables in a state-space model. More recently, Dertimanis and Koulocheris (2011) investigate the interconnection between VAR and state-space models.

financial data). To do so, we propose a methodology to estimate those models at monthly frequency within a standard state-space framework. The inflation rate is our variable of interest and is modeled together with other key macroeconomic and/or financial variables.

The main idea is to impose restrictions on model parameters, such that out-of-sample long-run forecasts of inflation can be anchored on survey-based inflation expectations. Since survey data usually report average (long-run) inflation expectations over multiple periods,[7] we assume the twelve-month inflation forecast from our proposed model is a proxy for the respective survey prediction. A similar strategy is adopted in Kozicki and Tinsley (2012). In other words, we impose the so-called *Shifting Endpoint (SE)* restriction in a standard VAR or VARX model, such that the inflation rate forecast accumulated in twelve months matches the survey-based inflation expectation at a desired (long-run) horizon.

The restricted VAR or VARX models are labelled VAR-SE and VARX-SE models, respectively, due to this long-run forecast anchoring.

2.1 VAR with shifting endpoint

In its basic form, a vector autoregression (VAR) model consists of a set of k endogenous variables $x_t = \begin{bmatrix} x_{1,t} & \cdots & x_{k,t} \end{bmatrix}'$. The VAR($p$) process is, then, defined as:

$$x_t = F_1 x_{t-1} + ... + F_p x_{t-p} + \phi + u_t, \tag{1}$$

where F_i are $k \times k$ coefficient matrices for $i = 1, .., p$, ϕ is a $k \times 1$ vector of intercepts, and u_t is a k-dimensional white noise process such that $\mathbb{E}(u_t) = 0$.

As is well known, the VAR(p) can be rewritten as a VAR(1), in the following way:

$$\xi_t = F\xi_{t-1} + c + v_t, \tag{2}$$

where $\xi_t = \begin{bmatrix} x_t' & x_{t-1}' & \cdots & x_{t-p+1}' \end{bmatrix}'$ is a $kp \times 1$ vector stacking all variables (and lags), $F = \begin{bmatrix} F_1 & F_2 & \cdots & F_{p-1} & F_p \\ I_k & 0 & \cdots & 0 & 0 \\ 0 & I_k & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_k & 0 \end{bmatrix}$ is a $kp \times kp$ coefficient matrix, and $c = \begin{bmatrix} \phi' & 0 & \cdots & 0 \end{bmatrix}'$ and $v_t = \begin{bmatrix} u_t' & 0 & \cdots & 0 \end{bmatrix}'$ are $kp \times 1$ vectors.

[7] For example, participants in the Livingston Survey in the U.S. are asked to give 6-month and 12-month forecasts of the CPI level. In turn, the participants in the Survey of Professional Forecasters (SPF) organized by the Federal Reserve Bank of Philadelphia report their long-run projections for inflation in annualized percentage points. The ECB SPF respondents also report their longer-term inflation expectations in annual percentage changes.

Now, consider the VAR model adjusted by *shifting endpoint*, herein called VAR-SE model, which can be written in a standard state-space format, as follows:

$$X_t = AX_{t-1} + B\varepsilon_t, \tag{3}$$

$$Y_t = CX_t + D\epsilon_t, \tag{4}$$

where X_t is the state vector, Y_t is the observable variables vector, ε_t and ϵ_t are vectors of innovations, and A, B, C and D are coefficient matrices.

To do so, let $Y_t = \begin{bmatrix} \mathbf{y}_t' & \mathbf{1}_t & f_{t+H|t} \end{bmatrix}'$, where $\mathbf{y}_t = \begin{bmatrix} y_{1,t} & y_{2,t} & ... & y_{k,t} \end{bmatrix}'$ is a vector with k observable variables, the first being the inflation rate, which is our variable of interest (i.e., $y_{1,t} = \pi_t$), $\mathbf{1}_t$ is equal to 1 for all t, and $f_{t+H|t}$ is the consensus survey-based inflation expectation (i.e., cross-section average of the individual expectations) for the annualized inflation rate H periods ahead. Also, let $X_t = \begin{bmatrix} \xi_t' & c_t & \mu_t \end{bmatrix}'$,[8] where c_t is a constant state associated with the intercept of the equations describing the dynamics of the observable variables, except for the inflation equation, and the random walk $\mu_t = \mu_{t-1} + \varepsilon_{\mu,t}$ replaces the intercept.[9]

Next, consider the so-called *Shifting Endpoint (SE) restriction*, which is an equation that connects the survey inflation expectation $f_{t+H|t}$ with the long-run forecast from the VAR model. We guarantee such relationship holds by imposing a set of constraints in the VAR parameters. The SE restriction is defined as follows:

$$f_{t+H|t} = \sum_{h=H-11}^{H} \mathbb{E}[\pi_{t+h} \mid \mathcal{F}_t], \tag{5}$$

where $\mathbb{E}(\cdot)$ denotes the conditional expectation from the VAR-SE model,[10] and $\mathcal{F}_t$ is the information set at period t. As will be later discussed in Proposition 1, this restriction can be rewritten as $f_{t+H|t} = \beta X_t$, where $\beta = J_1 \sum_{h=H-11}^{H} A^h$, which completes the VAR-SE model description.[11] This way, the survey-based inflation expectation perfectly matches the forecast from the VAR-SE model for the annualized inflation H periods ahead.[12] Next, we

[8] The model assumptions, later presented, will impose the first k states of ξ_t to be equal to the observable variables ($x_t = y_t$), whereas the remaining states are their respective lags.

[9] Kozicki and Tinsley (2012) point out that: *"In thinking about the dynamics of such long-horizon perceptions, note that if survey participants could forecast future changes to their perceptions of the level at which inflation would stabilize, then such changes would be immediately incorporated in their long-run perceptions. Consequently, changes in the endpoint should not be forecastable. This property is captured by assuming that the endpoint evolves according to a random walk."*

[10] Note the sum in equation (5) has 12 terms, which is due to our interest here in estimating the VAR-SE model in monthly frequency. For other frequencies, the VAR-SE model setup could be adjusted accordingly. The horizon H is a choice of the econometrician but also depends on the maximum horizon available in the survey.

[11] The vector J_1 is part of a family of selection vectors such that, for all $i = 1, ..., k$, J_i corresponds to the i-th row of an identity matrix of dimension $kp + 2$. These relationships are obtained by imposing appropriate constraints on the coefficient matrices A, B, C and D.

[12] Additional restrictions could be further considered in the model. For instance, matching the survey expectation with the model forecast at different horizons (short- or medium-run) or even considering the shifting endpoint restriction for other endogenous variables besides inflation. We leave this potential route as suggestion for future extensions of the proposed setup.

show some technical assumptions to write the VAR-SE model in a state-space setup:

Assumption A1 Let $Y_t = \begin{bmatrix} \mathbf{y}_t' & \mathbf{1}_t & f_{t+H|t} \end{bmatrix}'$ be a $(k+2) \times 1$ vector, such that $\mathbf{y}_t = \begin{bmatrix} \pi_t & y_{2,t} & \ldots & y_{k,t} \end{bmatrix}'$ and $f_{t+H|t}$ are covariance-stationary processes.

Assumption A2 Let $X_t = \begin{bmatrix} \xi_t' & c_t & \mu_t \end{bmatrix}'$ be a $(kp+2) \times 1$ vector, where $c_t = 1$ for all t, and μ_t is a random walk process.

Assumption A3 Let $A = \begin{bmatrix} F_1 & F_2 & \cdots & F_{p-1} & F_p & \phi & \psi \\ I_k & 0_k & \cdots & 0_k & 0_k & 0_{k\times 1} & 0_{k\times 1} \\ 0_k & I_k & \cdots & 0_k & 0_k & 0_{k\times 1} & 0_{k\times 1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0_k & 0_k & \cdots & I_k & 0_k & 0_{k\times 1} & 0_{k\times 1} \\ 0_{1\times k} & 0_{1\times k} & \cdots & 0_{1\times k} & 0_{1\times k} & 1 & 0 \\ 0_{1\times k} & 0_{1\times k} & \cdots & 0_{1\times k} & 0_{1\times k} & 0 & 1 \end{bmatrix}$ be a $(kp+2) \times (kp+2)$ matrix, where $\phi = [\ 0 \quad \phi_2 \quad \cdots \quad \phi_k\ ]'$ and $\psi = [\ 1 \quad 0 \quad \cdots \quad 0\ ]'$ are $k \times 1$ vectors.

Assumption A4 Let B be a $(kp+2) \times (kp+2)$ matrix with zeros, excepting the following elements (which are equal to one): (i) the i-th diagonal elements for $i = 1, ..., k$; and (ii) the $(kp+2)$-th diagonal element.

Assumption A5 Let $C = \begin{bmatrix} J_1' & \cdots & J_k' & \overline{J}' & \beta' \end{bmatrix}'$ be a $(k+2) \times (kp+2)$ matrix, where J_i is a $1 \times (kp+2)$ selection-vector filled with zeros, excepting the i-th element (which is equal to one), $\overline{J}$ is a $1 \times (kp+2)$ selection-vector filled with zeros, excepting the $(kp+1)$-th element, which is equal to one, and $\beta = J_1 \sum\limits_{h=H-11}^{H} A^h$ is a $1 \times (kp+2)$ row vector.

Assumption A6 Let D be a $(k+2) \times (k+2)$ matrix with zeros.

Assumption A7 Let $\varepsilon_t = \begin{bmatrix} v_t' & 0 & \varepsilon_{\mu,t} \end{bmatrix}'$ be a $(kp+2) \times 1$ vector of innovations, where $v_t = \begin{bmatrix} u_t' & 0 & \cdots & 0 \end{bmatrix}'$ is a $kp \times 1$ vector, and let ϵ_t be a $(k+2) \times 1$ vector of innovations. Assume both ε_t and ϵ_t are normally distributed.

Now, we provide our first theoretical result that presents the VAR-SE model in a state-space format and, most importantly, shows the shifting endpoint restriction (5) indeed holds. See Appendix B for an example of a simple VAR-SE model.

Proposition 1 *Consider a VAR model with p lags and k endogenous variables, such that the inflation rate π_t is the first variable of the VAR. Under Assumptions A1-A7, it follows that: (i) the VAR model can be rewritten in a state-space setup, as follows:*

$$\begin{aligned} X_t &= AX_{t-1} + B\varepsilon_t, \\ Y_t &= CX_t + D\epsilon_t, \end{aligned}$$

where X_t is a vector of latent states, Y_t contains observable variables, A, B, C and D are coefficient matrices, and $[\varepsilon_t, \epsilon_t]$ are innovations; and (ii) the Shifting Endpoint (SE) restriction holds, that is, $f_{t+H|t} = \sum_{h=H-11}^{H} \mathbb{E}[\pi_{t+h} \mid \mathcal{F}_t]$, where $\mathbb{E}(\cdot)$ denotes the conditional expectation from the restricted VAR, labelled VAR-SE model.

