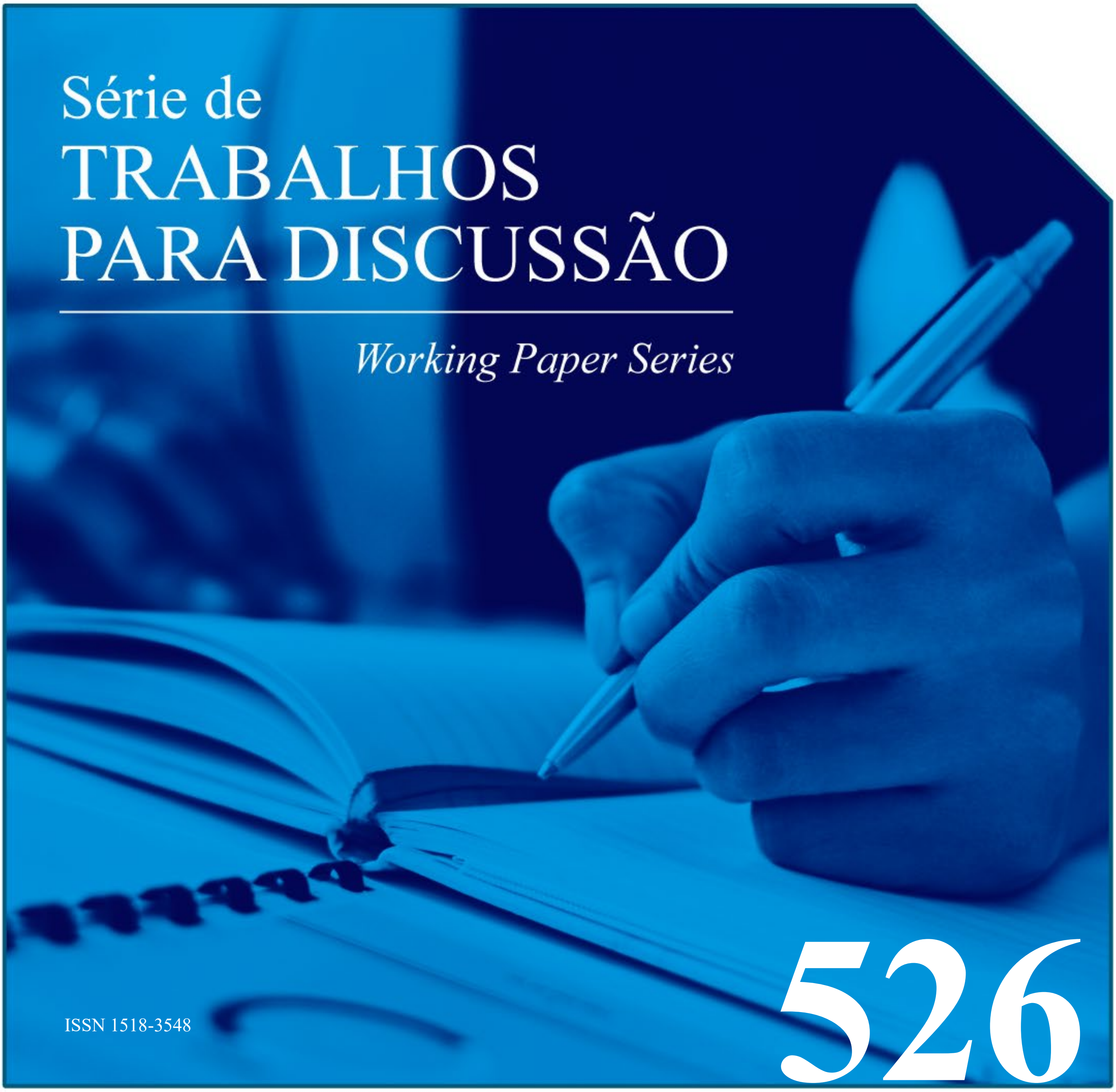

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Sumário Não Técnico

Este artigo apresenta uma ferramenta matemática para adaptar modelos do tipo DSGE (dinâmicos estocásticos de equilíbrio geral) para serem usados na análise de política macroprudencial. Os DSGE não modelam crises financeiras e por isso são inadequados para política macroprudencial. Essas políticas são justamente aquelas voltadas para minimizar a probabilidade de crises financeiras e sua severidade. Para adaptar os DSGE para esse fim, a ferramenta *RegGae* introduz crises financeiras nos DSGE por meio de mudanças de regime. Essas são alterações nas relações entre as variáveis econômicas. As mudanças de regime retratam o fato de que, durante crises financeiras, as relações entre as variáveis ficam diferentes. Algumas variáveis passam a importar mais enquanto outras passam a importar menos para afetar as demais. Uma crise financeira é uma mu-dança de regime probabilística, ou seja, uma crise acontece em cada período (o regime muda) com certa probabilidade. Essa probabilidade depende da situação da economia naquela data porque há circunstâncias mais suscetíveis do que outras a crises financeiras. Para modelar essa dependência, na ferramenta *RegGae* a probabilidade de haver uma mudança de regime é endógena: ela varia conforme a evolução da economia. Para se re-solver um DSGE e obter as equações que descrevem a evolução da economia modelada, é necessário adotar premissas sobre as expectativas dos agentes, o que os agentes esperam que ocorra, pois isso afeta como eles agem no presente e como a economia evolui. A solução matemática da *RegGae* adota a premissa de que os agentes acham que sabem ex-atamente se e quando o regime mudará para um regime de crise e de volta à normalidade. Por isso, diz-se que expectativas são determinísticas. Embora isso possa não ser verdade o tempo todo, pode ser uma premissa útil e válida para modelos de crises financeiras. Com efeito, muita gente acha que uma crise financeira não ocorrerá nunca, mesmo diante de condições propícias, e só muda de opinião com uma crise. Essa ferramenta contribui para se calibrar políticas macroprudencias, isto é, para decidir o melhor valor a ser escol-hido pela autoridade regulatória macroprudencial preparando uma revisão normativa. O me-lhor valor é aquele que equilibra o ganho de se reduzir a probabilidade e a severidade de crises financeiras com o custo da política para a sociedade.

Non-technical Summary

This article introduces a mathematical toolkit to adapt DSGE (dynamic stochastic general-equilibrium) models to be used in analyzing macroprudential policy. DSGEs do not model financial crises and therefore they are inadequate for macroprudential policy. Such poli-cies are precisely those for minimizing the probability of financial crises and their severity. To adapt DSGEs for this purpose, the *RegGae* toolkit embeds financial crises in DSGEs by means of regime changes. These are switches in the relationships among the economic variables. Regime changes represent the fact that, during financial crises, the relationships among economic variables are different. In such moments, some variables matter more whereas some variables matter less to affect the others. A financial crisis is a regime change of the probabilistic type, that is, a crisis happens in each period (the regime changes) with a certain probability. This probability depends on the situation of the economy at that moment as there are circumstances more susceptible than oth-ers to crises. To model such dependency, in *RegGae* the probability of a regime change is endogenous: it varies with the evolution of the economy. To solve *RegGae* and ob-tain the equations that describe the evolution of the model economy, it is necessary to make assumptions about agents' expectations, what they expect will happen, as their expectations affect how they behave in the present and how the economy evolves. The mathematical solution of *RegGae* makes the assumption that agents believe they know exactly if and when the regime will change into a crisis regime and back to normality. For this reason, it is said expectations are deterministic. While this may not be true at all times, it may be a useful and valid assumption for models of financial crises. Indeed, many people believe a financial crisis will never happen, even in the presence of conducive conditions, and only change opinion with a crisis. This toolkit contributes for calibrating macroprudential policy, that is, to decide the best value to be chosen by the macroprudential regulatory authority preparing to review a piece of regulation. The best value is the one that balances the benefit from reducing the probability and severity of financial crises with the cost of the policy to society.

RegGae: a toolkit for macroprudential policy with DSGEs*

Eduardo C. Castro†

Abstract

RegGae is a toolkit to adapt DSGE models for analyzing macroprudential policy. To be useful for macroprudential policy, a DSGE needs to have financial crises along the equilibrium path. *RegGae* embeds financial crises in DSGEs as regime switches, events that change the structural relationships in the economy. The solu-tion concept of *RegGae* is regime-wise linearization, a procedure that preserves the non-linearities of models of financial crises. The transition probabilities governing the switch are endogenous, conditional on the state variables. With the toolkit, DSGEs can be used to draw the distribution of variables in order to measure the expected welfare of macroprudential policy. This allows for calibrating macropru-dential tools to trade off mean and variance optimally. The toolkit unifies DSGE modeling with early warning (crisis prediction) methods. The endogeneity of the probability of regime switches reflects the fact that the probability of financial crises depends on the state of the economy while its timing cannot be forecasted. Because financial markets do not anticipate financial crises, *RegGae* assumes the typical per-fect foresight of the future sequence of regimes.

Keywords: DSGE, regime-switching, occasionally binding constraint, endogenous probability, financial crises, macroprudential policy, optimal policy

JEL Classification: C65, G01, G28.

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1 Introduction

RegGae is a toolkit designed to adapt DSGE models for analyzing macroprudential policy. The toolkit is dubbed *RegGae: Regime-Switching General Equilibrium with Endogenous Probabilities and Deterministic Expectations of Future Regimes*. It can be thought of as a toolkit or *framework*, rather than a specific model, as its high generality allows it to be applied to most existing DSGE models being used. The design choices were made with the specific objective of applying it for macroprudential policy. This article presents the toolkit, how it is setup and solved, and applies it to a simple two-equation DSGE with a regime-switch. Codes for Matlab/Dynare are made available to implement the toolkit. The contribution is both methodological and substantive. On the methodological side, the article characterizes closed-form solutions for regime switching DSGEs (RS-DSGEs) with deterministic expectations of future regimes (perfect foresight). On the substantive side, it provides an avenue to analyze macroprudential policy and financial stability with DSGE. These ideas will be developed in this article.

The overhaul in banking regulation that followed the 2008-09 global financial crisis created, or provided renewed interest in, policy tools that reduce the probability of financial crises or limit their severity. These policies were labeled "macroprudential" due to their nature that incentivizes prudent risk-taking by financial institutions that behave in ways that produce system-level, or macro, interactions. An example of such tools is Basel III's countercyclical capital buffer. The buffer level is to be managed by regulators along with the credit cycle under a macroprudential policy *rule*. But doubts remain on how to calibrate these rules: what level should be chosen by regulatory authorities at each moment? What are the optimal parameters of the reaction function of macroprudential policy? How should we explain policy trade-offs for policy-makers so that they are accountable to society? Can shock simulations help elaborating a narrative to communicate decisions to the public? These questions can be answered with DSGE models augmented for this purpose.

