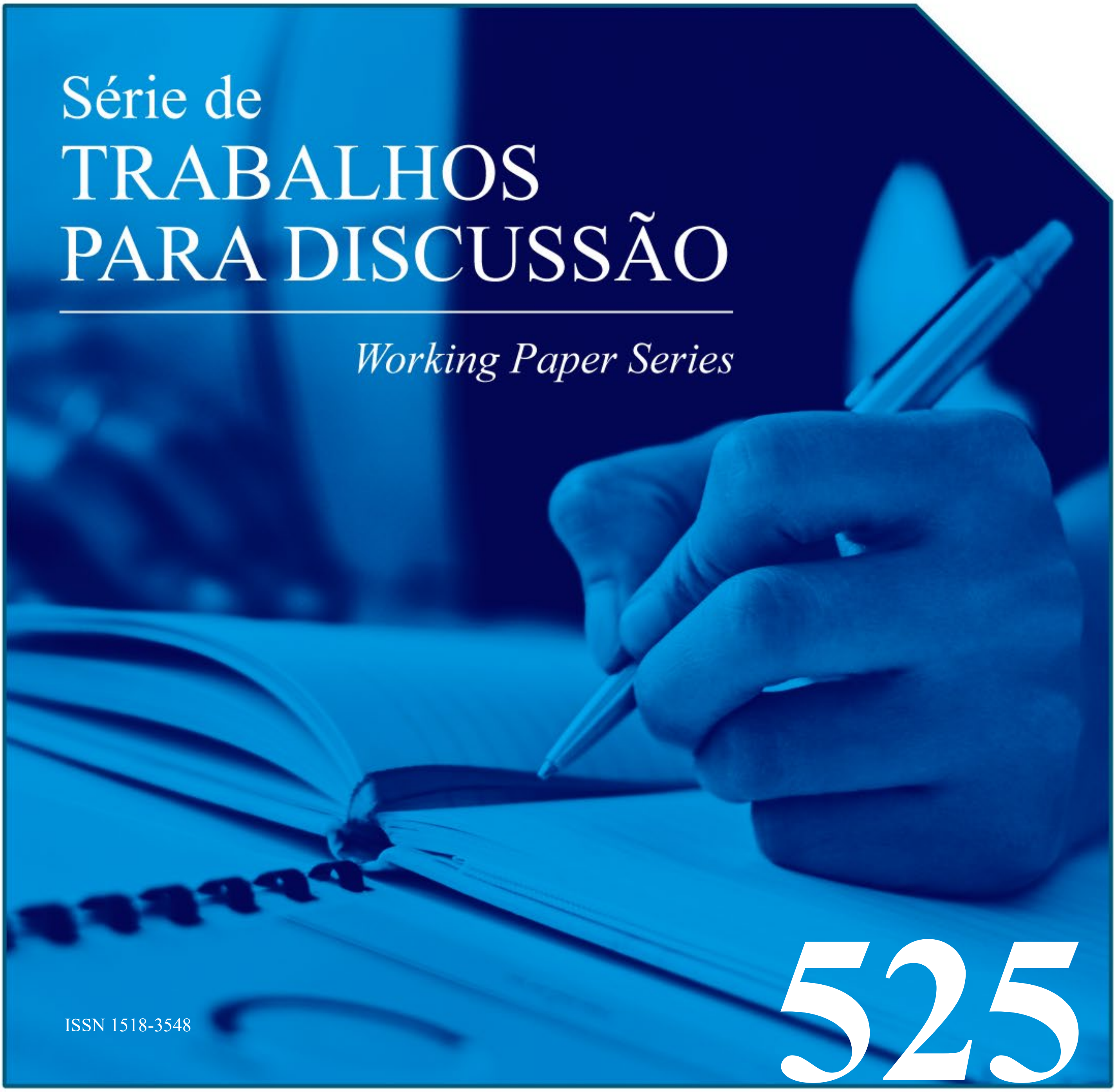

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Long-term stock returns in Brazil: volatile equity returns for U.S.-like investors

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Non-technical Summary

This paper studies the evolution of Brazilian stock market returns since the creation of the Ibovespa (the main Brazilian stock market index). We compute alternative measures for equity returns, comparing them with the ones related to the U.S. economy. In addition, we briefly review important episodes of the recent Brazilian economic history that are relevant for developments in stock markets.

Indeed, from 1968 to 2017, the arithmetic mean return of the Brazilian stock market is *21.5%* per year, with standard deviation of *67.8%*. In contrast, for the same period, the arithmetic mean return for the U.S. stock market is *8.0%* per year with a standard deviation of *16%*. Concerning geometric averages, the results are as follows. The Brazilian equity return is *6.50%* per year with a standard deviation of *51.9%*. With a much lower equity volatility of *18.40%*, the U.S. equity return is *6.15%* per year. In fact, the extreme volatility of returns in Brazil explains the huge difference between arithmetic and geometric means for this country compared with the U.S.

We assess the relative biases of arithmetic and geometric methods in these two countries with very different volatilities and compute an unbiased expected return estimator that penalizes expected returns for higher volatility and longer horizons due to the increasing imprecision of estimates. As the investment horizon increases, Brazilian equities mean annual returns decrease faster due to higher volatility.

Finally, the Jacquier et al. (2005) extension of Merton's (1969) "lifetime portfolio" model, which considers sampling uncertainty, rationalizes both Brazilian and U.S. numbers with similar risk aversions. Although equity returns have been higher in Brazil than in the U.S., the much higher Brazilian equity volatility discourages heavier investments in stocks. Hence, for similar relative risk aversion coefficient around six and horizons between 5 to 20 years, the model implies equity allocations close to *12%* and *32%* of financial wealth respectively in Brazil and in the U.S., matching the empirical evidence on the share of risky assets in households' portfolios across these two economies.

Sumário Não Técnico

Este artigo estuda a evolução do mercado de ações no Brasil a partir da criação do Ibovespa. Computam-se medidas alternativas para o retorno de ações, comparando-as com medidas semelhantes para os Estados Unidos. Além disso, este ensaio revisita importantes episódios da recente história econômica brasileira que são relevantes para entender as mudanças do mercado de acionário brasileiro.

Com efeito, entre 1968 e 2017, no Brasil, o retorno médio aritmético do índice de mercado acionário foi de *21,5%* por ano, com desvio padrão de *67,8%*. Nesse mesmo período, nos Estados Unidos, por seu turno, o retorno médio aritmético do índice de mercado acionário foi de *8,0%* por ano, com desvio padrão de *16%*. Com relação a retornos computados via média geométrica, os resultados foram os seguintes. Para o Brasil, o retorno acionário foi de *6,5%* por ano, com desvio padrão de *51,9%*. O retorno norte-americano foi de *6,15%* por ano, com volatilidade de apenas *18,40%*. De fato, no Brasil, a extrema volatilidade dos retornos é responsável pela diferença significativa entre os retornos médios aritmético e geométrico.