2.2 VARX with shifting endpoint

The vector autoregression model with exogenous variables (VARX) consists of a set of k *endogenous* variables $x_t = \begin{bmatrix} x_{1,t} & \cdots & x_{k,t} \end{bmatrix}'$ and a set of m *exogenous* (or independent) variables $z_t = \begin{bmatrix} z_{1,t} & \cdots & z_{m,t} \end{bmatrix}'$. Note z_t can naturally include lags of the exogenous variables as well.[13] The VARX(p) process is thus defined as:

$$x_t = F_1 x_{t-1} + ... + F_p x_{t-p} + \phi + \theta z_t + u_t, \tag{6}$$

where θ is a $k \times m$ matrix of coefficients associated with the exogenous variables z_t, and the other terms are the same discussed in the previous section.

The VARX(p) can be rewritten as a VARX(1), as follows:

$$\xi_t = F\xi_{t-1} + c + \delta z_t + v_t, \tag{7}$$

where F is a $kp \times kp$ matrix, c is a $kp \times 1$ vector, and $\delta = \begin{bmatrix} \theta_{k\times m} \\ 0_{(kp-k)\times m} \end{bmatrix}$ is a $kp \times m$ matrix. Successive substitutions for lagged ξ_t's give the following expression:[14]

$$\xi_{t+h} = F^h \xi_t + \sum_{i=0}^{h-1} F^i \left(c + \delta z_{t+h-i} + v_{t+h-i}\right). \tag{8}$$

Take the conditional expectation on both sides of the previous expression, as follows:

$$\mathbb{E}\left(\xi_{t+h} \mid \mathcal{F}_t\right) = F^h \xi_t + \sum_{i=0}^{h-1} F^i \left(c + \delta z_{t+h-i}\right). \tag{9}$$

Note that in order to predict the future values of the *endogenous* variables ξ_{t+h}, besides the model parameters present in matrices F, c and δ, the econometrician must also know the future paths of the *exogenous* variables, that is, from $t+1$ up to $t+h$. This way, define $z_t^* = \begin{bmatrix} z'_{t+1} & \cdots & z'_{t+H} \end{bmatrix}'$ as a $Hm \times 1$ vector, which stacks a finite set of future trajectories of z_t.

Now, we discuss the VARX model adjusted by *shifting endpoint*, herein labelled VARX-

[13] For instance, one might define $z_t = [z_{1,t}; z_{1,t-1}; \ldots; z_{1,t-q_1}; z_{2,t}; z_{2,t-1}; \ldots; z_{2,t-q_2}; \ldots; z_{m,t-q_m}]'$.

[14] See Baillie (1980) for a discussion on building forecasts using ARMAX models, and Lütkepohl (2005, p.403) using VARX models.

SE model, written in a state-space setup, as follows:

$$X_t^* = A^* X_{t-1}^* + B^* \varepsilon_t^*, \tag{10}$$

$$Y_t^* = C^* X_t^* + D^* \epsilon_t^*, \tag{11}$$

where X_t^* is the state vector, Y_t^* is the observable variables vector, ε_t^* and ϵ_t^* are vectors of innovations, and A^*, B^*, C^* and D^* are coefficient matrices. Next, consider the following technical assumptions:

Assumption B1 Let $Y_t^* = \begin{bmatrix} Y_t' & z_t^{*\prime} \end{bmatrix}'$ be a $(k+2+Hm)\times 1$ vector, such that Y_t and z_t^* are covariance-stationary processes.

Assumption B2 Let $X_t^* = \begin{bmatrix} X_t' & x_t^{*\prime} \end{bmatrix}'$ be a $(kp+2+Hm) \times 1$ vector, where $x_t^* = \begin{bmatrix} x_{1,t}^{*\prime} & \cdots & x_{H,t}^{*\prime} \end{bmatrix}'$ is a $Hm \times 1$ vector of states.

Assumption B3 Let $A^* = \begin{bmatrix} A_{(kp+2)\times(kp+2)} & \theta^*_{(kp+2)\times m} & 0_{(kp+2)\times(H-1)m} \\ 0_{(H-1)m\times(kp+2)} & 0_{(H-1)m\times m} & I_{(H-1)m} \\ 0_{m\times(kp+2)} & 0_{m\times m} & 0_{m\times(H-1)m} \end{bmatrix}$ be a $(kp+2+Hm)$ matrix, where $\theta^*_{(kp+2)\times m} = \begin{bmatrix} \theta_{k\times m} \\ 0_{(kp+2-k)\times m} \end{bmatrix}$.

Assumption B4 Let B^* be a $(kp+2+Hm)$ matrix with zeros, excepting the following elements (which are equal to one): (i) the i-th diagonal elements for $i = 1, ..., k$; (ii) the $(kp+2)$-th diagonal element; and (iii) the $(kp+2+(H-1)\,m+l)$-th diagonal elements for $l = 1, ..., m$.

Assumption B5 Let $C^* = \begin{bmatrix} J_1^{*\prime} & \cdots & J_k^{*\prime} & \overline{J^*}' & \beta^{*\prime} & \gamma' \end{bmatrix}'$ be a $(k+2+Hm)\times(kp+2+Hm)$ matrix, where J_i^* is a $1\times(kp+2+Hm)$ selection-vector filled with zeros, excepting the i-th element (which is equal to one), $\overline{J^*}$ is a $1\times(kp+2+Hm)$ selection-vector filled with zeros, excepting the $(kp+1)$-th element (equal to one). Let $\beta^* = J_1^* \sum_{h=H-11}^{H} A^{*h}$ be a $1\times(kp+2+Hm)$ vector. Also, let $\gamma = \begin{bmatrix} 0_{Hm\times(kp+2)} & I_{Hm} \end{bmatrix}$ be a $Hm\times(kp+2+Hm)$ matrix.

Assumption B6 Let D^* be a $(k+2+Hm)$ matrix with zeros.

Assumption B7 Let $\varepsilon_t^* = \begin{bmatrix} \varepsilon_{t\ (kp+2)\times 1} \\ 0_{(H-1)m\times 1} \\ \varepsilon_{z,t\ m\times 1} \end{bmatrix}$ be a $(kp+2+Hm)\times 1$ vector of innovations, and let $\epsilon_t^* = \begin{bmatrix} \epsilon_{t\ (k+2)\times 1} \\ 0_{Hm\times 1} \end{bmatrix}$ be a $(k+2+Hm)\times 1$ vector of innovations. Assume both ε_t^* and ϵ_t^* are normally distributed.

Next, we show our second theoretical result that presents the VARX-SE model casted in a state-space format and, more importantly, ensures the shifting endpoint restriction (5) is valid when building forecasts using a VARX(p) model. See Appendix B for an example of a simple VARX-SE model.

Proposition 2 *Consider the VARX model (6) with p lags, k endogenous variables (such that the inflation rate π_t is the first variable), and m exogenous variables. Under Assumptions B1-B7, it follows that: (i) the VARX model can be rewritten in a state-space setup, as follows:*

$$\begin{aligned} X_t^* &= A^* X_{t-1}^* + B^* \varepsilon_t^* \\ Y_t^* &= C^* X_t^* + D^* \epsilon_t^* \end{aligned}$$

where X_t^* is a vector of latent states, Y_t^* contains observable variables, A^*, B^*, C^* and D^* are coefficient matrices and $[\varepsilon_t^*, \epsilon_t^*]$ are innovations; and (ii) the Shifting Endpoint (SE) restriction holds, that is, $f_{t+H|t} = \sum_{h=H-11}^{H} \mathbb{E}^*[\pi_{t+h} \mid \mathcal{F}_t]$, where $\mathbb{E}^*(\cdot)$ denotes the conditional expectation from the restricted VARX, labelled VARX-SE model.

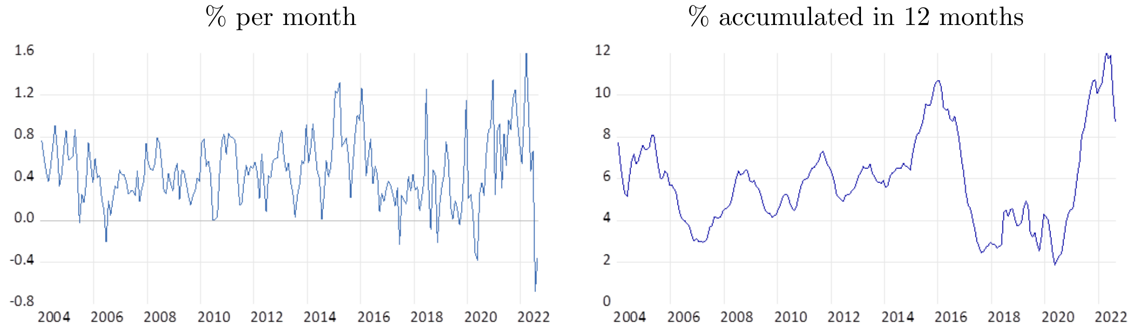
3 Empirical Exercise

3.1 Data

Our goal is to forecast the Brazilian monthly inflation, as measured by the Extended National Consumer Price Index (IPCA), published by the Brazilian Institute of Geography and Statistics (IBGE). The IPCA inflation (see Figure 1) is used as the official inflation measure and the target of monetary policy in Brazil.

The sample period spans over roughly 20 years of data, from January 2003 to August 2022 ($T = 236$ observations). The sample starts almost a decade after the Brazilian monetary stabilization plan in mid-1994.[15]

Figure 1 - Inflation rate (IPCA)



[15] The limited sample size is a well-known constraint in Brazilian applied macroeconomic studies, particularly focused on inflation dynamics, where different policy regimes have been the case in past decades. This way, by starting the sample in 2003 we avoid large structural breaks (see Machado and Portugal, 2014).

From a theoretical standpoint, numerous models with information rigidities have been proposed in the literature along the past decades to explain the *inflation dynamics* and the expectations formation process.[16] Two prominent approaches are the sticky-information approach of Mankiw and Reis (2002) and the noisy-information model of Sims (2003).[17]

From an empirical perspective, one of the main drivers of the inflation dynamics observed in emerging economies is the inflationary inertia (or degree of persistence).[18] Besides past inflation, other predictors suggested in the empirical literature to forecast inflation usually include economic slack measures (e.g., recall the Phillips curve), measures of real aggregate activity other than unemployment (Stock and Watson, 1999[19]), financial variables[20] (Forni et al., 2003), and surveys of inflation expectations (Ang et al., 2007; Faust and Wright, 2013), among others.[21] See Araujo and Gaglianone (2022) for a recent discussion on variable importance and top predictors of inflation in Brazil, extracted from a large dataset of macroeconomic and financial variables. According to these authors, the set of variables selected by machine learning methods to predict monthly inflation (e.g., elastic net and adaptive lasso) often includes past inflation (inertial inflationary dynamics) and variables related to the real economy (e.g., commercial electricity consumption). Regarding the 12-month inflation rate, the set of most frequent variables also includes interest rates, fiscal variables, external sector variables (current account, dollar index, oil price, CRB), besides some variables not traditionally used to forecast inflation, such as the temperature of the Pacific Ocean, due to the role that the El Niño and La Niña might play in food inflation. In this paper, we use the (ad hoc) set of macroeconomic and financial variables presented in Table 1 to explain the inflation dynamics in Brazil.