At its most basic level, policy analysis is weighing the costs *and* benefits of each option for policy-makers to pick the best. The tools for this exercise should provide both sides of the coin. DSGE models of the constant-parameter family can quantify the *costs* of policy tightening — macroprudential policy, monetary or other — in terms of output forgone. This is commonly done with impulse response functions displaying the time trajectory of output and other welfare-relevant variables following a policy shock or innovation. The cost is the difference between the trajectories of the welfare relevant variables with and without the policy action. However, the *benefits* of macroprudential policy with constant-parameter DSGEs cannot be quantified with impulse responses. Macroprudential policy is geared towards ensuring the smooth, uninterrupted functioning of the financial system. It does so by strengthening the system's resilience, thereby lowering the likelihood of the realization of a destabilizing event sufficiently severe so as to tip the system into a financial crisis. The exogenous shocks introduced in constant-parameter DSGEs do not represent financial crises: financial crises are not "just" extraordinarily large shocks, periods of greater volatility. Instead, financial crises are much graver events, distinctive in both nature and magnitude, with shifts in the fundamental relationships governing the economy. In the absence of financial crises in constant-parameter DSGEs, the macroprudential tools appear in these DSGEs simply as an alternative for *monetary* policy, not for financial stability policy. This is because macroprudential policy operates through many of the same channels of monetary policy and have effects on aggregate demand, the output gap, and prices. Constant-parameter DSGEs allow for measuring the demand-management impact of macroprudential tools but not their financial stability benefits. The absence of financial crises in these typical DSGEs implies there is no financial stability metric against which to perform cost-benefit assessment of policy.

To augment DSGEs for macroprudential policy, we introduce regime-switches to function as financial crises inside DSGEs. A financial crisis can be seen as a regime-switch, a change, a self-confirming change, in the system's parameters (such as market liquidity) when over-leveraged institutions face credibility problems and when stabilizing forces no longer dominate (Borio and Zhu, 2012). The switch often occurs when agents realize they had been making mistakes. Under these circumstances, a new equilibrium emerges in which financial intermediation is severely disrupted and a credit crunch ensues. The regime switches considered in this article correspond, but are not limited, to the switches caused by the occasionally binding constraint ("OccBin" or "OBCC", if the constraint is a collateral constraint) method to embed financial crises into DSGE (Guerrieri and Iacoviello, 2015; Bianchi and Mendoza, 2018; Laséen et al. 2017; IMF, 2017). Because of this correspondence and because regime switches in financial crises are typically seen as resulting from binding constraints, this article will use the terms "regime switch" and "constraint binding" interchangeably, unless the regime change in question is not strictly a binding constraint — it can be, for example, a low technology regime in a growth DSGE model. A regime change is a reduced form representation of a fully microfounded model of triggers of financial crises. The occbin models in Laséen and al. (2017) and Bianchi and Mendoza (2018) microfound to some extent what happens in a typical financial crisis — fire sales, asset price drops, liquidity hoarding — and endogenize its probability. They do not microfound why the regime switches, the crisis trigger itself. *RegGae* goes along the same lines and proposes a solution for any DSGE with reduced-form representations of financial crises. The disrupted-intermediation regime when the constraint binds — that is, when the rules (parameters and equations) governing the system change

— is consistent with a lower *equilibrium* output and prices that can be modeled through regime switches (Woodford, 2012).

In order to conduct macroprudential policy analysis, we augment RS-DSGEs by introducing also endogenous probabilities of regime changes. This is because shocks in RS-DSGEs are typically assumed to be exogenous, that is, their distributions are time-invariant, independent from the system's state. However, financial crises are shocks whose probability and magnitude vary with time and depend on the state of the system. In fact, "[t]he [2008] shock was endogenous..." (Stiglitz, 2018). For example, credit growth affects the probability of financial crises. To augment DSGE and thereby overcome the "exogeneity" inadequacy, we propose that the transition probabilities governing the regime-switch be endogenous (state-contingent). It is a generalization of the endogenous mechanism of binding collateral constraint of Bianchi and Mendoza (2017) and Laséen et al. (2016). This modeling choice would be consistent with what have come to be common knowledge in the financial stability field: while the timing of crises is unpredictable, the buildup of the vulnerabilities that make crises more likely can be observed.

We adopt an additional component that distinguishes *RegGae* from other RS-DSGE methods: the *deterministic* expectations of future regimes, *perfect foresight*, as opposed to a probabilistic forecast of regimes. We provide the solution for a general RS-DSGE under the assumption that agents expect a *certain* future path of regimes: agents believe they know the future regime sequence and form expectations accordingly. This assumption is used widely in most DSGE when applying shocks to technology, preferences, and other deep parameters in the model: agents are assumed to know the future trajectory of the variable receiving the shock. This assumption is used in OccBin models of Guerrieri and Iacoviello (2015) — GI, henceforth —, although not explicitly stated. While information may be available that financial vulnerabilities are building up and a crisis is more likely, agents may have other incentives to behave as if a financial crisis would never happen (Ajello et al., 2015; Aikman et al., 2018; Gennaioli et al., 2015). Indeed, bankers may incur moral hazard, compensation may be based on benchmarking, behavior may be overoptimistic, limited liability may cause risk underpricing, until the music stops. Indeed, such market failures justify the adoption of prudential regulation standards such as Basel III. For simplicity, we assume even the policy-makers, who implement the policy rules, have deterministic expectations of future regimes. It is a departure from fully rational expectations but it still implies agents are forward-looking and act rationally, consistently with (the deterministic and possibly wrong) expectations. We claim this assumption can be appropriate to study financial crises based on empirical evidence. Nevertheless, we admit the validity of this assumption is ultimately an empirical question and depends on the specifics of the DSGE being used and on the expectation formation protocol assumed by the analyst.

RegGae's adopts a solution strategy of regime-wise linearization: a first-order perturbation around each regime-specific steady state, equivalent to GI's solution. The difference between GI and *RegGae* is one of context, application and method, not of solution. GI is concerned with a constraint — such as $i_t = \max\{0, \text{Taylor Rule}\}$, which is the zero lower bound — that was binding today and they look for the policy function with a "guess-and-verify" algorithm: guess how long the constraint would bind, solve the model backwards starting from the last constrained period (guessed), and verify that the constraint would bind exactly the guessed duration (or try again).

RegGae provides a closed form solution for all time-varying coefficients, including for the variance-covariance matrices, which is absent in GI. This is key for applying shocks during any regime, which is necessary for macroprudential policy analysis but impossible to be derived with GI's method. *RegGae* provides the solution for each duration expected by agents the analyst may wish to assume. *RegGae* is concerned with switches in parameter values when financial crises erupt. It looks for the policy function that depends on what agents expect to happen next, and solve the model backwards from the starting period of the "s-infinite" regime (the regime expected to last forever). The solution in GI also assumes deterministic expectations of future regimes: agents do not weigh future outcomes with their probability and the "volatility paradox" also holds. *RegGae's* theorem formalizes sufficiency conditions for uniqueness, a "Long-Run Taylor Principle for Macroprudential Policy" of sorts. Our contribution is to shed new light on GI's OccBin, to bring up a different way to use it, applied to macroprudential policy analysis.

RegGae is to be used in a different way and it allows deriving optimal macroprudential policy and optimal coordination with monetary policy. Instead of focusing on impulse response functions as the typical DSGE exercise, *RegGae* is to be used simulating the hypothetical paths of the variables and deriving their distribution. It is a tool for macroprudential analysis thanks to the fact that it allows for identifying the financial stability risks involved in each policy choice or shock, including "GDP-at-risk" (GaR) and other "value-at-risk" (VaR) metrics (IMF, 2017; Anderson et al., 2018).

RegGae provides an avenue for further developments needed in DSGE modeling, a topic explored in an dedicated edition of the *Oxford Review of Economic Policy* (January 2018):

1. **Non-linearities:** *RegGae* provides a convenient way to introduce non-linearities in DSGEs while maintaining the linearized, usual structure of existing DSGEs. It does so to the extent that a regime-switch is a highly non-linear event. While each

regime is linear or linearized, the switch (or the expectation of a switch) introduces a non-linearity conveniently in an otherwise fully linearized system.

2. **Deterministic expectations of future regimes:** *RegGae* introduces an expectation formation structure that departs from full rationality while maintaining the principle that agents care about the future in the spirit called for by Blanchard (2018). *RegGae* assumes expectations are imperfectly rational in the sense that regime expectations are *deterministic*: in each period, agents assign a probability of zero or one to each future regime. Agents believe they know the future regimes and act accordingly. This puts DSGE modeling closer to the literature on behavioral economics and to the empirically observed behavior during bubbles, euphoria, and "irrational exhuberance" (Ajello et al., 2015; Aikman et al., 2018; Gennaioli et al., 2015). *RegGae* contains, as a special case, some elements of the heuristics of Woodford (2018): agents may be assumed to care about only a finite number of periods ahead.

3. **Endogeneity of shocks:** Pre-crisis DSGEs did not model properly the buildup of vulnerabilities to shocks, which are endogenous (Woodford, 2012; Stiglitz, 2018). *RegGae* addresses this concern frontally by assuming endogenous probabilities of financial crises.

In the next section we present how *RegGae* relates to previous literature. In section 3, we introduce *RegGae*, the general toolkit for RS-DSGEs with endogenous transition probabilities and deterministic expectations of future regimes. Finally, we outline the way for optimizing policy rules and provide a simple demonstration example with a 2-equation DSGE. The appendix contains the proofs.

2 Relation to previous literature

Regime-switching DSGEs are not new but only recently started to make their way into the study of financial stability. A series of theoretical studies have facilitated the introduction of RS-DSGEs into the financial stability realm. Starting from the seminal work by Hamilton (1989) came the work by Davig and Leeper (2007) and by Farmer, Waggoner and Zha (2009 and 2011), which characterized equilibrium and derived necessary conditions for existence and stability of RS-DSGEs. The idea that financial crises probabilities might be endogenous was put forth by Woodford (2012) and formalized and solved by Maih (2015) and Barthélemy and Marx (2017). Nevertheless, these works apply the technique to study monetary policy regimes where expectations are fully rational in that they take into account the true probability of future regime switches. Our application and, therefore, assumptions about expectations will differ from this line of work and be closer to GI's OccBin framework. We bypass the debate on system stability concepts given our focus in *deterministic* expectations justified by the specific application of interest.

We build on a stream of work that proposes the use of RS-DSGE models with endogenous probabilities to analyze financial stability and crises. Davig and Leeper (2009) mention that monetary policy changes during financial crises but do not model financial crises as regime changes explicitly. Woodford (2012) elaborates the idea and formalizes it but does not provide a general solution for infinite periods. Benes et al. (2014a, 2014b) come close to using regime switching to assess financial stability by modeling expectation reversals (deterministic) causing hard crashes. More recently, it was proposed the study of rational bubbles and financial stability would follow regime-switching models but with fixed probabilities of regime switches (Martin and Ventura, 2018). From the stream of literature of "occasionally binding constraint" used for analyzing the zero lower bound of interest rates emerged some work that applied the same technique for modeling nonlinearities and engineering financial crises. He and Krishnamurthy (2014), Laséen et al. (2017), and Bianchi and Mendoza (2018) introduce DSGEs with occasionally binding *collateral* constraints: a stochastic (auto-correlated) variable governing the activation of the constraint. In this line of work, the endogenous probability of regime switches emerges naturally from the model instead of being assumed.