O artigo também avalia o viés relativo entre o retorno médio aritmético e o geométrico para os dois países em questão, caracterizados por níveis de volatilidade distintos para seus respectivos mercados de ações. Adicionalmente, calcula-se um estimador sem viés para a média dos retornos. Esse estimador penaliza os retornos esperados de acordo com o nível de volatilidade e o horizonte de investimento, refletindo a imprecisão crescente das estimativas para o retorno esperado.

Finalmente, o artigo considera o problema de alocação intertemporal de portfólio estudado por Merton (1969), de acordo como a extensão feita por Jacquier et al. (2005). Tanto para o agente representativo brasileiro quanto para o norte-americano, preferências com coeficiente de aversão ao risco em torno de seis são capazes de replicar o percentual da riqueza alocada em ativos arriscados (*12%* no Brasil e *32%* nos Estados Unidos). De fato, apesar de apresentar retornos médios mais elevados, o mercado acionário brasileiro é muito volátil, o que desencoraja investimentos consideráveis em ações.

Long-term stock returns in Brazil: volatile equity returns for U.S.-like investors [*]

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Antônio Z. Sanvicente[****]

Abstract

This paper tells the history of Brazilian stock market returns since the creation of the Ibovespa (the main Brazilian stock market index). From 1968 to 2019, the arithmetic mean return of the Brazilian stock market is *21.3%* per year. The equity premium is *20.1%* per year, with a huge standard deviation of *67%*. Surprisingly, such numbers are compatible with investors' risk aversions that accommodate the very different U.S. market evidence, reinforcing the belief that national investors are similar in nature. The equity premium has been higher in Brazil than in the U.S., but the much higher Brazilian volatility discourages heavier investments in stocks.

Keywords: equity returns, equity risk premium, emerging market, lifetime portfolio selection
JEL Classification: E21, G10, G12

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1. Introduction

The documentation of the U.S. equity premium in the past century is comprehensive, and numbers like *8%* annual equity return, above *6%* annual equity premium, below *20%* volatility, and *0.40* Sharpe ratio are on the top of the head of every financial economist. Studies that contemplate other countries' experiences for 50 years or more also exist for other industrial economies (see Campbell 2003 and Dimson et al. 2008), but they are scant for emerging economies.

Estimates of long-term returns and appraisals of their magnitudes through the lens of theory in different environments are key inputs driving asset pricing research and portfolio management. A vast literature since Mehra and Prescott (1985) has investigated the hypothesis that these magnitudes could result from micro-founded theories of rational investment under uncertainty. In summary, researchers regard the U.S. historical market stock returns as puzzlingly high (see Mehra and Prescott 2003). [1]

The literature, however, has not dedicated enough attention to emerging markets and did not provide solid answers to the following questions: how do equity returns of a typical emerging

[1] For other industrial economies, Dimson et al. (2008) find high Sharpe ratios, though less impressive than in the U.S.; and Campbell (2003) concludes that implicit risk aversions from the consumption-based model are implausibly high in general.

economy compare with the U.S. from a half-century perspective? What are the challenges for explaining observed data for an emerging economy according to standard asset pricing theory? [2]

In this paper, we document the 1968-2019 equity premium in Brazil, where macroeconomic risks have been substantial, providing an opportunity to learn about asset pricing in emerging markets.[3] We gauge the Brazilian experience against the U.S., which serves as the benchmark economy. The comparative analysis encompasses several dimensions, concentrating on the following issues: (a) average returns and equity premium estimated under alternative methods; (b) performance measures that consider high order moments; (c) long-term expected returns over different investment horizons; and (d) long-run asset allocation between risky and riskless assets.

This multidimensional viewpoint gives a broader picture of the functioning of the stock market in a typical emerging economy, in contrast to industrial countries' experiences. [4] In short, our goal is to compare specific stock market's characteristics in these two economies, highlighting

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[2] Goetzmann and Ibbotson (2008) point out that "… Our understanding of the historical experience of investors is relatively limited once we step beyond a few well-studied markets."

[3] The high Brazilian stock market volatility pops out in international comparisons. For example, it is the third highest in Fama and French (1998), below Argentina. And it is the third highest in Rouwenhorst (1999), below Argentina and Venezuela.

[4] According to the World Factbook from the U.S. Central Intelligence Agency (https://www.cia.gov/library/publications/the-world-factbook/geos/br.html), in 2017, Brazil produced a GDP (purchasing power parity) of US$3.24 trillion (eighth-largest economy in the world) with a population of 207 million. The Brazilian market value of publicly traded shares was $642.5 billion on 31 December 2017 (nineteenth-largest in the world).

their implications for long-run financial decisions. However, we do not attempt to interpret our results through the lens of asset pricing equilibrium models nor discuss the puzzles associated with some specifications for preferences in these models. An additional limitation is our focus on historical long-run equity premium at the annual frequency, following the tradition in the literature since Mehra and Prescott (1985). In appendix A, to assess the robustness of the benchmark results displayed in Table I, we present our computations at different frequencies and samples, discuss the effect of expected inflation (a different choice for the deflator) to build real variables, on the equity premium, and appraise the time series characteristics of our annual data. The findings summarized in this appendix support the baseline results of Table I.

Indeed, our main contribution is on long-run issues, studying the Brazilian equity premium and returns during an extended period (from 1968 to 2019), which begins with the creation of the Ibovespa – the São Paulo Stock Exchange index. [5] Previous literature has explored the behavior of Brazilian equity returns (see alternative analysis in Fama and French 1998, Rouwenhorst 1999, Bonomo and Garcia 2001, Bonomo and Domingues 2002, Sampaio 2002, Cysne 2006, or Varga and Brito 2016).

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[5] The São Paulo Stock Exchange, Bovespa is nowadays part of B3 (in full, B3 - Brasil Bolsa Balcão S.A. or B3 - Brazil, Stock Exchange and Over-the-Counter Market). In 2008, the Bovespa and the Brazilian Mercantile and Futures Exchange (BM&F) merged, creating BM&FBOVESPA. In 2017, BM&FBOVESPA merged with CETIP, creating B3. There were 338 companies listed at Bovespa as of March 2017.

[6] Fama and French (1998) compute an average annual dollar return of the Brazilian market in excess of the U.S. T-Bill between 1987-1995 equal to *34.99%* (with 79.15% standard deviation). Rouwenhorst (1999) finds a Brazilian market return of *19.35%* (with 26.67% standard deviation) in local currency for the period 1982:Q1-1997:Q4, or of *4.27%* (with 20.17% standard deviation) in US Dollars. Bonomo and Garcia (2001) document an average equity premium of *28.82%* (with 70.49% standard deviation) between 1976:1-1992:12. Cysne (2006) presents an average

To some extent, these studies are limited by data availability and most of them use monthly or quarterly samples starting at the early 1980's or later; therefore, they do not explicitly provide the long-term perspective emphasized in this paper.