[16] Because many features of observed inflation expectations (e.g., disagreement and predictable forecast errors) are not compatible with the standard rational expectations model based on perfect information.

[17] See also Coibion and Gorodnichenko (2015), and Coibion et al. (2018) for an interesting discussion on inflation, information rigidity and the expectations formation process. More recently, Areosa et al. (2020) depart from the sticky-information model by allowing information to be also dispersed, whereas Areosa et al. (2021) use a similar approach to analyze how price setting changes when the interest rate is understood as a public signal that informs the view of the monetary authority on the current state of the economy.

[18] In Brazil, the relevance of past inflation has been vastly documented in applied studies. For instance, see Kohlscheen (2012), and Gaglianone, Guillén and Figueiredo (2018).

[19] In particular, an index of aggregate activity in the U.S. based on 168 economic indicators.

[20] An alternative to address the information deficiency problem of the econometric model is to include in the VAR a forward-looking series of asset prices, such as the Ibovespa stock price index. If financial markets are efficient, this series would incorporate relevant information available to agents and its inclusion would help to align the econometrician's information set with that of the agents (Kilian and Lütkepohl, 2017).

[21] Areosa and Areosa (2014) investigate the impact of a mega sports event (e.g., FIFA World Cup 2014 and the 2016 Olympic Games) in consumer inflation, and conclude the impact of such events on the Brazilian inflation (IPCA) is limited and transitory.

Table 1 - Macroeconomic and financial variables

Name	Description	Source	Unit	Nickname	Transform.
IPCA	consumer price index	IBGE	% p.m.	ipca	-
IPC-Fipe	consumer price index	Fipe	% p.m.	ipc-fipe	-
IGP-DI	general price index	FGV	% p.m.	igp-di	-
FX-rate	nominal exchange rate, R$/US$	Reuters	Units	fx-rate	$\Delta \ln(.)$
Interest rate	real interest rate (IPCA bond, 1 year)	Anbima	% p.a.	int-rate	$\Delta(.)$
IBC-BR	central bank economic activity index	BCB	Index	ibc-br	HP filter
Ibovespa	Brazilian stock exchange index	Reuters	Index	ibovespa	$\Delta \ln(.)$
Brent	oil price (Brent, Europe)	Reuters	US$/barrel	brent	$\Delta \ln(.)$
CRB	CRB all commodities index	Reuters	Index	crb	$\Delta \ln(.)$
Dollar index	U.S. dollar index (DXY)	Reuters	Index	dxy	$\Delta \ln(.)$
U.S. interest rate	U.S. Treasury 3-month yield	Reuters	% p.a.	t-bill	$\Delta(.)$
Energy	commercial consumption of electricity	Eletrobras	GWh	energy	$\Delta \ln(.)$

Note the set of variables includes past inflation (IPCA, IPC-Fipe, IGP-DI), exchange rate (which is a central variable regarding the pass-through of imported inflation to domestic inflation), output gap[22] and real interest rate (both series representing the traditional channel of monetary policy based on aggregate demand), besides some usual sources of macroeconomic shocks (e.g., commodity prices and foreign interest rate). Based on this monthly dataset, we first estimate ten different VAR and VARX models, without imposing the shifting endpoint (SE) restriction, and considering $p = 1$ lag.[23] Table 2 summarizes the model specifications. Note they have from 4 to 6 endogenous variables, and from 0 to 11 exogenous variables.

Table 2 - VAR and VARX models

Model	Endogenous	Exogenous
1	ipca, fx-rate, int-rate, ibc-br	-
2	ipca, fx-rate, int-rate, ibc-br, ipc-fipe, igp-di	-
3	ipca, fx-rate, int-rate, ibc-br, ibovespa	-
4	ipca, fx-rate, int-rate, ibc-br	brent
5	ipca, fx-rate, int-rate, ibc-br	crb
6	ipca, fx-rate, int-rate, ibc-br	dxy
7	ipca, fx-rate, int-rate, ibc-br	t-bill
8	ipca, fx-rate, int-rate, ibc-br	energy
9	ipca, fx-rate, int-rate, ibc-br	brent, crb, dxy, t-bill, energy
10	ipca, fx-rate, int-rate, ibc-br	seasonal dummies

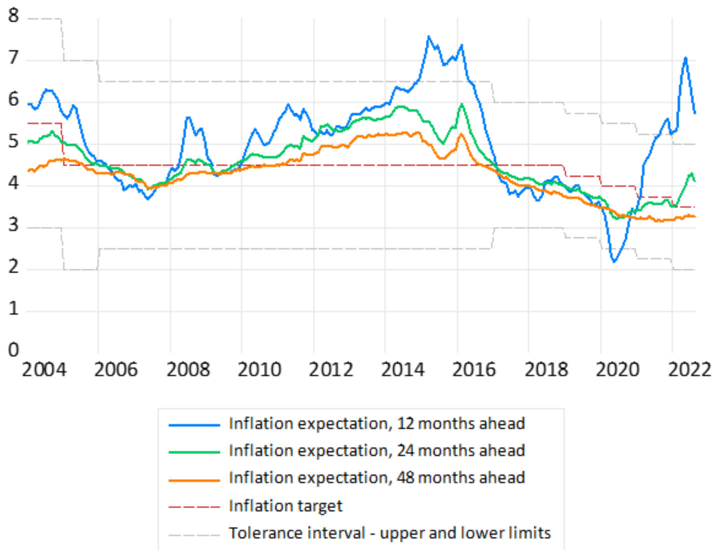
[22] Proxied by the HP-filtered IBC-BR series.

[23] According to the Schwarz information criterion and diagnostic testing, considering the maximum of 8 lags in the tests, $p = 1$ is the optimal lag for model 1 (VAR without exogenous variables) with no SE restriction. Future research could consider an optimal lag for each model specification.

Next, in order to estimate the VAR-SE and VARX-SE models, using the state-space approach discussed in previous sections, we must choose a proxy for long-term inflation anchoring. To do so, we use inflation expectations from the *Focus* survey of professional forecasts, which is a panel database put together by the Banco Central do Brasil. The survey covers more than 100 professional forecasters (e.g., banks, asset management firms, consulting firms and relevant non-financial institutions), which are followed throughout time with a reasonable turnover. Also, the forecasts are supplied over different horizons and for a large array of macroeconomic series; see Gaglianone et al. (2022) for further details.

In our empirical exercise, we use the (consensus) average of individual IPCA inflation forecasts across all survey participants, considering a fixed forecast horizon of 4 years ($H =$ 48 months). For comparison purposes, we also consider $H = 12$ or 24 months. These three measures based on fixed forecast horizons are displayed in Figure 2, together with the inflation target, and are computed from linear interpolation of calendar end-of-year (Focus survey) inflation forecasts.

Figure 2 - Inflation target and Focus survey inflation expectations (% accum. in 12 months)



Note the consensus inflation forecast four years ahead (orange line in Figure 2) fluctuates around 4.5% per year (p.y.) after 2004, achieves its highest value of 5.3% p.y. by the end of 2014, and gradually declines toward 3.0% p.y. since 2016. Possible reasons for this recent decline observed in long-term inflation expectations are central bank credibility gains obtained in recent years[24] coupled with lower inflation targets set after 2018, as shown in Table 3.[25]

[24] For instance, see Val et al. (2017), Issler and Soares (2019), and Oliveira and Gaglianone (2020).
[25] Source: https://www.bcb.gov.br/en/monetarypolicy/historicalpath

Table 3 - Inflation targeting track record (since 2016)

Year	Target (%)	Tolerance Interval (± p.p.)	Actual Inflation (IPCA % p.y.)
2016	4.50	2.00	6.29
2017	4.50	1.50	2.95
2018	4.50	1.50	3.75
2019	4.25	1.50	4.31
2020	4.00	1.50	4.52
2021	3.75	1.50	10.06
2022	3.50	1.50	-
2023	3.25	1.50	-
2024	3.00	1.50	-
2025	3.00	1.50	-

This way, using the three different measures of inflation anchoring discussed above, we estimate three additional versions of the ten models presented in Table 2. Hereafter, we label SE-12, SE-24 and SE-48 the VAR (or VARX) models imposing the shifting endpoint restriction, and using the Focus survey (interpolated) forecast of inflation 12, 24 and 48 months ahead, respectively. In these cases, model parameters are estimated using a Kalman filter with maximum likelihood estimation.

In total, we estimate an amount of 40 models: 10 models without shifting endpoint, besides 3×10 models with the SE restriction as described above. Based on these 40 models, we conduct an out-of-sample forecasting exercise, with forecast horizon (h) ranging from 1 to 48 months. Following the usual procedure, the first part of the sample ($t = 1, ..., T_1$) is used to estimate the models, whereas the second part of the sample ($t = T_1 + 1, ..., T_2$) is reserved for genuine out-of-sample forecast comparison.

All models are re-estimated every month in a *recursive* estimation scheme (i.e., expanding sample size), as we incorporate every new time-series observation, one at a time. In this context, each model is initially estimated using the first T_1 observations and the out-of-sample point forecasts are generated. Then, we add an additional observation at the end of the estimation sample, re-estimate all models, and generate again out-of-sample forecasts. This process is repeated along the remaining data.

Regarding the exogenous variables, a *random walk* approach is used to set the future path of such variables[26] beyond the considered estimation sample ($t = 1, ..., T_1$), that is, $\mathbb{E}\left(z_{T_1+h} - z_{T_1} \mid \mathcal{F}_{T_1}\right) = 0$, for all $h = 1, ..., H$.

For the sake of completeness, we also considered the case of *perfect foresight*, where inflation forecasts are built using the *actual* figures of the exogenous variables.

As a final robustness check, we estimate all models considering the full sample as well as a pre-COVID-19 sample, to disregard the more recent macroeconomic shocks due to the impact of the pandemic in Brazil since March 2020.

[26] After considering the transformations described in last column of Table 1.

3.2 Results

In this section, we present the results of the out-of-sample forecasting exercise. Figures 3 and 4 show monthly inflation rates observed up to December 2015, together with inflation forecasts of selected models, with forecast horizons from 1 up to 48 months. Note the difference in long-run forecasts between models without shifting endpoint compared to those models imposing such long-term restriction.