Our paper also contributes to the debate on the welfare benefits of leaning-against-the-wind (LAW, using monetary policy to counter financial stability risks). Svensson (2016 and 2017) argues LAW does not improve welfare. However, the method adopted in those articles is a cross-section comparison of the welfare of different states weighed by probabilities, not a dynamic weighing of regimes' welfares as we do here. As indicated by Laséen et al. (2017), it applies to a surprise episode of LAW, not to a systematic LAW rule. Regarding techniques to measure welfare, *RegGae* is better equipped to assess welfare than other studies that build in, into welfare functions, financial stability objectives. This is because it takes full account of crises probabilities and the post-crisis welfare developments and normalization of regime. Indeed, articles such as Angelini et al. (2014) and Laureys and Meeks (2018) build in welfare functions volatilities and deviations from "sustainable" levels of financial stability-relevant variables (respectively). This is because our toolkit takes full account of the benefits of macroprudential policy by reducing crises probabilities. The framework for welfare assessment in Laséen et al. (2017) is equivalent to ours. We proceed now to present *RegGae*.

3 *RegGae*, the mathematical toolkit

RegGae has three elements: a regime-switching DSGE, an expectation formation protocol, and an endogenous probability function for regimes. In this section we introduce one of these elements at a time, starting with the RS-DSGE. We present the solution after the first two elements to highlight that the third element is not needed for the solution.

3.1 The regime-switching DSGE

Consider a generic RS-DSGE where some parameter values and some equations may change every period according to the realization of a random variable, the regime variable, s_t. The RS-DSGE can be represented by:

$$E_t[g_{s_t}(x_{t+1}, x_t, x_{t-1}, \Gamma(s_t), \Upsilon(s_t), \varepsilon_t)] = 0 \tag{1}$$

where:

- $E_t[\cdot] \equiv E[\cdot|\Omega_t]$ is the expectation operator conditional on the information set known at time t;
- $g_{s_t}(\cdot)$ is a regime-specific system of difference equations in x_t, its lead and lag;
- x_t is the vector of endogenous variables;
- $\Gamma(s_t)$ is a vector of regime-specific parameters of g_{s_t} outside the control of the social planner. It can contain (and be thought of as) both parameters that enter $g_{s_t}(\cdot)$ and actually shift the equations as well as *sunspot* variables that coordinate agents to play a particular equilibrium (in multiple equilibrium settings);
- $\Upsilon(s_t)$ is a vector of (possibly) regime-contingent parameters chosen by the social planner;
- ε_t is a vector of exogenous shocks.

The assumptions about the properties of the distributions of the regime variable s_t and the exogenous shocks ε_t will be introduced as we present the toolkit.

Examples of regimes may include a normal regime, a bubble/credit boom regime, a crisis regime, a zero lower-bound regime, a post-crisis regime, and variety of intermediary regimes. The toolkit allows for modeling crises of different severities by assuming each crisis severity corresponds to a different regime such as "severe crisis" regime or "mild crisis" regime.

Each regime may possess a regime-specific steady state, $\bar{x}(s_t)$. The regime-specific steady state is the steady state that would prevail if there was only that regime. The regime-specific steady state solves:

$$g_{s_t}(\bar{x}(s_t), \bar{x}(s_t), \bar{x}(s_t), \Gamma(s_t), \Upsilon(s_t), 0) = 0 \tag{2}$$

The system can be linearized around each regime-specific steady state, regime-wise. Then, the generic regime-wise linear RS-DSGE can be rearranged as:[1]

$$A(s_t) \begin{bmatrix} x_t^p \\ E_t[x_{t+1}^j] \end{bmatrix} = B(s_t) \begin{bmatrix} x_{t-1}^p \\ x_t^j \end{bmatrix} + C(s_t) + D(s_t)\varepsilon_t \tag{3}$$

where

- x_{t-1}^p is a column vector, a partition of x_{t-1}, of p predetermined variables;
- x_t^j is a column vector, the complementary partition of x_t, of j non-predetermined, forward-looking variables ("jumpers");
- $A(s_t)$, $B(s_t)$, and $D(s_t)$ are conformable matrices corresponding to the Jacobians of the DSGE[2] and the variance-covariance matrix of shocks. They are functions of $\Gamma(s_t)$ and $\Upsilon(s_t)$.
- The non-homogenous term $C(s_t)$ is a regime-specific vector of constants resulting from the steady state of each regime. Specifically,

$$C(s_t) \equiv [A(s_t) - B(s_t)]\bar{x}(s_t) \tag{4}$$

  Naturally, the regime-specific steady states $\bar{x}(s_t)$ are also functions of $\Gamma(s_t)$ and $\Upsilon(s_t)$.

We normalize the fundamental shocks vector ε_t to unit variance. Thus, $D(s_t)$ is a regime-specific matrix of standard deviations and correlations. Therefore, *RegGae* comprehends the stochastic volatility case.

With this specification, all elements of this regime-wise linearized, general DSGE are regime-dependent: the matrices that govern the system's motion direction, $A(s_t)$ and $B(s_t)$, the system's level, $C(s_t)$, and shock variance-covariances, captured via $D(s_t)$.

[1] In this representation, we wrote off the "static variables" (those which appear only in period t) with the QR decomposition, but they can easily be brought back in.

[2] The Jacobians can be obtained with Dynare for each regime-specific steady state.

We adopt the end-of-period timing convention for state-variables and expectations, in keeping with Dynare: x_t^p is the value at the *end* of period t, *after* all decisions in t are taken and all t-indexed random variables are realized.

Equation (3) is the expectational linear system to be solved. To that end, we need to know the expectation formation protocol.

3.2 Deterministic Expectations of Future Regimes

Solving equation (3) requires specifying the expectation formation protocol. The model solution depends on the sequence of regimes that materialized previously and the regimes expected in the future. In constant-parameter DSGEs, this is not an issue as there is a single regime and thus expectations conform with the true process. But in regime switching DSGEs there is a need to specify beliefs about future regimes. Solving the model with rational expectations and endogenous probabilities of regime switch is a technical challenge that only recently has been overcome (Barthélemy and Marx, 2017; Maih, 2015). Notwithstanding these technical advances and because the application of interest here is financial crises, a different route is taken by *RegGae*. We provide a solution for the toolkit under the assumption all agents, including the policy authority, do not incorporate in their expectations the true probability the regime may switch.[3] As in perfect foresight models, agents are assumed to believe the current regime and their future path is *certain*: they believe they know the current regime and their future sequence and attribute only probabilities 0 or 1 to current and future regimes, never a probability strictly between 0 and 1. This assumption is also imposed by GI together with the additional requirement that the expected duration until return to baseline regime depends on the tightness of the occasionally binding constraint. How long one expects to be in a regime may depend on the state vector. But conditional on the state vector, expectations are deterministic. Agents do assess regime duration properly as they form beliefs about the future regime sequence. In a sense, this presents one answer to the question asked in Blanchard (2018): "How can we deviate from rational expectations, while keeping the notion that people and firms care about the future?".

To provide a clear understanding, consider a DSGE model with a production function where TFP is subject to a shock. Typically, it is modeled assuming TFP A_t is an endogenous variable and its law of motion is one of the equations of the DSGE:

$$\ln A_t = \rho \ln A_{t-1} + \sigma \varepsilon_t \text{ where } |\rho| < 1 \text{ and } \varepsilon_t \sim N(0,1) \tag{5}$$

[3]Note the social planner is not an agent in the toolkit. Its role is only to choose only once, at the beginning of times, the reaction function to be implemented by the policy authority during the evolution of the system.

A typical exercise to analyze the dynamics of a DSGE model is to derive the impulse response functions. We perform this exercise in a simple DSGE in the next section. It assumes initially the system is in steady state $A_0 = 1$. By applying a productivity shock of one standard deviation, $\ln A_1 = \sigma$, productivity is knocked out of its steady state and starts an asymptotic return at a rate ρ. This implies the agents in such typical DSGE are assumed to know exactly the future trajectory of A_t (perfect foresight). The entire dynamics is derived based on a perturbation around a single steady state, $A_0 = 1$.

Under *RegGae*, TFP may be considered not an endogenous variable but a parameter in the system. Each value of TFP corresponds to a different regime $A(s_t)$ and its law of motion provides the future sequence of regimes expected by agents. Therefore, the assumption of perfect foresight in *RegGae* is the same as the most common DSGEs in the literature. The difference is in the regime-wise linearization, which yields a different solution that respects the non-linearities. The next section presents an example where *RegGae* solution is superior, closer to the true non-linear solution.

Putting formally the expectations of the future sequence of regimes, for each node in the regime history tree $(..., s_{t-2}, s_{t-1}, s_t) \equiv \Im_t$, for each history of endogenous variables $(..., x_{t-2}, x_{t-1}) \equiv X_{t-1}$ and current exogenous shock ε_t, agents form expectations that the regime will be $\Re(\Im_t, X_{t-1}, \varepsilon_t) = (s_t, s_{t+1}, s_{t+2}, ...)$ where $\Re(\cdot)$ is an ordered set-valued mapping from past and present histories of regimes $\Im_t$ to expected present and future regimes. The toolkit allows the modeler to assume expectations deviate from the true current regime. This flexibility differentiates *RegGae* from OccBin's algorithm in which expectations abide by regime feasibility. We specify expectations with the following assumption:

Assumption 1. ***Deterministic expectations of future regimes:*** *Let R be the set of regimes, $\mathbb{R}$ be the real line, n be the number of endogenous variables, n_e be the number of exogenous shocks. Let there be a mapping $\Re$ from past and present histories to present and future sequence of regimes*

$$\Re : \prod_{-\infty}^{t} R \times \{\mathbb{R}^n\}_{-\infty}^{t-1} \times \mathbb{R}^{n_e} \rightarrow \prod_{t}^{\infty} R$$

For any history up to end of period $t-1$ plus the random variables realized in t, namely, (s_t, ε_t), denoted $(\Im_t, X_{t-1}, \varepsilon_t)$, agents assign probability 1 that the current regime and its future sequence will be $\Re(\Im_t, X_{t-1}, \varepsilon_t)$, that is,

$$E_t[s_t, s_{t+1}, s_{t+2}...|\Im_t, X_{t-1}, \varepsilon_t] \equiv \Re(\Im_t, X_{t-1}, \varepsilon_t)$$

We call $\Re(\Im_t, X_{t-1}, \varepsilon_t)$ an *expectation formation protocol.* There is a number of possibilities to explore with this specification, which were not considered in GI. One example, in the context of zero lower bound for interest rates, is to assume agents expect the central bank will comply with the commitment to keep interest rates at zero for a certain number of periods k even after the zero-lower bound constraint ceases to bind. The number of periods k can be set optimally so that inflation rises faster. Then there is a third regime — not considered by GI — where interest rates are zero but the zero-lower bound is slack (the reference regime monetary policy rule would call for a positive interest rate). The expected future regime sequence is revised every period according to the realization of random variables.