Concerned with ex-ante returns, the arithmetic mean real return of the Brazilian stock market is *21.3%* per year from 1968 to 2019. The equity premium over the savings account interest rates is *20.1%* per year, with a huge standard deviation of *67%*, implying a Sharpe ratio of *0.30*. For the same period, the arithmetic mean return for the U.S. stock market is *8.2%* per year. The U.S. equity premium over the 1-year Treasury Bills is *6.4%* per year, with a standard deviation of *17%*, implying a Sharpe ratio of *0.38*. Alternatively, continuous compounding reveals a Brazilian geometric mean equity return of *6.8%* per year with *49.2%* annual volatility and a mean risk-free rate of *0.86%* per year. We compared those values to a U.S. geometric mean equity return of *6.3%* per year with *17.8%* annual volatility, and a mean risk-free rate is *1.59%* per year.[7]

We additionally analyze percentiles, higher moments and the Aumann and Serrano (2008) riskiness index (AS index henceforth) to better assess how the returns distributions differ in moments that matter to investors. The AS indices clearly indicate the much higher risk of the Brazilian markets relative to their U.S. equivalents. However, we surprisingly conclude the skewness and kurtosis effects on both countries AS indices are not sizeable and are similar. In sum, the main difference of these countries' stock markets riskiness is in the enormous variance of Brazilian returns, with significant implications for expected returns and asset allocation.

Brazilian market return of *31.33%* and equity premium of *15.92%* over the Brazilian interbank rate between 1992-2004. Varga and Brito (2016) show an average monthly market return of *1.08%* (with 7.84% standard deviation) between 1999:7-2015:6.

[7] Section 2.4 explains how the arithmetic and geometric means can provide deceptively so different pictures. Anticipating, given lognormality of the discrete gross return rates, R: $lnE(R) = E(lnR) + 0.5\sigma^2(lnR)$.

While researchers understand well the difference between the arithmetic and geometric population means, the bias generated in compounding with one or the other sample average depends on the interaction between market volatility and investment horizon, with non-trivial effects on expected long-term returns. Using the unbiased estimator suggested in Jacquier et al. (2003), we account for the impact of the mean parameter uncertainty under the distinct Brazilian and U.S. volatility environments. [8] The much higher Brazilian volatility considerably penalized long-term expected stock returns.

Finally, we show the dissimilar Brazilian and U.S. stock market returns can result from the demands of investors that handle risk similarly. Although there are striking differences in the macroeconomic environments and resulting volatilities that impact on stock holdings and cost of equities, Brazilians and U.S. investors can be depicted alike. The extension of Merton's (1969) "lifetime portfolio" model, which takes sampling uncertainty into account like in Jacquier et al. (2005), rationalizes both Brazilian and U.S. numbers with similar risk aversions. Although the equity premium and equity return have been higher in Brazil than in the U.S., the much higher Brazilian equity volatility discourages heavier investments in stocks. For similar relative risk aversions, around 4 to 6, the model implies equity allocations close to *12%* and *32%* of financial wealth respectively in Brazil and the U.S., matching income tax data. [9]

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[8] Given our historical perspective of the equity premium, we abstract from the literature on predictability (Campbell and Thompson 2008, Welch and Goyal 2008), learning (Barberis 2000), or model-implied forward-looking premium (Jagannathan et al. 2000, Fama and French 2002).

[9] If instead of Merton's (1969) terminal wealth perspective, we choose the consumption-based asset pricing approach of Mehra and Prescott (1985), we find the equity premium is as puzzlingly high in Brazil as it is in the U.S. for reasonable degrees of risk aversion. In other words, both risk aversions have to be very high to accommodate such

In the following section, we present brief histories of stock returns in Brazil and the U.S. during the past fifty-two years from the perspective of three estimators: arithmetic mean, geometric mean, and an unbiased alternative. In section 3, we analyze both markets through the lens of Merton's (1969) lifetime portfolio model with expected returns uncertainty. We conclude in section 4.

2. Two 50-year Histories

2.1 Data

We study the Ibovespa, a total return index of the São Paulo Stock Exchange (BOVESPA), from its creation in December of 1967 until December of 2019. The Brazilian market-return series are nominal, and we deflate all of them by the General Price Index (*Índice Geral de Preços – Disponibilidade Interna*, IGP-DI). Concerned with the domestic investor view, we compute returns in the local currency. [10]

We choose the return on the Savings Account, called *Caderneta de Poupança,* as the Brazilian "riskless" short-term real interest rate series. The *Caderneta de Poupança* is (imperfectly) inflation-indexed. It is the most popular financial investment vehicle in Brazil and regarded as the least risky investment by individuals. We presented this variable in the tables below

premia, what is another sign of the similarity between Brazilian and U.S. investors. This "short-term investment" perspective is available upon request in a longer working paper version.

[10] Because we are interested in real returns, we do not report nominal pictures. For those curious about nominal returns, we can provide the respective tables upon request.

under the label of *Short-term real interest*. Alternative interest rates used in the Brazilian literature are not available for this extended period and embed varying bank spreads. [11]

For the U.S., like Dimson et al. (2008), we use the capitalization-weighted CRSP Index of all NYSE stocks from 1968 to 1970. Thenceforth, from 1971 to 2019, we employ the Wilshire 5000 Index, which contains over 7,000 U.S. stocks, including those listed on Nasdaq. We deflate U.S. nominal series by the Producer Price Index for All Commodities. As for the short-term interest rate, we use the 1-year constant maturity U.S. Treasury Bill rate.

To keep an eye on countries' real activity, we also present annual real Gross Domestic Product (in constant LCU) growth rates. Both the Brazilian and U.S. series are from the World Bank.

2.2 Arithmetic averages

How much are the expected real equity and short-term interest returns in Brazil? In addition, how does the Brazilian equity premium compare with that of the U.S.?

[11] In a previous version of this paper, we combined two short-term interest rate series: (i) the return on the Savings Account; and (ii) a merge of the Brazilian Treasury Obligations (*Obrigações do Tesouro Nacional and Obrigações Reajustáveis do Tesouro Nacional* until 1974) with the Brazilian interbank rate (*SELIC* after 1974). Such composite series results in a higher average short-term interest rate. However, presentations and discussions made us forgo this option. These previous results are available upon request with the same conclusions of this current version. Facing a similar choice between U.S. government and U.S. municipals in the 19th century, Goetzmann and Ibbotson (2008) choose the minimum yield between yearly U.S. government and U.S. municipals as a measure of the (nearly) riskless rate.