For instance, models without SE produce long-run monthly inflation forecasts (Figure 3) around 0.48% per month (close to 0.49% p.m., which is the sample average of inflation up to December 2015), whereas the same models now imposing the SE restriction produce long-run forecasts of 0.40% p.m. The long-run forecasts accumulated in 12 months (Figure 4) for the same models without SE and imposing SE are, respectively, around 5.90% and 4.87% per year. As expected, the inflation anchoring in these cases (i.e., Focus inflation expectation, 48 months ahead, available in December 2015) is exactly 4.87% p.y.

Figure 3 - Inflation and out-of-sample forecasts (% p.m.)

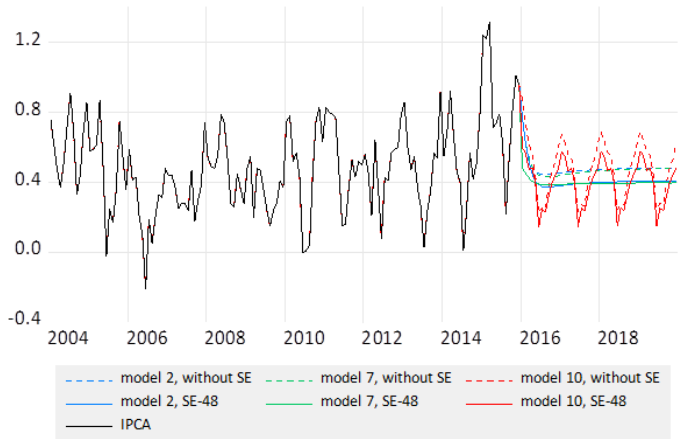
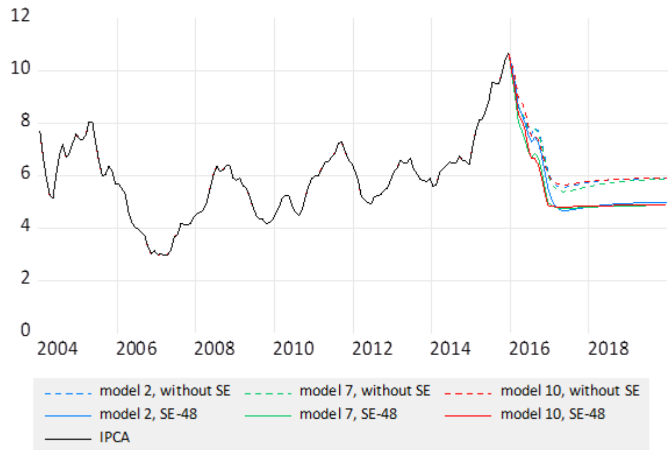


Figure 4 - Inflation and out-of-sample forecasts (% accum. in 12 months)



Figures 5 and 6 show, for illustrative purposes, the IPCA inflation and respective forecasts of model 2, estimated with sample ending in different periods, corresponding to the months of December in 2013, 2014, 2015, 2016, 2017 and 2018. It is worth noting in Figure 5 that, in all cases, the long-term projections are very close to 0.47% p.m. level, illustrating the difficulty of the traditional VAR modeling in generating long-term forecasts that capture recent structural changes in the dynamics of inflation.

In contrast, in the case of the VAR-SE model that imposes the SE restriction, Figure 6 shows the long-term projections fall as the sample incorporates more recent periods, characterized by the gradual decrease in the long-term inflation expectations from the Focus survey, as shown in Figure 2.

Figure 5 - IPCA (% p.m.) and forecasts from a VAR model, without shifting endpoint (SE)

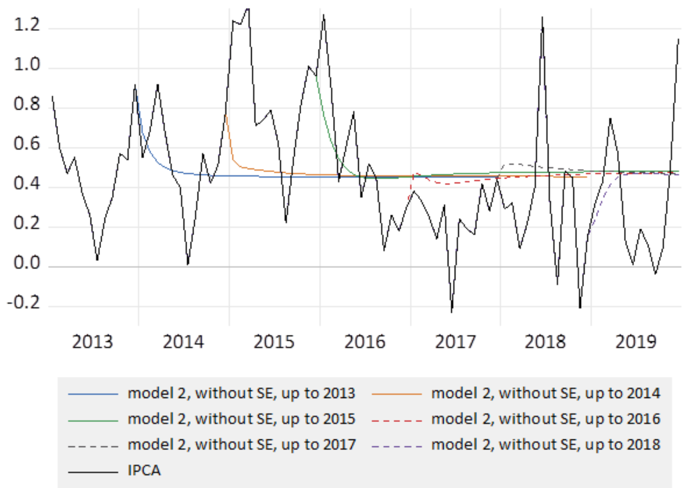
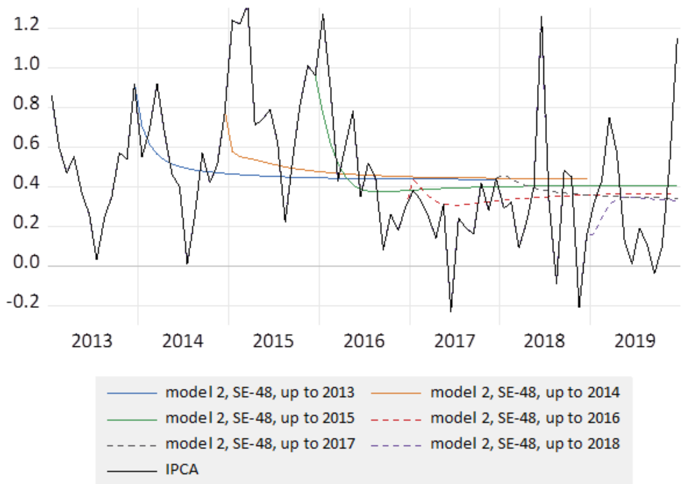


Figure 6 - IPCA (% p.m.) and forecasts from a VAR-SE model



Note: SE-48 denotes the shifting endpoint restriction, based on 48 months ahead inflation expectation.

Table 4 presents the accuracy results of the forecasting exercise in terms of mean squared error (MSE) for selected forecast horizons.[27] For each horizon h, we compare the results of the traditional VAR or VARX model (i.e., without the SE restriction) with the same specification imposing the SE-48 restriction (i.e., using the Focus survey forecast of inflation 48 months ahead as long-run anchoring). In addition, we present in the last column (SE-PF) the results of the same SE restriction but together with a *perfect foresight* assumption for the exogenous variables. The yellow cells represent the models with the lowest MSEs in a given horizon, whereas the green cells denote the highest MSEs. To save space here, we only show the results for 3 model specifications. In Appendix C, we present the results of all models described in Table 2 considering different proxies of inflation anchoring as well (i.e., SE-12, SE-24, and SE-48).

Table 4 - Mean Squared Error (MSE)

Panel A: Pre-COVID-19 sample

$h = 1$

Model	Without SE	SE - Focus	SE - PF
1	0.078	0.077	0.077
5	0.078	0.131	0.124
10	0.081	0.136	0.126

$h = 6$

Model	Without SE	SE - Focus	SE - PF
1	0.119	0.112 *	0.112 *
5	0.119	0.130	0.125
10	0.107	0.141	0.120

$h = 12$

Model	Without SE	SE - Focus	SE - PF
1	0.122	0.111 **	0.111 **
5	0.118	0.134 *	0.129
10	0.114	0.137	0.127

$h = 48$

Model	Without SE	SE - Focus	SE - PF
1	0.109	0.102	0.102
5	0.101	0.091 ***	0.092
10	0.121	0.091 ***	0.095 ***

Panel B: Full sample

$h = 1$

Model	Without SE	SE - Focus	SE - PF
1	0.113	0.116	0.116
5	0.115	0.221	0.217
10	0.115	0.217	0.216

$h = 6$

Model	Without SE	SE - Focus	SE - PF
1	0.179	0.196	0.196
5	0.186	0.191	0.188
10	0.163	0.186	0.182

$h = 12$

Model	Without SE	SE - Focus	SE - PF
1	0.178	0.194	0.194
5	0.180	0.194	0.189 *
10	0.167	0.190	0.190

$h = 48$

Model	Without SE	SE - Focus	SE - PF
1	0.197	0.214	0.214
5	0.195	0.189 *	0.192
10	0.196	0.189	0.184 **

Notes: Pre-COVID-19 evaluation sample ranges from Feb/2013 to Feb/2020 (85 observations for h=1 and 38 for h=48). Evaluation based on full sample ranges from Feb/2013 to Aug/2022 (115 observations for h=1 and 68 for h=48). ***, ** and * indicate rejection at the 1%, 5% and 10% levels, respectively, using the test of Clark and West (2007). In each row, the model in the first column is the benchmark. Yellow cells indicate the lowest MSEs in a given horizon h and green cells indicate the highest MSEs.

[27] Future research could investigate forecast bias (mean forecast error), besides considering other accuracy measures, such as the mean absolute error (MAE).

In Panel A of Table 4, first note the yellow cells, in general, belong to unrestricted models in the short-run ($h = 1$), whereas in the long-run ($h = 48$) they tend to belong to restricted models (SE-Focus or SE-PF), thus confirming the efficacy of the shifting endpoint approach to improve VAR forecasts at longer horizons. Furthermore, note the SE restriction indeed improved the forecast accuracy of model 1 (VAR without exogenous variables), in all considered horizons, when compared to the same model without the SE restriction. Regarding the use of exogenous variables (e.g., seasonal dummies), note they seem to improve the forecast accuracy, in some cases, considering the pre-COVID-19 sample. In turn, the forecasting exercise considering a *perfect foresight* assumption to generate the future path of exogenous variables along the out-of-sample exercise (instead of the *random walk* assumption) confirms the previous findings besides delivering (as expected) lower MSEs in many cases.[28] Finally, note the predictive capacity comparison test of Clark and West (2007) statistically confirms the superiority of the VARX-SE models for $h = 48$ (excepting model 5, with SE-PF), when compared to the VARX counterparts without SE.

Panel B of Table 4 shows the same results now considering the full sample, that is, up to August 2022. First note the cells exhibit much higher figures compared to the ones from Panel A. This results comes from the fact that more recent observations (i.e., after the COVID-19 impact in Brazil) are characterized by large forecast errors, due to sizeable unanticipated inflationary shocks. Once again, the empirical results indicate that the SE restriction helps improving the forecast accuracy at longer horizons, whereas the unrestricted models do a better job at shorter horizons.[29, 30] Considering the results of the full sample for all model specifications presented in Appendix C (Tables C.2 and C.4), note the models without SE usually provide lower MSEs compared to their SE counterparts. One explanation is that the long-term anchoring, pointing to lower inflation levels (around 3% p.y.), has become temporarily ineffective during this recent *high-uncertainty* regime after the COVID-19 pandemic, in which inflation rate outliers prevail.[31] Nonetheless, long-term anchoring proxies (properly reflecting the new macroeconomic environment) will likely restore the usefulness of SE models in the near future.