RegGae allows sufficient flexibility for the researcher to calibrate the expectation formation protocol for the specific application of interest. Whether expectations are deterministic is ultimately an empirical question. The assumption may be valid for some specific DSGEs capturing some specific features of the real world but not for others. The extent of the difference it makes relative to the fully rational expectations equilibrium may also depend on the specific DSGE being used and on the specific expectation formation protocol. Because the validity of this assumption is DSGE-specific and $\Re$-specific, we do not compare its results with a fully rational regime-switching framework.

We note that, while this assumption may appear strong,[4] a closer look shows it is justified for the specific objective of modeling financial crises and macroprudential policy. First, there is empirical evidence that suggests it. This assumption would be consistent with findings from the behavioral economics literature about how agents factor in infrequent events such as financial crises. This assumption can represent the stylized fact that individual lenders and borrowers do not account for financial stability risks when deciding whether to lend and borrow (remember the subprime?). After a hiatus without crises, investors fall into believing "this time is different" and neglect crises risks (Gennaioli et al., 2015). As stated by Stiglitz (2018), "banks engage in contracts with each other that may be individually rational, but result in greater systemic risk...". Borio and Zhu (2012) argue that "even when risks are recognized, it may sometimes be difficult for market participants to withdraw from the fray, as the short-term pain is not seen as offset by future potential gains". Ajello et al. (2015) find evidence in forecasters survey that the subjective crisis probability was close to zero in early 2008, just months before the most acute moment of the crisis in the fall of the year.

[4]GI also argues that, while this assumption may appear "draconian", it is routinely imposed when solving DSGE models by standard first-order perturbation.

Perhaps the most compelling empirical observation suggesting that agents behave as if financial crises are a zero probability event is the fact that financial markets are typically calm (low spreads and low volatility) until the crisis eve, when vulnerabilities are already in place ("volatility paradox"). In fact, it is understood that long periods of low volatility and low spreads may increase risk appetite, fuel disregard for risk buildup, and actually increase the vulnerabilities and, thus, the probability of crisis (Borio and Zhu, 2012; Anderson et al., 2018; Adrian et. al, 2019). Central banks and financial stability authorities actually monitor *low* volatilities, not high, as leading indicators of financial crises (Aikman et al., 2017). He and Krishnamurthy (2014) suggest that lack of information about leverage and of repo market transparency can be blamed for the *lack of market anticipation* of the 2007-09 crisis. They state categorically that "*is not possible to construct a model in which spreads are low ex-ante, as in the data, and yet the probability of a crisis is high. [...] our model offers little advance warning of the crisis that followed. That is, without the benefit of hindsight, in both the model and data the probability of the 2007-2009 crisis is low.*" These findings provide empirical support for our assumption that expectations of financial crises are deterministic. With this assumption, we provide a toolkit for constructing such models He and Krishnamurthy (2014) state are impossible to be built.

Second, there are game theoretical foundations for this modeling choice. In any equilibrium of a repeated game with multiple equilibria, players do not expect the equilibrium path will switch to a different one during the play of an equilibrium (or it would not be an equilibrium). Thus, when an equilibrium shift does happen, it is fully unexpected. Naturally, the regime switch under *RegGae* is a closed-form solution that "black-boxes" a microfounded game governing the equilibrium path. This argument is valid for unexpected regime switches, not for expected regime switches. For expected switches, the effects of the switch are fully built in current actions.

And third, Assumption 1 does *not* imply the probability of crises plays no role. In fact, if the social planner chooses the reaction function to be implemented by the policy authority taking into account the distribution of s_t, then by implementing the macroprudential policy rule, the policy authority causes agents to react indirectly (via the policy rule) and to internalize the probability of crises in their actions and expectations, albeit not explicitly. For example, agents may not expect a financial crisis but they may expect the activation of the countercyclical capital buffer (CCyB). If they act based on the expectation of the activation of the CCyB, they act in line with the probability of crises.

3.3 Solution of *RegGae*

Equation (3) together with an expectation formation protocol $\Re(\Im_t, X_{t-1}, \varepsilon_t)$ can be solved by regime-wise linearization under certain conditions yielding a solution in state-space representation in the following format:

$$\begin{bmatrix} x_t^p \\ x_t^j \end{bmatrix} = \begin{bmatrix} M_\iota \\ V_\iota \end{bmatrix} x_{t-1}^p + \begin{bmatrix} H_\iota \\ Y_\iota \end{bmatrix} + \begin{bmatrix} K_\iota \\ W_\iota \end{bmatrix} \varepsilon_t \tag{6}$$

where the general solution for block-matrices M_ι, H_ι, K_ι, V_ι, Y_ι and W_ι is presented below and derived in Appendix A. The upper block of matrix equation (6) corresponds to the state-equation while the lower-block corresponds to the signal-equation. The quality of the regime-wise linear solution has already been established by GI. These matrices are grouped into history node types, denoted by ι. A node type is defined by the sequence of regimes expected going forward at that node. For example, it can be a node type where regime s is expected forever, or where regime s will last k periods and then the regime will switch to $E_t[s_{t+k+1}] = s'$, etc. The set of node types depends on $\Re$.

Thus the solution, if one exists at a specific history node, depends on the expectation formation protocol but not on the actual process governing the regime switch as, by assumption, this does not affect expectations. Let the set of matrices that solve the model at the nodes when a solution exists be denoted by $\mathbb{S}$ and the set of node types be denoted by $\mathbb{I}(\Re)$, that is,

$$\mathbb{S}(\Re) \equiv \{M_\iota, H_\iota, K_\iota, V_\iota, Y_\iota, W_\iota | \iota \in \mathbb{I}(\Re)\} \tag{7}$$

An expectation formation protocol may culminate in history nodes which are type *s-infinite*. In an s-infinite history node type, *agents expect regime $s \in R$ to last forever.* It is an identical situation to the well known, constant parameter, single regime case. This definition will be convenient when characterizing the solution (Appendix A).

Definition 1. ***s-infinite history types:*** *History $(\Im_t, X_{t-1}, \varepsilon_t)$ is s-infinite if and only if, for some $s \in R$,*

$$\Re(\Im_t, X_{t-1}, \varepsilon_t) = (s, s, s, ...)$$

In the following propositions we present the regime-wise linear solutions for two types of history nodes (using the notation of Appendix A).

Proposition 1 (Linear solution in s-infinite history nodes). *Let the node $(\Im_t, X_{t-1}, \varepsilon_t)$ be an s-infinite history node and let $\mathbb{S}(\Re(\Im_t, X_{t-1}, \varepsilon_t))$ indicate the linear solution there:*

$$\mathbb{S}(\Re(\Im_t, X_{t-1}, \varepsilon_t)) \equiv \{M, H, K, V, Y, W\, | s \in \mathbb{I}(\Re(\Im_t, X_{t-1}, \varepsilon_t))\}$$

Let the Blanchard-Kahn conditions hold in regime s. Then $\mathbb{S}(\Re(\Im_t, X_{t-1}, \varepsilon_t))$ is equal to:

$$V = -Z_{22}^{-1} Z_{21} \tag{8}$$

$$Y = -Z_{22}^{-1} S_{22}^{-1} (I_j - T_{22} S_{22}^{-1})^{-1} \tilde{C}_j \tag{9}$$

$$W = -Z_{22}^{-1} S_{22}^{-1} \tilde{D}_j \tag{10}$$

$$M = (T_p Z_1 + T_p Z_2 V)^{-1} (S_p Z_1 + S_p Z_2 V) \tag{11}$$

$$H = (T_p Z_1 + T_p Z_2 V)^{-1} \left[(S_p Z_2 - T_p Z_2) Y + \tilde{C}_p \right] \tag{12}$$

$$K = (T_p Z_1 + T_p Z_2 V)^{-1} \left[(S_p Z_2 W + \tilde{D}_p) \right] \tag{13}$$

Proof. The proof is Appendix A. □

Of course, this is the (first order) solution for constant-parameter DSGEs computed with Dynare, present in DSGE text-books. The only novelty thus far is the closed form solution for the non-homogeneous terms H and Y. The next proposition introduces a novelty: the closed form solution for finite node types, that is, where the regime is expected to change eventually at least once.

Proposition 2 (**Linear solution at finite histories**). *Let s and s' be two regimes, not necessarily the same. Let $(\Im_t, X_{t-1}, \varepsilon_t)$ be a history node before the expected switch from s to s', $\Re(\Im_t, X_{t-1}, \varepsilon_t) = (s, s', s_{t+2}, ...)$. Let $\mathbb{S}(\Re(\Im_t, X_{t-1}, \varepsilon_t))$ denote the linear solution at that node:*

$$\mathbb{S}(\Re(\Im_t, X_{t-1}, \varepsilon_t)) \equiv \{M_\iota, Q_\iota, S_\iota, V_\iota, Z_\iota, W_\iota | \iota \in \mathbb{I}(\Re(\Im_t, X_{t-1}, \varepsilon_t))\}$$

Let $E_t[\Re(\Im_{t+1}, X_t, \varepsilon_{t+1})] \equiv (s', s_{t+2}, ...)$ and let there exists a unique expected solution for the next expected history $(\Im_{t+1}, X_t, \varepsilon_{t+1})$:

$$E_t[\mathbb{S}(\Re(\Im_{t+1}, X_t, \varepsilon_{t+1}))] = \{M_{\iota'}, Q_{\iota'}, S_{\iota'}, V_{\iota'}, Z_{\iota'}, W_{\iota'} | \iota' \in \mathbb{I}(E_t[\Re(\Im_{t+1}, X_t, \varepsilon_{t+1})])\}$$