Academics, concerned with ex-ante expected returns, advocate using the arithmetic mean. For example, Mehra and Prescott (2008) define V_H as the value H periods into the future:

$$V_H = V_0 \prod_{t=1}^{H} R_t = V_0 \prod_{t=1}^{H} (1 + a_t), \tag{1}$$

where V_0 is the amount invested today and $R_t = (1 + a_t)$ is the realized return in period t.

If one takes expectations and assumes uncorrelated returns:

$$E[V_H] = V_0 \prod_{t=1}^{H} (1 + E[a_t]) = V_0(1 + \bar{a})^H = V_0 e^{\bar{\alpha}H}, \tag{2}$$

where $\bar{a}$ is the arithmetic mean rate of return and $\bar{\alpha}$ is its equivalent, measured as a continuously compounded rate of return.

In Table I, we present arithmetic sample averages $(\bar{a})$ of real returns and other summary statistics of key financial and macro variables for the 1968-2019 period. The average real returns on stocks are high. The mean return of the Brazilian stock market is *21.3%* per year with volatility of *66.6%*, while the U.S. mean stock market return is *8.2%* per year with volatility of *18.1%*. Figures 1.1 and 1.2 illustrate how different are the histograms of equity returns, identifying each year return in the distributions.

Appendix A assesses the robustness of our benchmark computations in Table I. There, we present our computations at different frequencies and samples. We also show the results concerning the use of the expected inflation, an alternative deflator to construct real variables, on the equity premium. The measures that employ expected inflation, limited by data availability, comprise small annual samples. Finally, we explore statistical characteristics of the annual time series, focusing on the discussion of autocorrelation and conditional heteroscedasticity test results and their implications for the computation of real risk free and equity returns, as well as the equity premium.

The main findings in the appendix are:

a) The means and standard deviations of the real risk free and equity returns, and the equity premium, using quarterly and monthly time series (in annualized terms), are similar in magnitudes to the values reported in Table I (see Table A.I in appendix A);

b) Concerning the mean equity premium in subsamples, we show a somewhat stable value for the U.S., but wide oscillations in the Brazilian premium over time, which becomes much smaller in the last two decades (see Table A.II and Figures A.1 and A.2 in appendix A).

c) The use of expected inflation as the deflator has significant effects in computing real risk free returns, especially for the U.S., but small effects on both equity premia (see Table A.III in appendix A);

d) Concerning the statistical characteristics of the time series for the real risk free and equity returns, and the equity premium, we did not detect heteroscedasticity for the six time series. We find signs of autocorrelation only for the U.S. risk free series. For this specific time series, the unconditional mean based on a statistical model fitted to the data was not statistical different from the historical mean, according to a Wald test. Therefore, for all annual series investigated, the historical means of Table I are indeed good measures concerning the first two statistical moments.

Table I - Annual returns in Brazil and U.S. - 1968-2019

	Mean	Std. Dev.	Minimum	Maximum	N. of negatives	Longest run of negatives *
	(1)	(2)	(3)	(4)	(5)	(6)
			Brazil			
Equity	0.2127	0.6656	-0.7411 ('90)	3.1638 ('91)	25	6 ('10)
Short-term interest	0.0115	0.0749	-0.2368 ('80)	0.2230 ('95)	22	4 ('99)
Equity premium	0.2012	0.6704	-0.7605 ('87)	3.2059 ('91)	24	3 ('72;'78;'00;'13)
GDP growth	0.0388	0.0426	-0.0439 ('81)	0.1398 ('73)	8	2 ('15)
Inflation	2.2656	5.2850	-0.0143 ('09)	27.0817 ('93)	2	
			U.S.			
Equity	0.0817	0.1811	-0.4070 ('74)	0.3750 ('91)	18	3 ('00)
Short-term interest	0.0174	0.0534	-0.1126 ('74)	0.1228 ('01)	21	5 ('09)
Equity premium	0.0642	0.1696	-0.4232 ('08)	0.3281 ('13)	15	3 ('00)
GDP growth	0.0284	0.0194	-0.0278 ('09)	0.0726 ('84)	7	2 ('74; '08)
Inflation	0.0360	0.0512	-0.0685 ('15)	0.2089 ('74)	9	

Notes : Annually compounded rates per year in the respective local currency. Returns are real returns, except for inflation. Equity premium is the Equity return minus the return on Short-term interest. Std.Dev. is the standard deviation of the annually compounded returns. Computed using *52* yearly observations. Number in parentheses indicates the year of occurence.
* In column (9), the number in parentheses indicates the first year of the sequence of years.

Figure 1.1 - Histogram of annual Brazilian real equity returns, 1968-2019

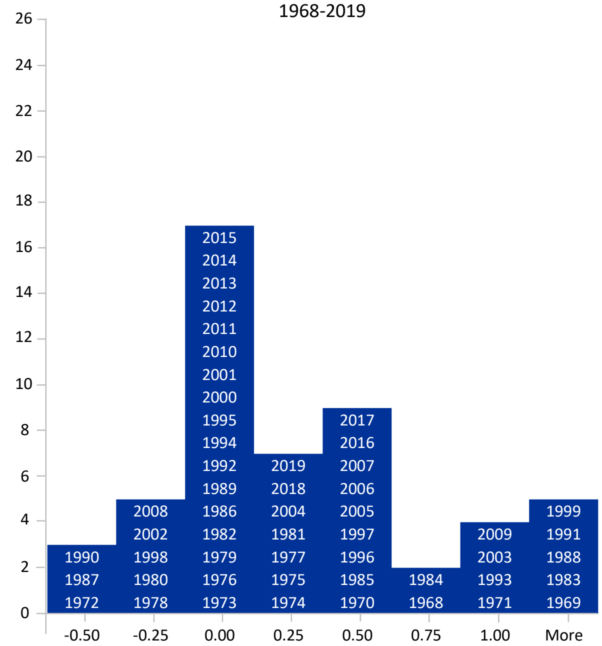
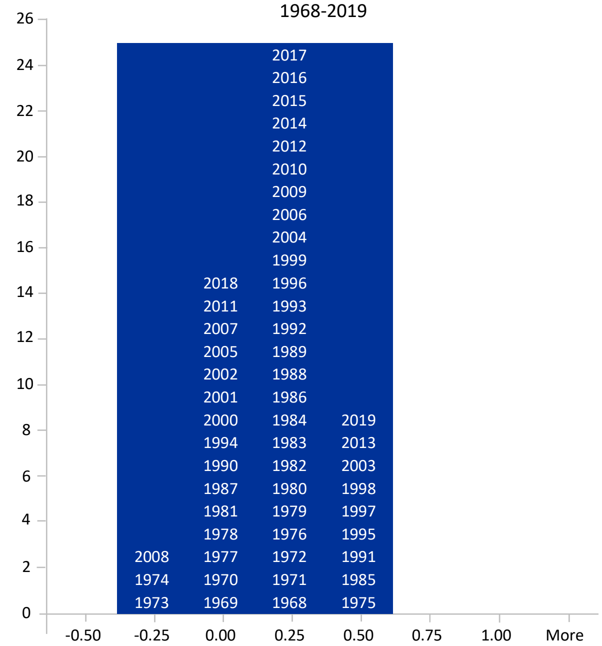


Figure 1.2 - Histogram of annual US real equity returns, 1968-2019



Relative to stocks, the short-term real interest rates are low and much less volatile. From those numbers, a Brazilian equity premium of *20.1%* per year emerges, with a standard deviation of *67.0%* and a Sharpe ratio of *0.30*. The U.S. equity premium is *6.4%* per year with a standard deviation of *17.0%* and a Sharpe ratio of *0.38*.