[28] For horizons up to one year, and considering an inflation anchoring of 24 or 48 months ahead (SE-24 or SE-48), the *perfect foresight* exercise (Table C.3) delivered lower MSEs compared to the *random walk* approach (Table C.1) for models 4 to 10 in the majority of cases. In turn, model 8 with no anchoring (without SE) reduced their MSEs in all considered horizons.

[29] When considering the full sample, some exogenous variables do improve the forecast accuracy compared to the models only based on endogenous variables. Excepting the case for $h = 1$, the orange cells in Tables C.2 and C.4 always belong to model specifications with exogenous variables (such as, models 5, 8 and 10).

[30] Again, the *perfect foresight* approach delivered lower MSEs, in several cases, compared to the *random walk* approach. For horizons up to one year, and considering no inflation anchoring (without SE), the *perfect foresight* exercise (Table C.4) delivered lower MSEs compared to the *random walk* approach (Table C.2) for models 4 and 5, in all cases. In turn, for horizons up to two years, and considering an inflation anchoring of 48 months (SE-48), the *perfect foresight* produced lower MSEs, compared to the *random walk*, for models 4 to 10, in several cases.

[31] Considering post-COVID-19 observations, inflation consistently surprised to the upside (i.e., inflation was generally higher than forecasts). Because the unconstrained models have an upward bias (as illustrated in Figures 3 to 6), they tend to perform relatively better in the full sample compared to the constrained ones, but for the wrong reasons.

It is worth mentioning our results are in line with a relatively recent discussion in applied macroeconomics about model estimation after the COVID-19 pandemic. According to Frank Diebold[32] (in his blog entry of July 27, 2020): *"The point is that the pandemic recession is in many respects a massive outlier, so that one has to think hard about what to do with it in estimation. That is, as always one wants to fit signal, not noise, and the pandemic recession is in certain respects a massive burst of noise, capable of severely distorting parameter estimates and hence forecasts and nowcasts."*

In the same way, Lenza and Primiceri (2022) report that: *"The COVID-19 pandemic is causing unprecedented variation in many key macroeconomic variables."* According to the authors, the usual (ad hoc) strategy of dropping the observations corresponding to the pandemic period may be acceptable for the purpose of parameter VAR estimation. However, the authors point out that disregarding such recent data can be inappropriate for forecasting purposes, since it vastly underestimates uncertainty.

In the context of Bayesian VARs, Cascaldi-Garcia (2022) reports a very low number of extreme COVID-19 pandemic observations bias the estimated persistence of macroeconomic variables, affecting forecasts and giving a myopic view of the economic effects after a structural shock. To deal with these extreme episodes, the author proposes the so-called *Pandemic Priors* as an extension of the Minnesota Prior with time dummies.[33]

4 Conclusions

This paper proposes an econometric model that allows *anchoring* the long-term VAR and VARX forecasts using survey data, thus promptly incorporating structural changes in the economy as perceived by survey respondents.

Long-run forecasts of VAR models, by design, converge to the unconditional mean of their variables, which is directly linked to the intercept present in each equation of the model. However, changes in the economy that affect the long-term perspective of macroeconomic variables are not easily incorporated into traditional VARs, especially in the case of recent structural changes.

In fact, in a traditional VAR, changes that affect long-run inflation behavior do not have an immediate impact on model projections. In these cases, it is necessary to wait for a significant increase in the sample size, so that these changes can affect the unconditional mean of inflation and, consequently, the long-term forecasts. In addition, in an environment with more than one regime, the coefficients of the VAR would only reflect the average behavior of the dynamics across all regimes of each variable.

In contrast to the inertial behavior of VARs, changes in the conduct of monetary policy, or the setting of inflation targets at values other than those usually defined, can be quickly captured by agents' expectations for long-term inflation, since those expectations tend to

[32] https://fxdiebold.blogspot.com/2020/07/the-pandemic-recession-as-giant-outlier.html

[33] According to Cascaldi-Garcia (2022), the *Pandemic Priors* succeed in recovering historical relationships and the proper identification and propagation of structural shocks.

reflect structural changes in the level of inflation. Timely incorporating these expectations into VARs is an effective way to improve projections in situations where there are structural changes in the level of endogenous variables.

We illustrate our methodology using Brazilian data, where credibility gains and the convergence process of expectations to target in the most recent period are important changes to the dynamics of inflation. Moreover, after 14 years with an annual inflation target of 4.50%, the National Monetary Council (CMN) set lower targets for inflation: 4.25% for 2019, which gradually decreases to 3.00% for 2024 and 2025. These changes have been reflected in the behavior of analysts' expectations captured by the *Focus survey* conducted by the Banco Central do Brasil. However, the longer-term inflation projections of VAR models, in general, have distanced themselves from those derived from other models and analysts' expectations.

To correct this distortion, we propose a state-space model that incorporates VAR and VARX dynamics, coupled with a *shifting endpoint* (SE) restriction, where the intercept of the equation describing the inflation dynamics is time variant, following a random walk process. This way, long-run inflation forecasts match survey predictions. With this approach, two benefits are obtained: (i) the model becomes less inertial due to the introduction of long-term survey expectations; and (ii) the level of forecasts becomes adjustable over the long term.

The empirical results for Brazil using a pre-COVID-19 sample (up to March 2020) confirm the advantages of our approach over traditional models. Overall, the use of survey information in VAR and VARX models improves forecast accuracy in terms of MSE at longer horizons.[34] The Clark and West (2007) test often confirms the superiority of the proposed models compared to the traditional methods.

In turn, the results considering the full sample (up to August 2022) exhibit much higher MSEs when compared to the pre-COVID-19 sample, especially because recent observations are characterized by large forecast errors, due to sizeable and unanticipated macroeconomic shocks that occurred after the arrival of the pandemic in Brazil. Also, models without SE tend to provide lower MSEs compared to the SE counterparts. One explanation is that long-run anchoring driven by survey data (e.g., pointing to an annual inflation rate around 3% p.y.) has become temporarily ineffective after the COVID-19 impact in Brazil, where inflation rate outliers prevail together with greater macroeconomic uncertainty. Nonetheless, long-term anchoring from survey data, properly reflecting the new macroeconomic environment after the pandemic, will likely restore the usefulness of SE models in the near future.

Altogether, these findings represent a valuable contribution to academics, practitioners and policymakers interested in macroeconomic forecasting using VAR models with exogenous variables and survey information, especially with interest in Brazilian inflation.

[34] In some cases, the improvement in forecast accuracy due to the SE restriction also occurs in the short and medium terms.

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Appendix A. Proofs of Propositions

Proof of Proposition 1. (i) First, we show the proposed state-space framework can reproduce the VAR(p) process. Since $X_t = AX_{t-1} + B\varepsilon_t$, it follows that:

$$X_t = \begin{bmatrix} x_t \\ x_{t-1} \\ \vdots \\ x_{t-p+2} \\ x_{t-p+1} \\ c_t \\ \mu_t \end{bmatrix} = \begin{bmatrix} F_1 & F_2 & \cdots & F_{p-1} & F_p & \phi & \psi \\ I_k & 0_k & \cdots & 0_k & 0_k & 0_{k\times 1} & 0_{k\times 1} \\ 0_k & I_k & \cdots & 0_k & 0_k & 0_{k\times 1} & 0_{k\times 1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0_k & 0_k & \cdots & I_k & 0_k & 0_{k\times 1} & 0_{k\times 1} \\ 0_{1\times k} & 0_{1\times k} & \cdots & 0_{1\times k} & 0_{1\times k} & 1 & 0 \\ 0_{1\times k} & 0_{1\times k} & \cdots & 0_{1\times k} & 0_{1\times k} & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ x_{t-2} \\ \vdots \\ x_{t-p+1} \\ x_{t-p} \\ c_{t-1} \\ \mu_{t-1} \end{bmatrix}$$
$$+ \begin{bmatrix} I_k & 0_k & \cdots & 0_k & 0_{k\times 1} & 0_{k\times 1} \\ 0_k & 0_k & \cdots & 0_k & 0_{k\times 1} & 0_{k\times 1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0_k & 0_k & \cdots & 0_k & 0_{k\times 1} & 0_{k\times 1} \\ 0_{1\times k} & 0_{1\times k} & \cdots & 0_{1\times k} & 0 & 0 \\ 0_{1\times k} & 0_{1\times k} & \cdots & 0_{1\times k} & 0 & 1 \end{bmatrix} \begin{bmatrix} u_t \\ 0 \\ \vdots \\ 0 \\ 0 \\ \varepsilon_{\mu,t} \end{bmatrix}.$$

This way, it follows that $x_t = F_1 x_{t-1} + ... + F_p x_{t-p} + \phi c_{t-1} + \psi \mu_{t-1} + u_t$; $x_{t-s} = x_{t-s}$ for all $s = 1, ..., p-1$; $c_t = c_{t-1} = 1$ for all t, and $\mu_t = \mu_{t-1} + \varepsilon_{\mu,t}$. Regarding the first variable of the VAR (i.e., inflation), note that: $x_{1,t} = F_{1,1}x_{1,t-1} + ... + F_{1,p}x_{1,t-p} + F_{2,1}x_{2,t-1} + ... + F_{2,p}x_{2,t-p} + ... + \mu_{t-1} + u_{1,t}$, that is, inflation follows an autoregressive process, plus other terms, and a random walk (instead of the usual intercept). With respect to the other variables, it follows that: $x_{i,t} = F_{i,1}x_{i,t-1} + ... + F_{i,p}x_{i,t-p} + ... + \phi_i + u_{i,t}$ for all $i = 2, ..., k$. Therefore, the proposed state-space model is able to reproduce the original VAR dynamics, but considering a random walk term in the first equation (inflation) instead of the usual intercept.