Then, the solution $\mathbb{S}(\Re(\Im_t, X_{t-1}, \varepsilon_t))$ is:

$$V_\iota = \left\{ I_j - (S_{22} Z_{22})^{-1} [T_{22} Z_{21} + T_{22} Z_{22} V_{\iota'}][T_p Z_1 + T_p Z_2 V_{\iota'}]^{-1} S_p Z_2 \right\}^{-1} (S_{22} Z_{22})^{-1} \left\{ [T_{22} Z_{21} + T_{22} Z_{22} V_{\iota'}][T_p Z_1 + T_p Z_2 V_{\iota'}]^{-1} S_p Z_1 - S_{22} Z_{21} \right\} \tag{14}$$

$$Y_\iota = \left\{ I_j - (S_{22}Z_{22})^{-1}[T_{22}Z_{21} + T_{22}Z_{22}V_{\iota'}][T_pZ_1 + T_pZ_2V_{\iota'}]^{-1}S_pZ_2 \right\}^{-1}$$

$$(S_{22}Z_{22})^{-1}\left\{[T_{22}Z_{21} + T_{22}Z_{22}V_{\iota'}][T_pZ_1 + T_pZ_2V_{\iota'}]^{-1}\big[-T_pZ_2Y_{\iota'} + \tilde{C}_p\big] + T_{22}Z_{22}Y_{\iota'}]\right\} \quad (15)$$

$$W_\iota = \left\{ I_j - (S_{22}Z_{22})^{-1}[T_{22}Z_{21} + T_{22}Z_{22}V_{\iota'}][T_pZ_1 + T_pZ_2V_{\iota'}]^{-1}S_pZ_2 \right\}^{-1}$$

$$(S_{22}Z_{22})^{-1}\left\{[T_{22}Z_{21} + T_{22}Z_{22}V_{\iota'}][T_pZ_1 + T_pZ_2V_{\iota'}]^{-1}\tilde{D}_p - \tilde{D}_j\right\} \quad (16)$$

$$M_\iota = (T_pZ_1 + T_pZ_2V_{\iota'})^{-1}(S_pZ_1 + S_pZ_2V_\iota) \quad (17)$$

$$H_\iota = (T_pZ_1 + T_pZ_2V_{\iota'})^{-1}[-T_pZ_2Y_{\iota'} + S_pZ_2Y_\iota + \tilde{C}_p] \quad (18)$$

$$K_\iota = (T_pZ_1 + T_pZ_2V_{\iota'})^{-1}[S_pZ_2W_\iota + \tilde{D}_p] \quad (19)$$

Proof. The proof is Appendix A. □

Proposition 2 shows the solution today depends on the expected solution tomorrow. It implies existence and uniqueness today is ensured by expected existence and uniqueness tomorrow. With that we can state the theorem of *RegGae:*

Theorem 1 (***RegGae*: Global Existence and Uniqueness**). *Let $\Re(\cdot)$ be such that, in all attainable regime history nodes $(\Im_t, X_{t-1}, \varepsilon_t)$, agents expect the system to culminate in s-infinite histories in a globally bounded number of periods. Let R_∞ be the subset of the regime set R of all regimes s for which s-infinite histories are expected somewhere in the attainable history tree. Let also the Blanchard-Kahn conditions hold for all regimes in R_∞. Then a unique solution $\mathbb{S}(\Re)$ exists everywhere in the attainable history tree.*

Proof. Let $(\Im_t, X_{t-1}, \varepsilon_t)$ be an attainable history node. Then, by assumption, $\Re(\Im_t, X_{t-1}, \varepsilon_t)$ is expected to culminate in s-infinite history for some regime $s \in R_\infty$ from time $t+n+1$ onwards, that is, $\Re(\Im_t, X_{t-1}, \varepsilon_t) = (s^e_t, s^e_{t+1}, s^e_{t+2}, ..., s^e_{t+n}, s, s, s...)$ where superscript e denotes the specific regime expected. Let $\mathbb{S}(\Re(\Im_t, X_{t-1}, \varepsilon_t))$ denote the subset of the solution set at history node $(\Im_t, X_{t-1}, \varepsilon_t)$. Let's show a unique $\mathbb{S}(\Re(\Im_t, X_{t-1}, \varepsilon_t))$ exists:
$s \in R_\infty \Rightarrow$ the Blanchard-Kahn conditions hold for s
$\Rightarrow \exists$ unique $E_t[\mathbb{S}(\Re(\Im_{t+n+k}, X_{t+n+k-1}, \varepsilon_{t+n+k}))]$ for $k \geq 1$ (by Proposition 1)
$\Rightarrow \exists$ unique $E_t[\mathbb{S}(\Re(\Im_{t+n}, X_{t+n-1}, \varepsilon_{t+n}))]$ (by Proposition 2)
$\Rightarrow \exists$ unique $E_t[\mathbb{S}(\Re(\Im_{t+n-1}, X_{t+n-2}, \varepsilon_{t+n-1}))]$ (by Proposition 2)
... (by backward induction)
$\Rightarrow \exists$ unique $E_t[\mathbb{S}(\Re(\Im_t, X_{t-1}, \varepsilon_t))] = \mathbb{S}(\Re(\Im_t, X_{t-1}, \varepsilon_t))$ (by Proposition 2) □

The theorem follows from the fact that, at any s-infinite history node, the Blanchard-Kahn (BK) conditions ensure expected existence and uniqueness at that history node

(Blanchard and Kahn, 1980). Intuitively, if expectations at all history nodes culminate in s-infinite histories after a finite, bounded number of periods, then the unique solution at that s-infinite history node can be rolled back to any starting point.

The main idea of Theorem 1 is also stated (informally) in GI. Theorem 1 simply states that the BK conditions are needed only for those regimes for which there are s-infinite histories. Regimes that are expected to last a bounded number of periods do not have to satisfy BK. But agents must expect that the history of regimes will eventually reach s-infinity at some node down the regime history tree. It is the deterministic expectation parallel of the result of Davig and Leeper (2007) that enlarged the determinacy set beyond the set where all regimes satisfy BK conditions, a "Long-Run Taylor Principle for Macroprudential Policy" of sorts. Note Theorem 1 is a sufficiency result for global determinacy. Other configurations of expectations — such as cyclical regime switch sequences or asymptotic convergence of regimes — can be investigated for node-wise and global sufficiency. Note also we applied the Law of Iterated Expectations. Therefore, the theorem would not apply if expectations violate this law.[5]

3.4 The endogenous probability of regime-switch

We now turn to the third component of *RegGae*. To model the fact that switching to a financial crisis regime depends on the state of the economy, we assume the probability of a switch is a function of the system's variables up to the previous period X_{t-1}, and of the history of regime realizations up to s_{t-1}, denoted $\Im_{t-1} \equiv (..., s_{t-2}, s_{t-1})$:

$$Pr[s_t = s|X_{t-1}, \Im_{t-1}] \equiv p(s|X_{t-1}, \Im_{t-1}) \tag{20}$$

where a Markov process is a special case.

This modeling strategy where crises are probabilistic events is consistent with the current understanding of financial crises. It is understood that it is possible to identify vulnerabilities and other conditions that make financial crises more likely and more severe but it is not possible to predict when a crisis will be triggered. For example, credit growth and indebtedness, which can be modeled with DSGEs, are usually considered vulnerabilities with superior early warning properties as crises predictors (BCBS, 2010; Gonzalez et al. 2017). As credit is booming and households and firms are more indebted, crises are more likely. However, the exact timing of financial crises cannot be predicted

[5] An example of expectations that violate the Law of Iterated Expectations would be the case where agents expect the current regime to last one additional period: $\Re(\Im_t, X_{t-1}, \varepsilon_t) = (s_t, s_t, s^e_{t+2}, s^e_{t+3}, ...)$ and $\Re(\Im_{t+1}) = (s_t, s_t, s^e_{t+3}, s^e_{t+4}, ...)$. In such a case, agents expect today something different from what they expect to expect tomorrow if expectations today are confirmed.

as the immediate crisis trigger may be a political or other exogenous event.

In many OBCC models, this crisis probability function emerges naturally from the model and do not need to be assumed (Laseen et al., 2017; Bianchi and Mendoza, 2018). In these models, the constraint slackness is a metric for crisis probability. This is because, the smaller the slackness, the more likely there will be a shock larger than the slackness — which would bind the constraint and tip the economy into a crisis.

Central banks, multilateral organizations, and other financial stability watchdogs already monitor proxies for function $p(\cdot)$ through a variety of means. Examples are the IMF's Vulnerability Exercise (Basu, Chamon, Crowe, 2017) and the "Financial System Stability Monitor" of the US's Office of Financial Research.[6] Function $p(\cdot)$ can be interpreted as the *vulnerabilities*, that is, the *unconditional* probability of a regime switch, unconditional on relevant factors not modeled in the DSGE. In actual policy-making, this unconditional probability is confronted with environmental *risks*, that is, outside knowledge about the environment and judgment to form a subjective *conditional* probability of a regime switch. Outside factors include political and external *risks*. *RegGae* allows modeling this idea precisely with a DSGE where credit and indebtedness are among the variables by assuming $p(crisis|\cdot, \Im_{t-1})$ is an increasing function of credit and indebtedness.

Note the probability of crises depends on the system in the previous period, not contemporaneously. As it has been shown, it would make no logical sense if the system's variables after a crisis determined the probability of triggering it in the first place (Barthelemy and Marx, 2017).

4 Analyzing and optimizing policy with *RegGae*

RegGae provides a natural way to search for the optimal policy. The model's solution $\mathbb{S}(\Re)$ combined with the process governing the regime switch $p(s_t|X_{t-1}, \Im_{t-1})$ can be simulated multiple times to draw the distribution of each variable in each date, conditional on a starting point. With some welfare metric, the expected welfare of each tuple $< \mathbb{S}, p >$ can be approximated by the Law of Large Numbers. Each value for $\Upsilon(s_t)$ yields a different distribution. This procedure allows for conducting a "GDP-at-risk" exercise (GaR) where, say, the 5-th percentile of the GDP distribution is mapped to each $\Upsilon(s_t)$. It allows also for conducting policy analysis by applying a shock to a policy tool and measuring the implied shift in the distribution of welfare. To conduct a macroprudential stress-test,

[6] https://www.financialresearch.gov/financial-vulnerabilities

it suffices to apply the desired shock, such as a credit boom, and draw the distribution of outcomes.

With the result of the simulations at hand, the social planner can then choose the value of the parameters $\Upsilon(s_t)$ of the reaction functions to be implemented by the policy authority in order to trade off mean and variance and thereby maximize expected welfare and achieve optimal coordination of monetary and macroprudential policy. In a linear model, the parameters $\Upsilon(s_t)$ are those rows of matrices $A(s_t)$ and $B(s_t)$ under the control of the policy authority. These may include (depending on the model) the parameters of the monetary and macroprudential policy rules — including the intermediate targets for inflation and financial stability.

The model's solution depends on the parameters of the policy rules and, to make this dependence explicit, we will write $\mathbb{S}(\Re, \Upsilon(s_t))$. Welfare depends on the sequence of realizations of the system, $\{x_{t+k}\}_{k=1}^{\infty}$. The optimization problem to be solved is:

$$\max_{\Upsilon(s_t)} E_t[U(\{x_{t+k}\}_{k=1}^{\infty})|\Im_t, p(\cdot), \mathbb{S}(\Re, \Upsilon(s_t))] \tag{21}$$

The constraints are built in the expectations of the social planner, including the BK conditions for s-infinite regimes. We assume the social planner knows the true process governing the regime switch, $p(\cdot)$, defined in equation (20). This models the idea that the conditions and vulnerabilities contained in x_t affecting the crisis probability are known by the social planner. The social planner also knows agents' expectation formation protocol, $\Re(\cdot)$, which embeds the cognitive biases of agents, including those of the macroprudential and monetary policy authority. The social planner acts once and for all by choosing $\Upsilon(s_t)$, thereby affecting agents expectations, actions, and the model's solution $\mathbb{S}(\Re, \Upsilon(s_t))$. The optimal choice takes agents' imperfectly rational expectations into account in the sense that the optimum depends on $\Re$. Thus, while agents form imperfectly rational expectations, their actions are affected by the rational expectations of the social planner through the choice of the parameters of the policy-makers reaction functions $\Upsilon(s_t)$.