Readers aware of the Brazilian fixed-income market reputation for paying high real interest rates may question our picture with an average annual "riskless" real return of *1.15%*, which is below the notoriously low U.S. annual average of *1.74%*. Another concern is about the certainty (or riskiness) of real short-term interest in the face of the Brazilian high inflation experience. We point that, although unpredictable shocks to inflation are more important in Brazil than in the U.S., the *-0.17* correlation between real interest rate and inflation in Brazil is weaker than the respective correlation of *-0.78* in the U.S. (correlation numbers not presented in the tables). We argue this lower real return-weaker inflation correlation configuration of the Brazilian interest rates is sensible given the inflation-indexed nature of the Brazilian Savings Account, which, although imperfectly, insures against the significant inflation risk, thus paying a lower return. Particularly empirically convenient, this indexation considerably offsets the Brazilian high-inflation distortions on a real interest rate that one should perceive as "riskless".

Farther looking at the real economic activity, the Brazil x U.S. differences in terms of GDP growth and inflation are also evident in the rows of Table I. Both GDP growth rate averages are lower than their respective average stock returns and higher than their short-term interest rates. The Brazilian GDP growth rates have been higher on average and much more volatile than those rates for the American economy.

In this fifty-two-year period, Brazil lived years of high economic volatility with the exhaustion of a cycle of high growth accrued from its industrialization, mainly funded by public savings. Deadlocks in the simultaneous re-democratization process and lack of consensus over the macroeconomic agenda degenerated into a severe fiscal crisis, and more than a decade to tame a very high and persistent inflation. [12] The average inflation in this half-century was *227%* per year, mostly accrued in the 1980s and early 1990s – the 1980-1994 average is *746%* per year, with annual rates as high as *1783%* in 1989, *1477%* in 1990 and *2708%* in 1993. [13] There were seven major stabilization plans between 1986 and 1994, which tried measures like price controls,

[12] Between 1968 and 2019, Brazil had twelve presidents: four Army generals (until 15-Mar.1985) and one civilian all selected indirectly (until 15-Mar.1990), and five elected in general democratic elections. Among the five latter, two were impeached (Fernando Collor on 29-Dec.1992 and Dilma Rousseff on 17-Apr.2016) and succeeded by their vice- presidents.

[13] Brazil had six monetary reforms in the 1968-2019 period. The local currencies were Cruzeiro Novo (13-Feb.1967), Cruzeiro (15-May.1970), Cruzado (28-Feb.1986), Cruzado Novo (16-Jan.1989), Cruzeiro (16-Mar.1990), Cruzeiro Real (1-Ago.1993) and Real (since 1-Jul.1994).

external debt moratorium, financial assets freeze, indexed contracts prohibition, government spending controls, and exchange rate anchor. [14]

This long inflation struggle had marked real effects. The Brazilian stock market had its worst years in 1987 and 1990, down by *74.11%* and *73.85%*, respectively, coinciding with the failures of two main inflation stabilization attempts: “Plano Cruzado” and “Plano Collor”. Primarily a recovery from the 1990’s stocks fire sale and partially due to the worldwide increase in business optimism, 1991 was the Brazilian stock market best year, with an impressive return of *316.38%*.

In comparison, the U.S. extreme years were the consequence of real shocks. The worst year was 1974, down by *40.70%* attributed to a combination of the 1973 oil crisis and the collapse of the Bretton Woods system over the previous years. [15] Corroborating the worldwide increase in business optimism, the U.S. also had its best year in 1991, going up by *37.50%*.

Table I additionally details how the higher Brazilian volatility outshines realized equity returns. The Brazilian minimum and maximum respectively in columns (3) and (4), as well as the number of years with negative returns in column (5) illustrate the much higher risk in Brazilian equities. Out of the fifty-two years studied, the Brazilian and U.S. stock markets had respectively *25* and *18* negative real returns. The longest sequence of negative stock returns lasted six years in Brazil, from 2010 to 2015, and three years in the U.S., from 2000 to 2002, in column (9).

[14] Before Plano Real on 28-Feb.1994, which finally reduced inflation to one-digit on average (the average inflation between 1995-2017 was 8.32% per year), there were Plano Cruzado (28-Feb.1986), Plano Cruzado 2 (22-Nov.1986), Plano Bresser (12-Jun.1987), Plano Verão (12-Jan.1989), Plano Collor 1 (16-Mar.1990) and Plano Collor 2 (31-Jan. 1991). After 1995, inflation remains low and stable (the average inflation from 1995 to 2019 is around 6%).

[15] The U.S. stock market went down by *29.51%* in 1973.

Despite the much higher Brazilian volatility, the correlations of *0.23* between countries' GDPs and of *0.33* between countries' stock returns are evidence of an important common world business activity factor (correlation numbers not presented in the tables).