(ii) Next, we show the shifting endpoint restriction holds. Since $Y_t = CX_t + D\epsilon_t$, it follows that: $\begin{bmatrix} y_{1,t} & \cdots & y_{k,t} & \mathbf{1}_t & f_{t+H|t} \end{bmatrix}' = \begin{bmatrix} J_1' & \cdots & J_k' & \overline{J}' & \beta' \end{bmatrix}' X_t \therefore y_{i,t} = x_{i,t}$ for all $i = 1, ..., k$; $\mathbf{1}_t = 1$ for all t, and $f_{t+H|t} = \beta X_t = \sum_{h=H-11}^{H} J_1 A^h X_t$. On the other hand, due to the AR(1) structure of the X_t process, it follows that the h-step ahead forecast of X_t is $\mathbb{E}(X_{t+h} \mid \mathcal{F}_t) = A^h X_t$, where $\mathcal{F}_t$ is the information set at period t. Since J_1 selects the first variable of X_t, one can select π_t in X_t as follows: $\pi_t = J_1 X_t \therefore \pi_{t+h} = J_1 X_{t+h}$. Thus, applying the conditional expectation in the previous expression, it follows that: $\mathbb{E}[\pi_{t+h} \mid \mathcal{F}_t] = \mathbb{E}[J_1 X_{t+h} \mid \mathcal{F}_t] = J_1 \mathbb{E}[X_{t+h} \mid \mathcal{F}_t] = J_1 A^h X_t$. Thus, $\sum_{h=H-11}^{H} \mathbb{E}[\pi_{t+h} \mid \mathcal{F}_t] = \sum_{h=H-11}^{H} J_1 A^h X_t = \beta X_t$, where $\beta \equiv \sum_{h=H-11}^{H} J_1 A^h$. Therefore, $f_{t+H|t} = \sum_{h=H-11}^{H} \mathbb{E}[\pi_{t+h} \mid \mathcal{F}_t]$. ∎

Proof of Proposition 2. (i) First, we show the proposed state-space framework can reproduce the VARX(p) process. Since $X_t^* = A^* X_{t-1}^* + B^* \varepsilon_t^*$, it follows that:

$$X_t^* = \begin{bmatrix} x_t \\ x_{t-1} \\ \vdots \\ x_{t-p+2} \\ x_{t-p+1} \\ c_t \\ \mu_t \\ x_{1,t}^* \\ \vdots \\ x_{H-1,t}^* \\ x_{H,t}^* \end{bmatrix} = \begin{bmatrix} F_1 & F_2 & \cdots & F_{p-1} & F_p & \phi & \psi & \theta & 0 & 0 & 0 \\ I_k & 0_k & \cdots & 0 & 0 & 0 & 0 & 0 & \vdots & \vdots & \vdots \\ 0_k & I_k & \cdots & 0 & 0 & 0 & 0 & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I_k & 0 & 0 & 0 & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \vdots & 0 & \vdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & 0 & 0 & \vdots & 1 & 0 & \vdots \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ x_{t-2} \\ \vdots \\ x_{t-p+1} \\ x_{t-p} \\ c_{t-1} \\ \mu_{t-1} \\ x_{1,t-1}^* \\ \vdots \\ x_{H-1,t-1}^* \\ x_{H,t-1}^* \end{bmatrix}$$

$$+ \begin{bmatrix} I_k & 0_k & \cdots & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0_k & 0_k & \cdots & 0 & 0 & 0 & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} u_t \\ 0 \\ \vdots \\ 0 \\ 0 \\ \varepsilon_{\mu,t} \\ 0 \\ \vdots \\ 0 \\ \varepsilon_{z,t} \end{bmatrix}.$$

This way, it follows that $x_t = F_1 x_{t-1} + ... + F_p x_{t-p} + \phi c_{t-1} + \psi \mu_{t-1} + \theta x_{1,t-1}^* + u_t$; $x_{t-s} = x_{t-s}$ for all $s = 1, ..., p-1$; $c_t = c_{t-1} = 1$ for all t, $\mu_t = \mu_{t-1} + \varepsilon_{\mu,t}$, $x_{i,t}^* = x_{i+1,t-1}^*$ for $i = 1, ..., H-1$, and $x_{H,t}^* = \varepsilon_{z,t}$. Regarding the first endogenous variable of the VARX (inflation), it follows that: $x_{1,t} = F_{1,1} x_{1,t-1} + ... + F_{1,p} x_{1,t-p} + F_{2,1} x_{2,t-1} + ... + F_{2,p} x_{2,t-p} + ... + \mu_{t-1} + \theta z_t + u_{1,t}$, since from the state-space equation of observable variables, it follows that: $z_t^* = \gamma X_t^* = x_t^* \therefore x_{1,t-1}^* = z_{1,t-1}^* = z_t$. In words, inflation follows an autoregressive process, plus other terms, and a random walk (instead of the usual intercept, as the case of the other endogenous variables). Therefore, the proposed state-space model is able to reproduce the original VARX dynamics, but considering a random walk term in the first equation (inflation) instead of the usual intercept, besides incorporating exogenous variables.

(ii) Next, we show the shifting endpoint restriction holds in the VARX setup. Since $Y_t^* = C^* X_t^* + D^* \epsilon_t^*$, it follows that: $\begin{bmatrix} y_{1,t} \\ \vdots \\ y_{k,t} \\ \mathbf{1}_t \\ f_{t+H|t} \\ z_t^* \end{bmatrix} = \begin{bmatrix} J_1^* \\ \vdots \\ J_k^* \\ \overline{J^*} \\ \beta^* \\ \gamma \end{bmatrix} X_t^* \therefore y_{i,t} = x_{i,t}$ for all $i = 1, ..., k$; $\mathbf{1}_t = 1$ for all t, $f_{t+H|t} = \beta^* X_t^* = J_1^* \sum_{h=H-11}^{H} A^{*h} X_t^*$, and $z_t^* = \gamma X_t^* = x_t^*$. On the other hand, due to the AR(1) structure of the X_t^* process, it follows that the h-step ahead forecast of X_t^* is $\mathbb{E}(X_{t+h}^* \mid \mathcal{F}_t) = A^{*h} X_t^*$, where $\mathcal{F}_t$ is the information set at period t. Since J_1^* selects the first variable of X_t^*, one can select π_t in vector X_t^*, as follows: $\pi_t = J_1^* X_t^* \therefore \pi_{t+h} = J_1^* X_{t+h}^*$. Thus, applying the conditional expectation in the previous expression, it follows that: $\mathbb{E}[\pi_{t+h} \mid \mathcal{F}_t] = J_1^* \mathbb{E}[X_{t+h}^* \mid \mathcal{F}_t] = J_1^* A^{*h} X_t^*$. Thus, $\sum_{h=H-11}^{H} \mathbb{E}[\pi_{t+h} \mid \mathcal{F}_t] = \sum_{h=H-11}^{H} J_1^* A^{*h} X_t^* = \beta^* X_t^*$, where $\beta^* = \sum_{h=H-11}^{H} J_1^* A^{*h}$. Therefore, $f_{t+H|t} = \sum_{h=H-11}^{H} \mathbb{E}[\pi_{t+h} \mid \mathcal{F}_t]$. ■

Appendix B. Examples of Models with Shifting Endpoint

Example 1: VAR-SE model

Assume the model has two variables ($k = 2$), one lag ($p = 1$), and the forecast horizon of interest is $H = 12$ months (i.e., the horizon at which we impose the shifting endpoint restriction). Let the vector of observable variables $Y_t = \begin{bmatrix} \pi_t & i_t & \mathbf{1}_t & f_{t+12|t} \end{bmatrix}'$ be a 4×1 vector, where π_t and i_t are two observable series (the IPCA monthly inflation rate and the Selic monetary policy interest rate) and $f_{t+12|t}$ is the *Focus* survey inflation expectation.

From assumptions A3-A6, it follows that: $A = \begin{bmatrix} a & b & 0 & 1 \\ c & d & \phi_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} J_1' & J_2' & \overline{J}' & \beta' \end{bmatrix}'$, and $D = \mathbf{0}_{4\times 4}$. From equation (4), it follows that:

$$\begin{bmatrix} \pi_t \\ i_t \\ \mathbf{1}_t \\ f_{t+12|t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & \beta & & \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \\ c_t \\ \mu_t \end{bmatrix} = \begin{bmatrix} x_{1,t} \\ x_{2,t} \\ c_t \\ \beta X_t \end{bmatrix}, \tag{12}$$

thus, $\pi_t = x_{1,t}$, $i_t = x_{2,t}$, $\mathbf{1}_t = c_t$, and $f_{t+12|t} = \beta X_t$. Now, from equation (3), it follows that:

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \\ c_t \\ \mu_t \end{bmatrix} = \begin{bmatrix} a & b & 0 & 1 \\ c & d & \phi_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,t-1} \\ x_{2,t-1} \\ c_{t-1} \\ \mu_{t-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{1,t} \\ u_{2,t} \\ 0 \\ \varepsilon_{\mu,t} \end{bmatrix} \tag{13}$$

$$= \begin{bmatrix} ax_{1,t-1} + bx_{2,t-1} + \mu_{t-1} + u_{1,t} \\ cx_{1,t-1} + dx_{2,t-1} + \phi_2 + u_{2,t} \\ c_{t-1} \\ \mu_{t-1} + \varepsilon_{\mu,t} \end{bmatrix}, \tag{14}$$

or, equivalently,

$$\pi_t = a\pi_{t-1} + bi_{t-1} + \mu_{t-1} + u_{1,t}, \tag{15}$$

$$i_t = c\pi_{t-1} + di_{t-1} + \phi_2 + u_{2,t}. \tag{16}$$

Note that (by design) the equation for π_t has a random walk term (μ_{t-1}) instead of the intercept, whereas the equation for i_t has the usual VAR intercept (ϕ_2).

Next, we check if the shifting endpoint restriction (5) holds. Due to the AR(1) structure of the X_t process, it follows that the h-step ahead forecast of X_t is given by $\mathbb{E}(X_{t+h} \mid \mathcal{F}_t) = A^h X_t$, where $\mathcal{F}_t$ is the information set at period t. Since, in our example, $J_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$ selects the first variable of X_t, one can select $x_{1,t} = \pi_t$ in X_t as follows: $\pi_t = J_1 X_t \therefore \pi_{t+h} = J_1 X_{t+h}$. Thus, applying the conditional expectation in previous expression, it follows that: $\mathbb{E}[\pi_{t+h} \mid \mathcal{F}_t] = \mathbb{E}[J_1 X_{t+h} \mid \mathcal{F}_t] = J_1 \mathbb{E}[X_{t+h} \mid \mathcal{F}_t] = J_1 A^h X_t$. The accumulated inflation forecast $H = 12$ months ahead is given by $\sum_{h=1}^{12} \mathbb{E}[\pi_{t+h} \mid \mathcal{F}_t] = \sum_{h=1}^{12} J_1 A^h X_t = J_1 \sum_{h=1}^{12} A^h X_t = \beta X_t$, where $\beta = J_1 \sum_{h=1}^{12} A^h$. Recall from the observable variables equation that $f_{t+12|t} = \beta X_t$. Therefore, $f_{t+12|t} = \sum_{h=1}^{12} \mathbb{E}[\pi_{t+h} \mid \mathcal{F}_t]$, which is exactly the shifting endpoint restriction, imposing the sum of monthly VAR forecasts to match the survey-based inflation expectation 12 months ahead.