4.1 A simple demonstration

In this section, we illustrate how to use *RegGae*. The example considers a situation where a costly policy can be deployed to mitigate the risk of negative shocks (regime switches) and their severity. With this example we show even simple models can be used to analyze macroprudential policy. This example rationalizes why it may be worth throwing resources away if it reduces the volatility of welfare of risk averse agents. Let there be the following DSGE:

$$W_t = E_t \sum_{t=0}^{\infty} \beta^t \ln C_t \tag{22}$$

$$Y_t = e^{z_t} K_{t-1}^{\alpha} \tag{23}$$

$$C_t + I_t = (1-\tau) Y_t \tag{24}$$

$$K_t = (1-\delta) K_{t-1} + I_t \tag{25}$$

$$z_t = \rho z_{t-1} + \sigma \varepsilon_t \text{ where } |\rho| < 1 \text{ and } \varepsilon_t \sim N(0,1) \tag{26}$$

where τ is an expropriation rate satisfying $0 \leq \tau \leq 1$. The expropriation rate is chosen once and for all by a benevolent dictator with a macroprudential objective of reducing the probability of negative shocks and mitigating their severity. This is really a DSGE on only two variables, consumption and capital, as output and investment are "static" (without leads and lags) and can be substituted away. The technology variable z_t is independent from (unaffected by) the other variables. The parameter values and the resulting steady state in the baseline regime ($z = 0$) are in Table 1.

Table 1: Baseline regime ($z = 0$): parameters and steady state

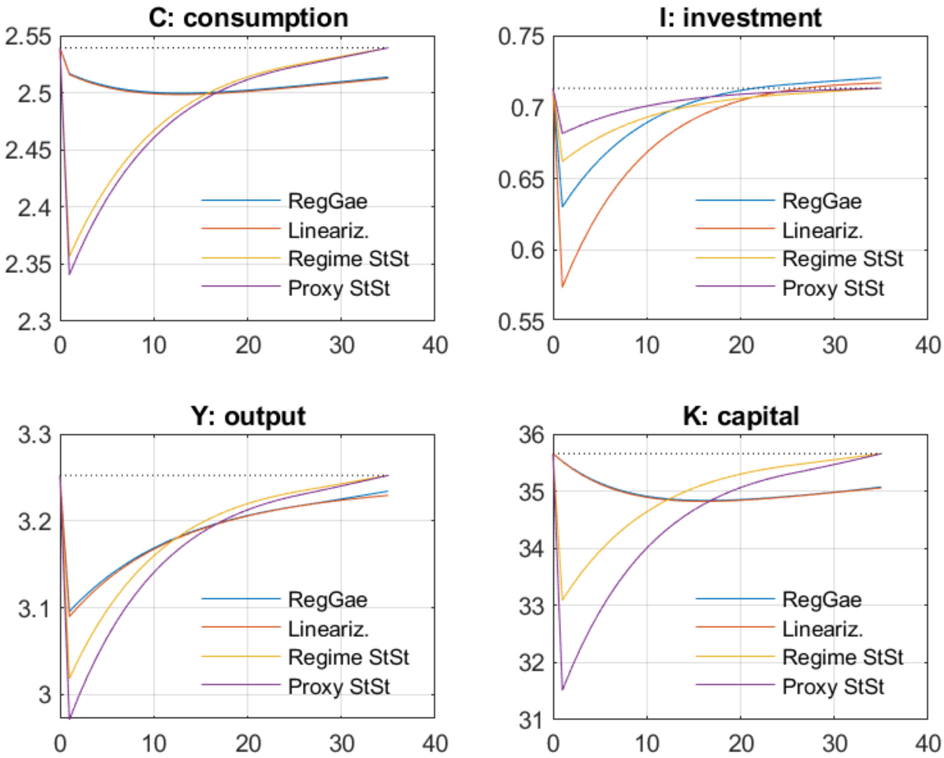
β	0.99
τ	0
α	0.33
δ	0.02
ρ	0.9
σ	0.05
$\bar{C}$	2.53928
$\bar{K}$	35.6565
$\bar{I}$	0.71313
$\bar{Y}$	3.25241

To set up this DSGE into *RegGae* format, drop the law of motion of the productivity variable z_t. Instead, call z_t the value of the productivity **parameter** in time t. Assume that expectation formation protocol is

$$\Re(\Im_t, X_{t-1}, \varepsilon_t) = \{z_t = \rho z_{t-1} + \sigma\varepsilon_t, \rho z_t, \rho^2 z_t, \rho^3 z_t, ..., \rho^k z_t, 0, ...\} \ \forall \ \Im_t \tag{27}$$

where k is the number of periods after which the regime parameter is arbitrarily close to the baseline regime, say, $\rho^k z_t < 0.0001$. This expectation formation protocol ensures this DSGE setup under *RegGae* is identical to the original DSGE.

Figure 1: IRF with RegGae: shock of $-\sigma$



It is possible then to perform the typical exercise of Impulse Response Functions of a negative shock and compare the results of *RegGae* with the linearized original DSGE (Figure 1). It suffices to construct the time-varying transition matrices with the formulas of Appendix A. To construct the transition matrices, start from the infinite history-type ($z_t = 0$) and move backwards with each value of z_t all the way until $z_t = -\sigma$ (about 35 periods). Then, to produce the IRF, start from the steady-state values of the variables, and roll the system forward with the sequence of transition matrices.

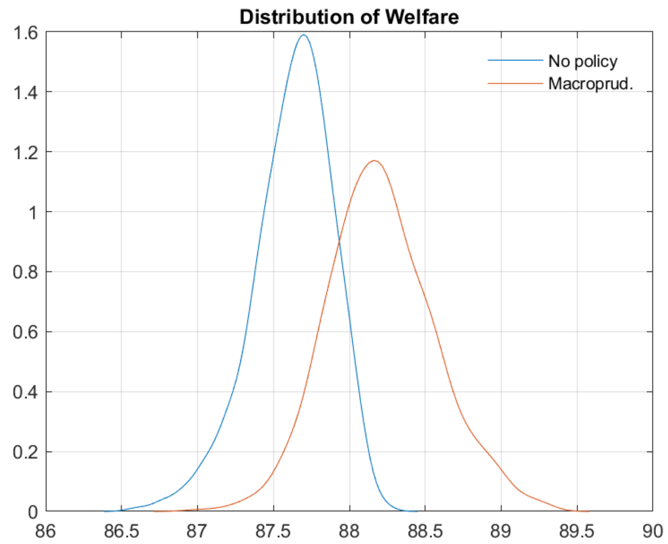
As expected, a negative productivity shock of one standard deviation (σ) causes a contraction in consumption, investment and output in both solution methods, standard linearization and *RegGae* (regime-wise linearization). However the contraction in investment is milder under *RegGae* than under the standard linearization (top-right chart of Figure 1). Indeed, the quality of *RegGae*'s solution is superior, closer to the true (non-linear) solution: the error in the resource constraint under *RegGae* is less than half of the linearized solution.

Let's now depart from the original DSGE and continue *RegGae*'s set up. Assume a negative shock represents a crisis and assume the distribution of z_t, denoted $p(z_t)$ is endogenous: it is conditional on the state of the economy contained in C_{t-1}:

$$p(z_t = -0.2 + 4\tau | \Im_{t-1}, C_{t-1}) = \frac{e^{\zeta_0 + \zeta_1 (C_{t-1} - 2.425)}}{1 + e^{\zeta_0 + \zeta_1 (C_{t-1} - 2.425)}} \tag{28}$$

and $p(z_t = \rho z_{t-1} | \Im_{t-1}, C_{t-1})$ with the complementary probability. We assume $\zeta_0 = \ln \frac{.005}{.0095}$ and $\zeta_1 = 75$. A crisis in this simple model is a negative productivity shock of size $-.2+4\tau$. The dependence of the shock size on τ (expropriation rate) can be thought of as capturing the channel whereby a (costly) macroprudential policy in good times dampens the severity of crises and represents a hedge against bad shocks. It is clearly a short-cut for a time-varying macroprudential policy where τ would be released when a shock hits and would be gradually built as consumption grows. Such a countercyclical macroprudential policy can be modelled with *RegGae* in an extension of this simple example.

Figure 2: Distribution of Welfare

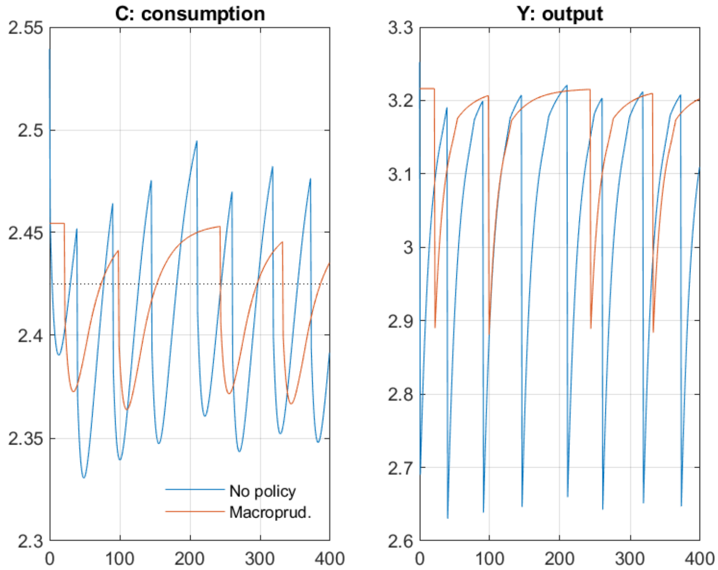


With these elements in place, we can now optimize over τ numerically to find the value that a benevolent dictator would choose to impose on society. Figure 2 depicts the distribution of welfare W drawn 1000 times with trails of 10,000 periods for $\tau = 0$ (no macroprudential policy) and $\tau = 0.0225$ (macroprudential policy).

$$W = \sum_{t=0}^{10,000} \beta^t \ln C_t$$

The resulting distributions are plotted in Figure 2. This procedure allows calibrating the macroprudential tool and comparing the outcomes of the system with and without macroprudential policy (Table 2).

Figure 3: A simulation with and without macroprudential policy



With these parameter values, expectation formation protocol, and crisis probability distribution, we find the result for this simple hypothetical setup:

1. The optimal macroprudential policy is approximately 0.0225. It is worth paying a cost in excess of 2% of output to reduce the probability and magnitude of crises.