2.3 Geometric average

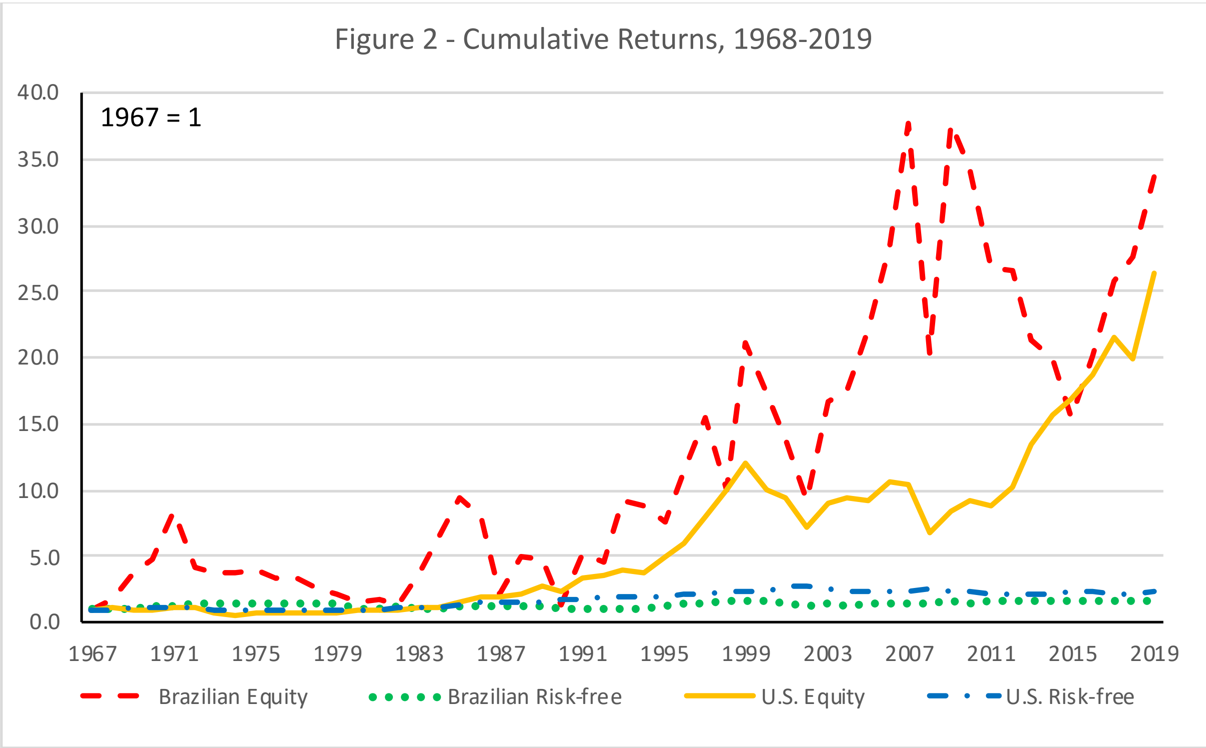
Practitioners prefer the simple intuition of compounding:

$$\left(\frac{V_H}{V_0}\right) = e^{\hat{r}H} \quad \Longrightarrow \quad \hat{r} = \frac{1}{H} ln\left(\frac{V_H}{V_0}\right), \qquad (3)$$

where $\hat{r}$ is the geometric average, which is the standard way to represent growth rates for past observed wealth. Additionally, geometric averages produce lower, or more conservative, long-term forecasts than arithmetic averages.

Though Figures 2, 3.1 and 3.2 are based on annual geometric returns, in Table II, departing from our baseline frequency of analysis, we consider quarterly time series to increase the data points in the sample, improving thus the accuracy of estimated higher moments of the statistical distributions for the investigated variables, which we need as input to compute the Aumann-Serrano riskness index, discussed below on pages 17 and 18.

Figure 2 displays cumulative returns during the past fifty-two years, and Table II presents descriptive statistics of geometric annualized returns, i.e., continuously compounded annualized rates for our variables of interest.



The first two rows of Table II present annualized geometric averages and standard deviations from quarterly data. The Brazilian geometric average equity return is *6.77%* per year with a very high *49.21%* as its measured volatility; and the short-term interest rate average is *0.86%* per year with *7.19%* as its measured volatility. With much lower equity volatility of *17.76%*, the U.S. geometric average equity return is *6.30%* per year; and the short-term interest rate has an average of *1.59%* per year and volatility of *4.09%*.

Table II - Geometric mean annual rates in Brazil and U.S. - 1968-2019

	Brazil			***U.S.***		
	Equity real return	Short-term real interest	Equity premium	Equity real return	Short-term real interest	Equity premium
	(1)	(2)	(3)	(4)	(5)	(6)
Mean	0,0677	0,0086	0,0591	0,0630	0,0159	0,0471
Std.Dev.	0,4921	0,0719	0,5009	0,1776	0,0409	0,1730
Skewness	-0,5062	-0,4228	-0,4162	-0,3735	0,4679	-0,4327
Kurtosis	4,7053	3,8488	4,4245	3,3672	5,5159	3,3373
25th-percentile	-0,3964	-0,0552	-0,4335	-0,0803	-0,0241	-0,1257
Median	0,0898	0,0230	0,0996	0,1028	0,0233	0,1027
75th-percentile	0,5624	0,0873	0,5462	0,2819	0,0576	0,2473
AS riskiness index	1,888	0,312	2,204	0,275	0,051	0,346
.5*(Std.Dev.^2)/Mean	1,789	0,301	2,122	0,251	0,052	0,320
Ratio	1,055	1,037	1,038	1,096	0,993	1,080
Minimum 5-year	-0,4027	-0,0759	-0,3759	-0,1333	-0,0375	-0,5062
Negatives in 5-yrs	19	16	22	15	19	13
Minimum 10-year	-0,1631	-0,0293	-0,1495	-0,0444	-0,0249	-0,0418
Negatives in 10-yrs	11	16	12	9	16	8
Minimum 20-year	-1,3308	-0,2955	-1,0901	0,6476	0,0031	0,2425
Negatives in 20-yrs	2	7	8	0	0	0
Minimum 25-year	0,0120	-0,0058	0,0107	0,0503	0,0069	0,0253
Negatives in 25-yrs	0	5	0	0	0	0
H_0: Autocorrelated	0,26	0,42	0,10	0,52	0,00	0,55

Notes: Continuously compounded annualized rates computed from quarterly data from 1968:Q1-2019:Q4 (*208* observations) in the respective local currency. Skewness is the third moment about the Mean divided by *Std.Dev.^3*. Kurtosis is the fourth moment about the Mean divided by *Std.Dev.^4*. *AS* is the Aumann-Serrano (2008) index of riskiness from the normal inverse Gaussian distribution and *.5*(Std.Dev.^2)/Mean* is the value to which the AS index degenerates when the data are normally distributed. Ratio is the former divided by the latter. The Cumby-Huizinga test for autocorrelation reports the *p*-value.

Skewness, kurtosis and percentiles in Table II provide complementary information for those forecasting future returns.[16] Both Brazilian and U.S. quarterly stock returns have/display negative skewness coefficients with positive excess-kurtosis (i.e., kurtosis of minus *3*). In Brazilian stocks, there is a 25% probability of getting a quarterly return lower than *-9.1%* (or, annualized *-39.6%*) and a 25% probability of a quarterly return higher than *14.1%* (or, annualized *56.2%*). With a less spread distribution for the U.S. stock returns, these numbers are -*2.0%* (or, annualized -*8.0%*) for the 25th-percentile and *7.0%* (or, annualized *28.2%*) for the 75th-percentile. Notice the Brazilian median is below the U.S. median. That is, with 50% probability, Brazilian stocks return less than *2.2%* per quarter (i.e., *9.0%* in annualized terms), while U.S. stocks return less than *2.6%* per quarter (i.e., *10.3%* in annualized terms).