Example 2: VARX-SE model

Assume the model has two endogenous variables ($k = 2$), one lag ($p = 1$), one exogenous variable ($m = 1$), and the forecast horizon of interest is again $H = 12$ months. Let the vector of observable variables be $Y_t = \begin{bmatrix} \pi_t & i_t & \mathbf{1}_t & f_{t+12|t} & z_t^{*\prime} \end{bmatrix}'$, where π_t, i_t and $f_{t+12|t}$ are the same variables of previous example. Now, let $z_t^* = \begin{bmatrix} oil_{t+1} & \cdots & oil_{t+12} \end{bmatrix}'$ be the vector of future (*exogenous*) oil prices. From assumptions B3-B6, it follows that:

$$A^* = \begin{bmatrix} a & b & 0 & 1 & \theta_1 & 0 & \cdots & 0 \\ c & d & \phi_2 & 0 & \theta_2 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, B^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$C^* = \begin{bmatrix} J_1^{*\prime} & J_2^{*\prime} & \overline{J^*}^{\prime} & \beta^{*\prime} & \gamma^\prime \end{bmatrix}^\prime$, and $D^* = \mathbf{0}_{16\times16}$.

From equation (11), it follows that:

$$\begin{bmatrix} \pi_t \\ i_t \\ \mathbf{1}_t \\ f_{t+12|t} \\ z_t^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & & \beta^* & & \\ & & & \gamma & & \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \\ c_t \\ \mu_t \\ x_t^* \end{bmatrix} = \begin{bmatrix} x_{1,t} \\ x_{2,t} \\ c_t \\ \beta^* X_t^* \\ x_t^* \end{bmatrix}. \tag{17}$$

This way, $\pi_t = x_{1,t}$, $i_t = x_{2,t}$, $\mathbf{1}_t = c_t$, $f_{t+12|t} = \beta^* X_t^*$, and $z_t^* = x_t^*$. Now, from equation (10), it follows that:

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \\ c_t \\ \mu_t \\ x_{1,t}^* \\ \vdots \\ x_{H-1,t}^* \\ x_{H,t}^* \end{bmatrix} = \begin{bmatrix} a & b & 0 & 1 & \theta_1 & 0 & \cdots & 0 \\ c & d & \phi_2 & 0 & \theta_2 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{1,t-1} \\ x_{2,t-1} \\ c_{t-1} \\ \mu_{t-1} \\ x_{1,t-1}^* \\ \vdots \\ x_{H-1,t-1}^* \\ x_{H,t-1}^* \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{1,t} \\ u_{2,t} \\ 0 \\ \varepsilon_{\mu,t} \\ 0 \\ \vdots \\ 0 \\ \varepsilon_{z,t} \end{bmatrix} \tag{18}$$

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \\ c_t \\ \mu_t \\ x^*_{1,t} \\ \vdots \\ x^*_{H-1,t} \\ x^*_{H,t} \end{bmatrix} = \begin{bmatrix} ax_{1,t-1} + bx_{2,t-1} + \mu_{t-1} + \theta_1 x^*_{1,t-1} + u_{1,t} \\ cx_{1,t-1} + dx_{2,t-1} + \phi_2 + \theta_2 x^*_{1,t-1} + u_{2,t} \\ c_{t-1} \\ \mu_{t-1} + \varepsilon_{\mu,t} \\ x^*_{2,t-1} \\ \vdots \\ x^*_{H,t-1} \\ \varepsilon_{z,t} \end{bmatrix}. \tag{19}$$

Since $x^*_t = z^*_t = \begin{bmatrix} oil_{t+1} & \cdots & oil_{t+12} \end{bmatrix}'$ and $x^*_{t-1} = z^*_{t-1} = \begin{bmatrix} oil_t & \cdots & oil_{t+11} \end{bmatrix}'$, it follows that $x^*_{1,t-1} = oil_t$ and $x^*_{1,t} = oil_{t+1} = x^*_{2,t-1}$. By combining previous results, it also follows that:

$$\pi_t = a\pi_{t-1} + bi_{t-1} + \mu_{t-1} + \theta_1 oil_t + u_{1,t}, \tag{20}$$

$$i_t = c\pi_{t-1} + di_{t-1} + \phi_2 + \theta_2 oil_t + u_{2,t}. \tag{21}$$

Note that (by design) the equation for π_t has again a random walk term (μ_{t-1}) instead of the intercept, whereas the equation for i_t has an intercept (ϕ_2). Also note both equations have now a term associated to the *exogenous* variable. Moreover, note that $c_t = c_{t-1} = 1$, $\mu_t = \mu_{t-1} + \varepsilon_{\mu,t}$, $x^*_{i,t} = x^*_{i+1,t-1}$ for $i = 1, ..., H-1$, and $x^*_{H,t} = \varepsilon_{z,t}$.

Next, we check whether the VAR shifting endpoint restriction (5) holds in this setup. Due to the AR(1) structure of the X^*_t process, it follows that the h-step ahead forecast of X^*_t is given by $\mathbb{E}(X^*_{t+h} \mid \mathcal{F}_t) = A^{*h} X^*_t$, where $\mathcal{F}_t$ is the information set at period t. Since $J^*_1 = \begin{bmatrix} 1 & 0 & ... & 0 \end{bmatrix}$ selects the first variable of X^*_t, one can select π_t in X^*_t as follows: $\pi_t = J^*_1 X^*_t \therefore \pi_{t+h} = J^*_1 X^*_{t+h}$. By applying the conditional expectation in previous expression, it follows that: $\mathbb{E}[\pi_{t+h} \mid \mathcal{F}_t] = J^*_1 \mathbb{E}[X^*_{t+h} \mid \mathcal{F}_t] = J^*_1 A^{*h} X^*_t$. The accumulated inflation forecast $H = 12$ months ahead is given by $\sum_{h=1}^{12} \mathbb{E}[\pi_{t+h} \mid \mathcal{F}_t] = J^*_1 \sum_{h=1}^{12} A^{*h} X^*_t = \beta^* X^*_t$, where $\beta^* \equiv J^*_1 \sum_{h=1}^{12} A^{*h}$. Recall from equation (11) that $f_{t+12|t} = \beta^* X^*_t$. Therefore, $f_{t+12|t} = \sum_{h=1}^{12} \mathbb{E}[\pi_{t+h} \mid \mathcal{F}_t]$, which is the shifting endpoint restriction, imposing the sum of monthly VARX forecasts to match the survey-based inflation expectation 12 months ahead.

Appendix C. Additional Results

Table C.1 - Mean Squared Error (MSE), *pre-COVID-19 sample*

$h = 1$

Model	Without SE	SE-12	SE-24	SE-48
1	0.078	0.072 **	0.075 *	0.077
2	0.070	0.068 *	0.067 **	0.067 **
3	0.080	0.073 **	0.078 *	0.078 *
4	0.078	0.120	0.127	0.127
5	0.078	0.120	0.132	0.131
6	0.078	0.119	0.126	0.128
7	0.078	0.119	0.123	0.125
8	0.079	0.120	0.125	0.126
9	0.078	0.123	0.133	0.134
10	0.081	0.116	0.120	0.136

$h = 3$

Model	Without SE	SE-12	SE-24	SE-48
1	0.106	0.094 ***	0.099 **	0.100
2	0.107	0.094 ***	0.097 **	0.097 **
3	0.106	0.094 ***	0.098 **	0.100 *
4	0.106	0.116	0.124	0.131
5	0.105	0.116	0.124	0.130
6	0.106	0.116	0.126	0.130
7	0.105	0.116	0.124	0.130
8	0.107	0.116	0.125	0.131
9	0.105	0.116	0.121	0.129
10	0.097	0.116	0.122	0.146

$h = 6$

Model	Without SE	SE-12	SE-24	SE-48
1	0.119	0.099 ***	0.108 **	0.112 *
2	0.119	0.099 ***	0.107 **	0.107 **
3	0.120	0.100 ***	0.109 **	0.112 *
4	0.120	0.122	0.120	0.130
5	0.119	0.122	0.120	0.130
6	0.119	0.122	0.122	0.131
7	0.118	0.122	0.121	0.134
8	0.120	0.122	0.122	0.134
9	0.117	0.122	0.120	0.131
10	0.107	0.122	0.122	0.141

$h = 12$

Model	Without SE	SE-12	SE-24	SE-48
1	0.122	0.121	0.112 **	0.111 **
2	0.125	0.119	0.112 ***	0.110 ***
3	0.123	0.120	0.112 **	0.112 **
4	0.122	0.135	0.128	0.133
5	0.118	0.135	0.128	0.134 *
6	0.122	0.135	0.128	0.133
7	0.118	0.135	0.128	0.136
8	0.122	0.135	0.128	0.135
9	0.113	0.134	0.128	0.136
10	0.114	0.135	0.128	0.137

$h = 24$

Model	Without SE	SE-12	SE-24	SE-48
1	0.140	0.152	0.130 **	0.129 **
2	0.140	0.148	0.130 **	0.129 **
3	0.141	0.149	0.129 **	0.130 **
4	0.141	0.159 **	0.146	0.149
5	0.138	0.158	0.146 **	0.149
6	0.141	0.159 **	0.146	0.150
7	0.138	0.159 **	0.146	0.150
8	0.142	0.159 ***	0.146	0.150
9	0.133	0.157 **	0.146	0.149
10	0.137	0.155	0.146	0.148

$h = 48$

Model	Without SE	SE-12	SE-24	SE-48
1	0.109	0.140 **	0.113	0.102
2	0.111	0.135 **	0.113	0.102 *
3	0.110	0.140 **	0.113	0.102 *
4	0.108	0.119 *	0.105	0.091 ***
5	0.101	0.117 *	0.104	0.091 ***
6	0.109	0.120 *	0.105 *	0.091 ***
7	0.112	0.120	0.106 ***	0.091 ***
8	0.121	0.120	0.106 ***	0.091 ***
9	0.114	0.115	0.104 ***	0.091 ***
10	0.121	0.106 *	0.103 *	0.091 ***

Notes: Evaluation sample ranges from Feb/2013 to Feb/2020 (85 observations for h=1 and 38 for h=48). ***, ** and * indicate rejection at 1%, 5% and 10% levels, respectively, using the test of Clark and West (2007). In each row, the model in the first column is the benchmark. Yellow cells indicate the best model (lowest MSE) in each row (orange denotes the best model in a given horizon h).