2. The macroprudential policy slashes the probability of crisis by about one half (bottom rows of Table 2). The unconditional crisis probability drops from 1.77% every period to 0.91%. Interestingly, the average crisis probability is smaller under the "no policy" as the more frequent crises drive the system to operate longer in post-crisis history-types, with low consumption. This can be seen in the example simulation of Figure 3 of one century of crises and recoveries under the two policies. The figure shows clearly the smaller consumption volatility with macroprudential policy.

3. The macroprudential policy produces a higher average consumption and a smaller consumption volatility. As a result, expected welfare is higher and the welfare at risk (5% percentile of welfare distribution) is higher.

Table 2: Effects of Macroprudential Policy

	$\tau = 0$	$\tau = 0.0225$
	Steady state	
$\bar{Y}$	3.2524	3.2162
$\bar{C}$	2.5393	2.4545
$\bar{I}$	0.7131	0.6893
$\bar{K}$	35.6565	34.4657
	Means	
Y_t	3.0478	3.1580
C_t	2.3951	2.4145
I_t	0.6902	0.6830
K_t	32.6401	33.6234
W	87.6276	88.1881
	Variances	
Y_t	0.0239	0.0059
C_t	0.019	0.0009
I_t	0.0061	0.0012
K_t	0.7028	0.3725
	Welfare-at-Risk (5%)	
W-a-R 5%	86.4948	86.7386
	Shock probability	
$p(\text{crisis})$	0.0177	0.0091
Average $p(\text{crisis}\|C_{t-1})$	0.0005	0.0023

5 Conclusion

Macroprudential policy analysis can be conducted with DSGE models with a regime switching toolkit. The switch introduces financial crises in the model, a critical missing piece in traditional DSGE. To that end, *RegGae* provides a *how-to* explanation of elements needed and formulas to be applied to augment a DSGE into a macroprudential policy-ready DSGE. *RegGae* can be applied to *any* specific DSGE, including non-linear models. Therefore, existing DSGEs being used in central banks for monetary policy can be seamlessly augmented to inform macroprudential policy as well. It suffices to collect the Jacobians and steady-state vectors of the DSGE (one set per regime), rearrange the matrices in the format of equation (3), figure out a reasonable expectation formation protocols and apply the formulas in this article.[7] With the crisis probability distribution evolving in parallel to the DSGE process, *RegGae* makes viable the derivation of values-at-risk metrics and the simulation of shifts in values-at-risk (vulnerabilities) caused by any of model's shocks.

[7] The Jacobian can be obtained with software packages such as Dynare.

The challenge to use *RegGae* is rather empirical and computational than technical. On the empirical side, it is difficult to calibrate reasonable parameter values for crises regimes, let alone estimate them. This is because financial crises are rare, temporary and therefore elusive. In addition, finding a tuple $< \mathbb{S}, p >$ requires devising an appropriate expectation formation protocol and a crisis probability function $p(\cdot)$. This is a direction toward where the early warning research could usefully evolve and, in this sense, *RegGae* unifies the research on DSGEs with that on early warning methods.

Performing the optimization may require non-usual computational capacity for the search algorithm. The lack of analytic solution for the optimization problem implies the need to simulate the model numerous times to obtain good estimates of the expected welfare of each policy rule.

Despite these challenges, *RegGae* can be a useful tool for research on financial stability and for assisting policy making. It provides DSGE models with a different procedure and repurposes DSGEs for a new application.

Appendix

A Proofs of Propositions 1 and 2

A.1 Introduction

The solution is the linear function $x_t = g_t(x^p_{t-1}, \varepsilon_t, \Im_t)$ from the regime-specific system

$$A(s_t)\begin{bmatrix} x^p_t \\ E_t[x^j_{t+1}] \end{bmatrix} = B(s_t)\begin{bmatrix} x^p_{t-1} \\ x^j_t \end{bmatrix} + C(s_t) + D(s_t)\varepsilon_t \tag{29}$$

and the expectation protocol $\Re(\Im_t, X_{t-1}, \varepsilon_t)$. The protocol matters as it pins down the sequence of regimes expected. The solution takes on the form

$$\begin{bmatrix} x^p_t \\ x^j_t \end{bmatrix} = \begin{bmatrix} M_\iota \\ V_\iota \end{bmatrix} x^p_{t-1} + \begin{bmatrix} H_\iota \\ Y_\iota \end{bmatrix} + \begin{bmatrix} K_\iota \\ W_\iota \end{bmatrix} \varepsilon_t \tag{30}$$

where ι denotes the node type, that is, the sequence of regimes expected from that history. With some abuse of notation, we will use time index for subscripts of block-matrices keeping in mind that each history t has its own history type given by the expected sequence of regimes going forward. To solve (29), start by QZ-decomposing the pair $(A(s_t), B(s_t))$:

$$A(s_t) = Q(s_t)T(s_t)Z(s_t)$$
$$B(s_t) = Q(s_t)S(s_t)Z(s_t)$$

Then equation (29) can be premultiplied by $Q'(s_t)$ yielding

$$T(s_t)Z(s_t)\begin{bmatrix} x^p_t \\ E_t[x^j_{t+1}] \end{bmatrix} = S(s_t)Z(s_t)\begin{bmatrix} x^p_{t-1} \\ x^j_t \end{bmatrix} + Q'(s_t)C(s_t) + Q'(s_t)D(s_t)\varepsilon_t \tag{31}$$

A.2 Special case: s-infinite regimes

A.2.1 Forward-looking variables

We adopt the strategy of recursive replacement of forward terms (Klein, 2000). Omit (s_t) and let $Q'C \equiv \tilde{C}$ and $Q'D \equiv \tilde{D}$. Expand matrices of equation (31) writing the eigenvalues in increasing value (in modulus) of their ratio $\frac{t_{ii}}{s_{ii}}$ along the diagonal of T and S and the eigenvectors accordingly in Z, so that :

$$\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}\begin{bmatrix} x^p_t \\ E_t[x^j_{t+1}] \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}\begin{bmatrix} x^p_{t-1} \\ x^j_t \end{bmatrix} + \tilde{C} + \tilde{D}\varepsilon_t \tag{32}$$

Let there be the auxiliary variables s_{t-1} and u_t denoting "stable" and "unstable" variables, respectively.

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} x^p_{t-1} \\ x^j_t \end{bmatrix} \equiv \begin{bmatrix} s_{t-1} \\ u_t \end{bmatrix} \tag{33}$$

Then from (32),

$$T_{22}E_t[u_{t+1}] = S_{22}u_t + \tilde{C}_j + \tilde{D}_j\varepsilon_t \tag{34}$$

Solve for u_t:

$$u_t = (S_{22})^{-1}(T_{22}E_t[u_{t+1}] - \tilde{C}_j - \tilde{D}_j\varepsilon_t) \tag{35}$$

Substitute forward once:

$$\Rightarrow u_t = S_{22}^{-1}(T_{22}\big\{S_{22}^{-1}(T_{22}E_t[u_{t+2}] - \tilde{C}_j - \tilde{D}_jE_t[\varepsilon_{t+1}])\big\} - \tilde{C}_j - \tilde{D}_j\varepsilon_t) \tag{36}$$

Develop:

$$\begin{aligned} \Rightarrow u_t = S_{22}^{-1}T_{22}S_{22}^{-1}T_{22}E_t[u_{t+2}] \\ - (S_{22}^{-1}T_{22}S_{22}^{-1} + S_{22}^{-1})\tilde{C}_j \\ - S_{22}^{-1}T_{22}S_{22}^{-1}\tilde{D}_jE_t[\varepsilon_{t+1}] - S_{22}^{-1}\tilde{D}_j\varepsilon_t \end{aligned} \tag{37}$$

Substitute forward again:

$$\begin{aligned} \Rightarrow u_t = (S_{22}^{-1}T_{22})^3E_t[u_{t+3}] \\ - [(S_{22}^{-1}T_{22})^2S_{22}^{-1} + S_{22}^{-1}T_{22}S_{22}^{-1} + S_{22}^{-1}]\tilde{C}_j \\ - (S_{22}^{-1}T_{22})^2S_{22}^{-1}\tilde{D}_jE_t[\varepsilon_{t+2}] - S_{22}^{-1}T_{22}S_{22}^{-1}\tilde{D}_jE_t[\varepsilon_{t+1}] - S_{22}^{-1}\tilde{D}_j\varepsilon_t \end{aligned} \tag{38}$$

Assume $E_t[\varepsilon_{t+k}] = 0\,\forall k$. Then, substitute forward *ad infinitum*. Assume the Blanchard-Kahn conditions hold. Then $lim_{t\to\infty}(S_{22}^{-1}T_{22})^t = 0$:

$$\Rightarrow u_t = -\sum_{t=0}^{\infty}(S_{22}^{-1}T_{22})^tS_{22}^{-1}\tilde{C}_j - S_{22}^{-1}\tilde{D}_j\varepsilon_t \tag{39}$$

$$\Rightarrow u_t = -(I_j - S_{22}^{-1}T_{22})^{-1}S_{22}^{-1}\tilde{C}_j - S_{22}^{-1}\tilde{D}_j\varepsilon_t \tag{40}$$

Plug in the lower rows of (33)

$$\begin{bmatrix} Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} x^p_{t-1} \\ x^j_t \end{bmatrix} = -(I_j - S_{22}^{-1}T_{22})^{-1}S_{22}^{-1}\tilde{C}_j - S_{22}^{-1}\tilde{D}_j\varepsilon_t \tag{41}$$

$$x^j_t = -Z_{22}^{-1}\Big[Z_{21}x^p_{t-1} + (I_j - S_{22}^{-1}T_{22})^{-1}S_{22}^{-1}\tilde{C}_j + S_{22}^{-1}\tilde{D}_j\varepsilon_t\Big] \tag{42}$$

If we let $x_t^j \equiv V x_{t-1}^p + Y + W\varepsilon_t$, then[8]

$$V = -Z_{22}^{-1} Z_{21} \tag{43}$$

$$Y = -Z_{22}^{-1}(I_j - S_{22}^{-1} T_{22})^{-1} S_{22}^{-1} \tilde{C}_j \tag{44}$$

$$W = -Z_{22}^{-1} S_{22}^{-1} \tilde{D}_j \tag{45}$$

where we need to keep in mind that matrices are regime-specific (but not necessarily from the same regime; see below).