Albeit the descriptive statistics listed in the above paragraphs make clear the Brazilian stock market is more volatile than the U.S., they do not provide an objective measure of riskiness. Another interesting issue is how these returns depart from the Normal distribution. Long-horizon continuously compounded returns converge to normal distributions, but that is not yet the case for quarterly returns (see Cont 2001, and Fama and French 2018b).

Aumann and Serrano (2008) propose an index of "riskiness" that addresses these two issues. The AS index enables an investor to assess which of two investments is *riskier* without referring to a specific utility function, thereby making comparisons easy. Although it is not our objective to put the Brazilian and U.S. stock markets as alternatives to the same investor, it is

[16] See Hughson et al. (2006) for an argument of why investors should be more interested in medians and percentiles than in the mathematical expectation. Harvey and Siddique (2000) and Dittmar (2002), among others, demonstrate the importance of skewness and kurtosis in investor preferences.

informative to assess the relative riskiness of the two markets through a riskiness index. Additionally, the AS index accounts for higher moments of the returns distributions and provides a measure of how they are far from Normal.

From the normal inverse Gaussian distribution, Homm and Pigorsch (2012) provide the following parametric formula for the AS index:

$$AS^{NIG}(\mu, \sigma^2, \chi, \kappa) = (3\kappa\mu - 4\mu\chi^2 - 6\chi\sigma + 9\sigma^2/\mu)/18, \qquad (4)$$

where μ is the mean, σ^2 is the variance, χ is the skewness, and κ measures excess-kurtosis. In case skewness and excess-kurtosis are zero, the return distribution converges to the normal distribution and the AS index becomes:

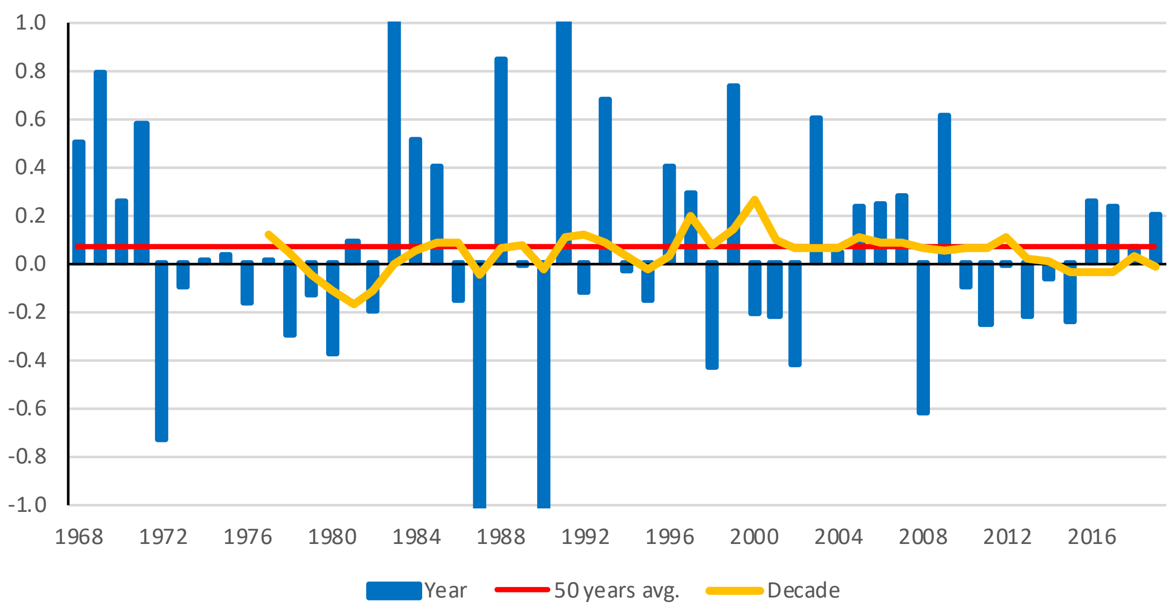
$$AS^{Nornal}(\mu, \sigma^2, \chi, \kappa) = (1/2)(\sigma^2/\mu). \qquad (5)$$

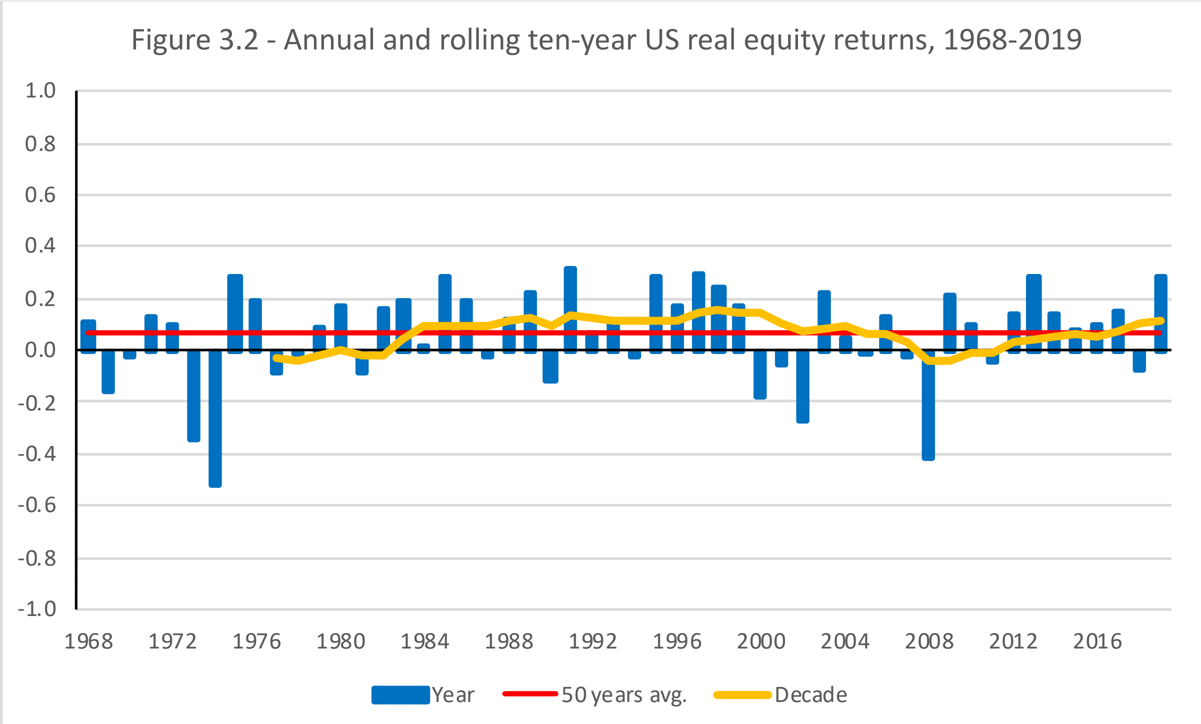
The AS indices in Table II confirm the Brazilian markets are much riskier than the U.S. markets. However, the Brazilian returns distributions are not further from the Normal distribution than the U.S. returns distributions. Indeed, the Brazilian *ratio* of equation (4) over (5) is closer to one than the same *ratio* for U.S. data, indicating the higher riskiness of the Brazilian markets derive mostly from its high variances.