Table C.2 - Mean Squared Error (MSE), *full sample*

$h = 1$

Model	Without SE	SE-12	SE-24	SE-48
1	0.113	0.115	0.114	0.116
2	0.099	0.106	0.099	0.098
3	0.113	0.114	0.114	0.115
4	0.114	0.196	0.204	0.220
5	0.115	0.196	0.207	0.221
6	0.113	0.197	0.210	0.222
7	0.113	0.196	0.204	0.220
8	0.113	0.196	0.208	0.223
9	0.114	0.194	0.199	0.218
10	0.115	0.191	0.191	0.217

$h = 3$

Model	Without SE	SE-12	SE-24	SE-48
1	0.159	0.165	0.167	0.170
2	0.157	0.165	0.163	0.165
3	0.158	0.166	0.164	0.169
4	0.160	0.198	0.179	0.201
5	0.161	0.198	0.177	0.197
6	0.159	0.198	0.179	0.201
7	0.160	0.198	0.179	0.201
8	0.160	0.197	0.181	0.204
9	0.164	0.197	0.173	0.190
10	0.149	0.194	0.166	0.195

$h = 6$

Model	Without SE	SE-12	SE-24	SE-48
1	0.179	0.177	0.187	0.196
2	0.177	0.179	0.185	0.193
3	0.177	0.177	0.185	0.195
4	0.183	0.203	0.183	0.193
5	0.186	0.203	0.182	0.191
6	0.179	0.203	0.184	0.194
7	0.183	0.203	0.184	0.196
8	0.180	0.203	0.185	0.196
9	0.191	0.204	0.181	0.190
10	0.163	0.202	0.178	0.186

$h = 12$

Model	Without SE	SE-12	SE-24	SE-48
1	0.178	0.206	0.191	0.194
2	0.179	0.206	0.191	0.192
3	0.178	0.206	0.191	0.194
4	0.180	0.202	0.193	0.196
5	0.180	0.202	0.192	0.194
6	0.178	0.202	0.193	0.196
7	0.178	0.202	0.193	0.197
8	0.178	0.202	0.193	0.197
9	0.180	0.203	0.192	0.194
10	0.167	0.208	0.191	0.190

$h = 24$

Model	Without SE	SE-12	SE-24	SE-48
1	0.196	0.222	0.207	0.210
2	0.196	0.218	0.207	0.209
3	0.196	0.220	0.207	0.210
4	0.198	0.227 *	0.215 *	0.205
5	0.199	0.227	0.215	0.205
6	0.196	0.227 *	0.215 *	0.206
7	0.199	0.227	0.215	0.206
8	0.194	0.227 *	0.215	0.206
9	0.203	0.228	0.215	0.204
10	0.189	0.233	0.216	0.202

$h = 48$

Model	Without SE	SE-12	SE-24	SE-48
1	0.197	0.248 ***	0.221	0.214
2	0.197	0.242 ***	0.220	0.214
3	0.197	0.247 ***	0.221	0.214
4	0.196	0.203	0.193	0.189
5	0.195	0.202	0.192	0.189 *
6	0.196	0.203	0.193	0.189
7	0.198	0.204	0.193	0.189
8	0.201	0.204	0.193	0.189
9	0.201	0.203	0.193 *	0.189 **
10	0.196	0.201	0.193	0.189

Notes: Evaluation sample ranges from Feb/2013 to Aug/2022 (115 observations for h=1 and 68 for h=48). ***, ** and * indicate rejection at 1%, 5% and 10% levels, respectively, using the test of Clark and West (2007). In each row, the model in the first column is the benchmark. Yellow cells indicate the best model (lowest MSE) in each row (orange denotes the best model in a given horizon h).

Table C.3 - Mean Squared Error (MSE), *pre-COVID-19 sample*

(perfect foresight assumption for the future path of the exogenous variables)

$h = 1$

Model	Without SE	SE-12	SE-24	SE-48
1	0.078	0.072 **	0.075 *	0.077
2	0.070	0.068 *	0.067 **	0.067 **
3	0.080	0.073 **	0.078 *	0.078 *
4	0.079	0.123	0.119	0.117
5	0.078	0.118	0.127	0.124
6	0.079	0.119	0.124	0.127
7	0.078	0.119	0.123	0.124
8	0.075	0.118	0.124	0.125
9	0.076	0.120	0.123	0.125
10	0.081	0.113	0.118	0.126

$h = 3$

Model	Without SE	SE-12	SE-24	SE-48
1	0.106	0.094 ***	0.099 **	0.100
2	0.107	0.094 ***	0.097 **	0.097 **
3	0.106	0.094 ***	0.098 **	0.100 *
4	0.108	0.120	0.116	0.120
5	0.107	0.115	0.122	0.125
6	0.107	0.116	0.124	0.131
7	0.106	0.116	0.123	0.129
8	0.107	0.116	0.125	0.132
9	0.107	0.117	0.115	0.123
10	0.097	0.104	0.106	0.119

$h = 6$

Model	Without SE	SE-12	SE-24	SE-48
1	0.119	0.099 ***	0.108 **	0.112 *
2	0.119	0.099 ***	0.107 **	0.107 **
3	0.120	0.100 ***	0.109 **	0.112 *
4	0.122	0.128	0.114 *	0.123
5	0.123	0.122	0.118	0.125
6	0.122	0.122	0.119	0.131
7	0.120	0.122	0.121	0.133
8	0.119	0.122	0.122	0.135
9	0.122	0.126	0.117 *	0.127
10	0.107	0.114	0.107	0.120

$h = 12$

Model	Without SE	SE-12	SE-24	SE-48
1	0.122	0.121	0.112 **	0.111 **
2	0.125	0.119	0.112 ***	0.110 ***
3	0.123	0.120	0.112 **	0.112 **
4	0.126	0.146	0.123	0.126
5	0.124	0.137	0.127	0.129
6	0.125	0.137	0.125	0.135
7	0.123	0.135	0.128	0.136
8	0.121	0.135	0.127	0.137
9	0.124	0.143	0.126	0.130
10	0.114	0.126	0.118	0.127

$h = 24$

Model	Without SE	SE-12	SE-24	SE-48
1	0.140	0.152	0.130 **	0.129 **
2	0.140	0.148	0.130 **	0.129 **
3	0.141	0.149	0.129 **	0.130 **
4	0.149	0.164	0.147	0.145
5	0.146	0.160	0.145	0.143
6	0.143	0.161	0.146	0.151
7	0.147	0.159 *	0.146	0.150
8	0.139	0.161 **	0.146	0.152
9	0.152	0.165	0.146	0.142
10	0.137	0.167	0.138	0.142

$h = 48$

Model	Without SE	SE-12	SE-24	SE-48
1	0.109	0.140 **	0.113	0.102
2	0.111	0.135 **	0.113	0.102 *
3	0.110	0.140 **	0.113	0.102 *
4	0.109	0.126 *	0.112	0.086 ***
5	0.096	0.122 *	0.109 *	0.092
6	0.111	0.123	0.110	0.095 ***
7	0.123	0.120	0.106 ***	0.091 ***
8	0.118	0.117	0.103 ***	0.091 ***
9	0.123	0.124	0.108 *	0.088 ***
10	0.121	0.136	0.117	0.095 ***

Notes: Evaluation sample ranges from Feb/2013 to Feb/2020 (85 observations for h=1 and 38 for h=48). ***, ** and * indicate rejection at 1%, 5% and 10% levels, respectively, using the test of Clark and West (2007). In each row, the model in the first column is the benchmark. Yellow cells indicate the best model (lowest MSE) in each row (orange denotes the best model in a given horizon h).

Table C.4 - Mean Squared Error (MSE), *full sample*

(perfect foresight assumption for the future path of the exogenous variables)

$h = 1$

Model	Without SE	SE-12	SE-24	SE-48
1	0.113	0.115	0.114	0.116
2	0.099	0.106	0.099	0.098
3	0.113	0.114	0.114	0.115
4	0.110	0.199	0.204	0.212
5	0.108	0.196	0.208	0.217
6	0.114	0.196	0.210	0.221
7	0.115	0.196	0.204	0.218
8	0.118	0.197	0.214	0.219
9	0.114	0.197	0.207	0.213
10	0.115	0.192	0.201	0.216

$h = 3$

Model	Without SE	SE-12	SE-24	SE-48
1	0.159	0.165	0.167	0.170
2	0.157	0.165	0.163	0.165
3	0.158	0.166	0.164	0.169
4	0.157	0.201	0.179	0.193
5	0.150	0.198	0.180	0.196
6	0.161	0.198	0.179	0.202
7	0.165	0.198	0.178	0.197
8	0.180	0.196	0.186	0.202
9	0.173	0.198	0.185	0.187
10	0.149	0.186	0.160	0.185

$h = 6$

Model	Without SE	SE-12	SE-24	SE-48
1	0.179	0.177	0.187	0.196
2	0.177	0.179	0.185	0.193
3	0.177	0.177	0.185	0.195
4	0.179	0.207	0.185	0.186
5	0.173	0.202	0.185	0.188
6	0.181	0.202	0.184	0.195
7	0.188	0.203	0.184	0.192
8	0.200	0.202	0.188	0.196
9	0.196	0.205	0.192	0.183
10	0.163	0.193	0.170	0.182

$h = 12$

Model	Without SE	SE-12	SE-24	SE-48
1	0.178	0.206	0.191	0.194
2	0.179	0.206	0.191	0.192
3	0.178	0.206	0.191	0.194
4	0.179	0.208	0.194	0.186
5	0.165	0.200	0.193	0.189 *
6	0.182	0.204	0.193	0.196
7	0.186	0.203	0.192	0.192
8	0.202	0.199	0.193	0.194
9	0.188	0.202	0.199	0.184
10	0.167	0.194	0.183	0.190

$h = 24$

Model	Without SE	SE-12	SE-24	SE-48
1	0.196	0.222	0.207	0.210
2	0.196	0.218	0.207	0.209
3	0.196	0.220	0.207	0.210
4	0.206	0.230	0.222	0.203
5	0.189	0.227	0.216 *	0.199
6	0.198	0.227	0.215 *	0.206
7	0.208	0.227	0.216	0.205
8	0.224	0.231	0.223	0.212
9	0.222	0.232	0.229	0.198 *
10	0.189	0.224	0.206 *	0.195

$h = 48$

Model	Without SE	SE-12	SE-24	SE-48
1	0.197	0.248 ***	0.221	0.214
2	0.197	0.242 ***	0.220	0.214
3	0.197	0.247 ***	0.221	0.214
4	0.208	0.210	0.200	0.188 ***
5	0.174	0.207	0.198 **	0.192
6	0.198	0.206	0.196	0.191
7	0.208	0.205	0.194	0.189 **
8	0.244	0.202 **	0.191 *	0.190 *
9	0.232	0.209	0.200 *	0.188 **
10	0.196	0.210	0.196	0.184 **

Notes: Evaluation sample ranges from Feb/2013 to Aug/2022 (115 observations for h=1 and 68 for h=48). ***, ** and * indicate rejection at 1%, 5% and 10% levels, respectively, using the test of Clark and West (2007). In each row, the model in the first column is the benchmark. Yellow cells indicate the best model (lowest MSE) in each row (orange denotes the best model in a given horizon h).