A.2.2 Predetermined variables

Let $T_p Z_1$ denote the multiplication of the upper p rows of T and first block-column (p columns) of Z (and adopt this notation rule going forward). Take equation (55) and use V, Y, and W twice:

$$T_p Z_1 x_t^p + T_p Z_2 (V x_t^p + Y) = S_p Z_1 x_{t-1}^p + S_p Z_2 (V x_{t-1}^p + Y + W\varepsilon_t) + \tilde{C}_p + \tilde{D}_p \varepsilon_t \tag{46}$$

Solve for x_t^p to find the law of motion of predetermined variables.

$$x_t^p = (T_p Z_1 + T_p Z_2 V)^{-1} \left[(S_p Z_1 + S_p Z_2 V) x_{t-1}^p + (S_p Z_2 - T_p Z_2) Y + \tilde{C}_p + (S_p Z_2 W + \tilde{D}_p)\varepsilon_t \right] \tag{47}$$

Therefore, if $x_t^p \equiv M x_{t-1}^p + H + K\varepsilon_t$, then

$$M = (T_p Z_1 + T_p Z_2 V)^{-1} (S_p Z_1 + S_p Z_2 V) \tag{48}$$

$$H = (T_p Z_1 + T_p Z_2 V)^{-1} \left[(S_p Z_2 - T_p Z_2) Y + \tilde{C}_p \right] \tag{49}$$

$$K = (T_p Z_1 + T_p Z_2 V)^{-1} \left[(S_p Z_2 W + \tilde{D}_p) \right] \tag{50}$$

A.3 General case: finite histories

In the general case, agents expect the regime will switch in the future. Let the expectation at t of forward-looking variables at time $t+1$ satisfy

$$E_t[x_{t+1}^j] \equiv V_{t+1} x_t^p + Y_{t+1} \tag{51}$$

[8]Naturally, we can also find Y by simply computing $Y = \bar{x}^j - V\bar{x}^p$ where bars indicate steady-state values.

for arbitrary matrices V_{t+1} and Y_{t+1}, where we assumed $E_t[\varepsilon_{t+1}] = 0$ for convenience only. To unclutter notation, we continue to omit (s_t) although matrices are regime-specific. Then equation (31) becomes:

$$\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} x_t^p \\ V_{t+1}x_t^p + Y_{t+1} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} x_{t-1}^p \\ x_t^j \end{bmatrix} + \tilde{C} + \tilde{D}\varepsilon_t \quad (52)$$

where the the Blanchard-Kahn conditions are not necessary. First, let's solve for the forward looking variables. Taking the lower block and solving for x_t^j

$$T_{22}Z_{21}x_t^p + T_{22}Z_{22}(V_{t+1}x_t^p + Y_{t+1}) = S_{22}Z_{21}x_{t-1}^p + S_{22}Z_{22}x_t^j + \tilde{C}_j + \tilde{D}_j\varepsilon_t \quad (53)$$

Solving (53) for x_t^j

$$\Rightarrow x_t^j = (S_{22}Z_{22})^{-1}\big[T_{22}Z_{21}x_t^p + T_{22}Z_{22}(V_{t+1}x_t^p + Y_{t+1}) - S_{22}Z_{21}x_{t-1}^p - \tilde{C}_j - \tilde{D}_j\varepsilon_t\big] \quad (54)$$

Note the term x_t^p needs to be eliminated out. To this end, let's solve for the pre-determined variables x_t^p with the p upper rows of equation (52).

$$T_pZ_1x_t^p + T_pZ_2(V_{t+1}x_t^p + Y_{t+1}) = S_pZ_1x_{t-1}^p + S_pZ_2x_t^j + \tilde{C}_p + \tilde{D}_p\varepsilon_t \quad (55)$$

$$x_t^p = (T_pZ_1 + T_pZ_2V_{t+1})^{-1}[-T_pZ_2Y_{t+1} + S_pZ_1x_{t-1}^p + S_pZ_2x_t^j + \tilde{C}_p + \tilde{D}_p\varepsilon_t] \quad (56)$$

Now, plug (56) in (54)

$$\begin{aligned} \Rightarrow x_t^j = (S_{22}Z_{22})^{-1}\Big\{&[T_{22}Z_{21} + T_{22}Z_{22}V_{t+1}][T_pZ_1 + T_pZ_2V_{t+1}]^{-1} \\ &\big[-T_pZ_2Y_{t+1} + S_pZ_1x_{t-1}^p + S_pZ_2x_t^j + \tilde{C}_p + \tilde{D}_p\varepsilon_t\big] \\ &+ T_{22}Z_{22}Y_{t+1} - S_{22}Z_{21}x_{t-1}^p - \tilde{C}_j - \tilde{D}_j\varepsilon_t\big]\Big\} \quad (57) \end{aligned}$$

Solve for x_t^j:

$$\begin{aligned} \Rightarrow x_t^j = {} & (S_{22}Z_{22})^{-1}\Big\{[T_{22}Z_{21} + T_{22}Z_{22}V_{t+1}][T_pZ_1 + T_pZ_2V_{t+1}]^{-1}S_pZ_2\Big\}x_t^j \\ &+ (S_{22}Z_{22})^{-1}\Big\{[T_jZ_1 + T_jZ_2V_{t+1}][T_pZ_1 + T_pZ_2V_{t+1}]^{-1} \\ &\big[-T_pZ_2Y_{t+1} + S_pZ_1x_{t-1}^p + \tilde{C}_p + \tilde{D}_p\varepsilon_t\big] \\ &+ T_{22}Z_{22}Y_{t+1} - S_{22}Z_{21}x_{t-1}^p - \tilde{C}_j - \tilde{D}_j\varepsilon_t\big]\Big\} \quad (58) \end{aligned}$$

$$\Rightarrow x_t^j = \left\{ I_j - (S_{22}Z_{22})^{-1}[T_{22}Z_{21} + T_{22}Z_{22}V_{t+1}][T_pZ_1 + T_pZ_2V_{t+1}]^{-1}S_pZ_2 \right\}^{-1}$$
$$(S_{22}Z_{22})^{-1}\left\{ [T_jZ_1 + T_jZ_2V_{t+1}][T_pZ_1 + T_pZ_2V_{t+1}]^{-1} \right.$$
$$\big[- T_pZ_2Y_{t+1} + S_pZ_1x_{t-1}^p + \tilde{C}_p + \tilde{D}_p\varepsilon_t]$$
$$\left. + T_{22}Z_{22}Y_{t+1} - S_{22}Z_{21}x_{t-1}^p - \tilde{C}_j - \tilde{D}_j\varepsilon_t] \right\} \quad (59)$$

Let $x_t^j \equiv V_tx_{t-1}^p + Y_t + W_t\varepsilon_t$. Then, from (59)

$$V_t = \left\{ I_j - (S_{22}Z_{22})^{-1}[T_{22}Z_{21} + T_{22}Z_{22}V_{t+1}][T_pZ_1 + T_pZ_2V_{t+1}]^{-1}S_pZ_2 \right\}^{-1}$$
$$(S_{22}Z_{22})^{-1}\left\{ [T_{22}Z_{21} + T_{22}Z_{22}V_{t+1}][T_pZ_1 + T_pZ_2V_{t+1}]^{-1}S_pZ_1 - S_{22}Z_{21} \right\} \quad (60)$$

$$Y_t = \left\{ I_j - (S_{22}Z_{22})^{-1}[T_{22}Z_{21} + T_{22}Z_{22}V_{t+1}][T_pZ_1 + T_pZ_2V_{t+1}]^{-1}S_pZ_2 \right\}^{-1}$$
$$(S_{22}Z_{22})^{-1}\left\{ [T_{22}Z_{21}+T_{22}Z_{22}V_{t+1}][T_pZ_1+T_pZ_2V_{t+1}]^{-1}\big[-T_pZ_2Y_{t+1}+\tilde{C}_p\big]+T_{22}Z_{22}Y_{t+1}-\tilde{C}_j] \right\} \quad (61)$$

$$W_t = \left\{ I_j - (S_{22}Z_{22})^{-1}[T_{22}Z_{21} + T_{22}Z_{22}V_{t+1}][T_pZ_1 + T_pZ_2V_{t+1}]^{-1}S_pZ_2 \right\}^{-1}$$
$$(S_{22}Z_{22})^{-1}\left\{ [T_{22}Z_{21} + T_{22}Z_{22}V_{t+1}][T_pZ_1 + T_pZ_2V_{t+1}]^{-1}\tilde{D}_p - \tilde{D}_j \right\} \quad (62)$$

To recover pre-determined variables, plug $x_t^j = V_tx_{t-1}^p + Y_t + W_t\varepsilon_t$ back into x_t^p in equation (56),

$$x_t^p = (T_pZ_1 + T_pZ_2V_{t+1})^{-1}[-T_pZ_2Y_{t+1} + S_pZ_1x_{t-1}^p + S_pZ_2(V_tx_{t-1}^p + Y_t + W_t\varepsilon_t) + \tilde{C}_p + \tilde{D}_p\varepsilon_t] \quad (63)$$

Let $x_t^p \equiv M_tx_{t-1}^p + H_t + K_t\varepsilon_t$. Then, from (63)

$$M_t = (T_pZ_1 + T_pZ_2V_{t+1})^{-1}(S_pZ_1 + S_pZ_2V_t) \quad (64)$$

$$H_t = (T_pZ_1 + T_pZ_2V_{t+1})^{-1}[-T_pZ_2Y_{t+1} + S_pZ_2Y_t + \tilde{C}_p] \quad (65)$$

$$K_t = (T_pZ_1 + T_pZ_2V_{t+1})^{-1}[S_pZ_2W_t + \tilde{D}_p] \quad (66)$$

A.4 Proxy steady states (specific to each node type)

Given the solution for each history type (equation 6),

$$\begin{bmatrix} x_t^p \\ x_t^j \end{bmatrix} = \begin{bmatrix} M_\iota \\ V_\iota \end{bmatrix} x_{t-1}^p + \begin{bmatrix} H_\iota \\ Y_\iota \end{bmatrix} + \begin{bmatrix} K_\iota \\ W_\iota \end{bmatrix} \varepsilon_t$$

it is useful to define a proxy for a steady state specific to each history type, denoted $\bar{x}_\iota = [\bar{x}_\iota^p \ \bar{x}_\iota^j]$', equal to the "steady state" of that history type ι.

$$H_\iota \equiv [I_p - M_\iota]\bar{x}_\iota^p \Rightarrow \bar{x}_\iota^p \equiv [I_p - M_\iota]^{-1} H_\iota \tag{67}$$

$$\bar{x}_\iota^j \equiv V_\iota \bar{x}_\iota^p + Y_\iota \tag{68}$$

These values will be helpful for characterizing the properties of the model being used.

A.5 A note on rationality, information and notation

By omitting (s_t) from the notation of matrices, it is implicitly assumed all block matrices refer to the same regime. This implies that agents' beliefs about the regime in effect and the actual regime in effect are the same. Nevertheless, the formulas would also apply, with adjusted notation, for the assumption that there is a cognitive discrepancy – due to some limitation on rationality or on information – between beliefs about which regime is in effect and the true regime in effect. It would suffice to apply the formulas using the lower block from one regime and the upper block from another. The lower block to solve for forward-looking variables corresponds to beliefs about current and future motion of the economy. The upper block corresponds to the true regime governing the motion of the economy. The modeler is able to assume even unfeasible trajectories can be expected, such as when an occasionally binding constraint is violated.[9]

[9] GI's OccBin assumes expectations respect feasibility, which is what the "guess-and-verify" algorithm is all about finding.

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