Table II also presents minimum cumulated returns in 5-, 10- and 25-year windows. Besides, it shows the numbers of rolling windows, within the 52 years studied, in which the investment resulted in negative cumulative returns after investing for that respective horizon. For example, in column (1) of row *Negatives in 10-yrs*, the "*11*" means there were eleven specific years in which Brazilian stocks produced cumulative losses after ten-year investments. One can identify those years along the yellow line in Figure 3.1. Respectively for Brazil and the U.S., Figures 3.1 and 3.2 illustrate the realized annual geometric equity returns for a rolling decade, the full 52-year period, and on a year-by-year basis.

While stocks have fewer years of negative real returns than short-term interest in the U.S., it is the opposite in Brazil, due to the latter high stock market volatility. Note, however, as the investment horizon increases, the equity risk of loss decreases relative to that of the short-term real interest rate in both markets.

Figure 3.1 - Annual and rolling ten-year Brazilian real equity returns, 1968-2019





Finally, Table II presents *p*-values of the Cumby-Huizinga test that do not reject the hypothesis of no autocorrelation. Uncorrelated returns is a key assumption to infer expected values from historical averages, as has been suggested in this paper.

2.4 An unbiased long-term mean return estimator

Assuming returns are lognormally distributed $lnR = r \sim N(\bar{r}, \sigma^2)$ – an assumption that gets better as the horizon increases, according to Cont (2001), and Fama and French (2018b) [17] – the Brazilian much higher volatility than that for the U.S. explains why the large difference

[17] Normality tests usually reject that quarterly continuously compounded returns are normally distributed. However, distributions of continuously compounded returns converge towards normal distributions, with horizon extension from one to 30 years, as demonstrated in Fama and French (2018b).

between their arithmetic means in Table I shrinks when we look at geometric returns in Table II. It should be:

$$(1+\bar{a}) = e^{\bar{\alpha}} = e^{\bar{r}+\frac{1}{2}\sigma^2} . \qquad (6)$$

where the $\frac{1}{2}\sigma^2$ term converts the expected return from a geometric mean to arithmetic mean. That is a Jensen's Inequality adjustment, since we are describing expectations of log returns: $lnE(R) = E(lnR) + 0.5\sigma^2(lnR)$. Note the Brazilian stock market volatility is so much higher than the U.S. that $(\bar{r}+\frac{1}{2}\sigma^2)/\bar{r} = 2.79$ in Brazil and $(\bar{r}+\frac{1}{2}\sigma^2)/\bar{r} = 1.25$ in the U.S.

The "Arithmetic" column of Table III computes average returns from Equation (6) and provides a sense of the consequences of lognormal assumption. The reported statistics are similar in magnitudes to their equivalents in column (1) of Table I, and thus henceforward we use the geometric average and standard deviation to build expected rates of return.

However, Jacquier et al. (2003) recall that mean estimates are subject to sampling variation. For lognormal returns, the geometric average estimate is:

$$\hat{r} = \bar{r} + \varepsilon\frac{\sigma}{\sqrt{T}} , \qquad \varepsilon \sim N(0,1), \qquad (7)$$

where T is the time span of the sample used in the estimation. Thus, the estimated return of an investment with horizon H is:

$$e^{(\hat{r}+\frac{1}{2}\sigma^2)H} = e^{\left(\bar{r}+\varepsilon\frac{\sigma}{\sqrt{T}}+\frac{1}{2}\sigma^2\right)H} = e^{(\bar{r}+\frac{1}{2}\sigma^2)H}e^{\left(\varepsilon\frac{\sigma}{\sqrt{T}}\right)H}, \qquad (8)$$

and the expected estimate is:

$$E\left[e^{\left(\hat{r}+\frac{1}{2}\sigma^2\right)H}\right] = e^{\left(\bar{r}+\frac{1}{2}\sigma^2\right)H} E\left[e^{\left(\varepsilon\frac{\sigma}{\sqrt{T}}\right)H}\right] = e^{\bar{\alpha}H} e^{\left(\frac{1}{2}\sigma^2\frac{H^2}{T}\right)}, \qquad (9)$$

showing that the arithmetic mean estimates are biased upward by the last term, $e^{\left(\frac{1}{2}\sigma^2\frac{H^2}{T}\right)}$.

Alternatively, one can write:

$$E\left[e^{\hat{r}H}\right] = E\left[e^{\left(\bar{r}+\varepsilon\frac{\sigma}{\sqrt{T}}\right)H}\right] = e^{\bar{r}H} e^{\left(\frac{1}{2}\sigma^2\frac{H^2}{T}\right)} = (1+\bar{\alpha})^H e^{\frac{1}{2}\sigma^2\left(\frac{H}{T}-1\right)H}, \qquad (10)$$

which indicates the geometric mean estimates are biased downward if $H<T$, and the bias increases with the volatility.

To remove such bias in the expected rates of return, Jacquier et al. (2003) suggest compounding at the unbiased mean rate of return estimator:

$$\hat{\alpha}^{unb} = \hat{r} + \frac{1}{2}\sigma^2\left(1-\frac{H}{T}\right), \qquad (11)$$

which has expectations:

$$E\left[e^{\left(\bar{r}+\varepsilon\frac{\sigma}{\sqrt{T}}+\frac{1}{2}\sigma^2\left(1-\frac{H}{T}\right)\right)H}\right] = e^{\left(\bar{r}+\frac{1}{2}\sigma^2\left(1-\frac{H}{T}\right)\right)H} e^{\left(\frac{1}{2}\sigma^2\frac{H^2}{T}\right)} = e^{\left(\bar{r}+\frac{1}{2}\sigma^2\right)H}. \qquad (12)$$

Jacquier et al. (2005) additionally propose an alternative "small-sample efficient" estimator, which minimizes the RMSE (root mean squared error) and presents significant efficiency gains. However, they abstract from its biased-side effect on the expected future portfolio value. Although the RMSE gain of the small-sample efficient estimator over the arithmetic mean is significant, the small- sample efficient estimator bias is as sizeable, the reason why we advocate for the unbiased estimator. [18]

In Table III, we present the unbiased mean returns for horizons from one to twenty-five years. As the horizon H increases, expected annual returns decrease faster in Brazilian equities, where the volatility is much higher. Note the Brazilian to U.S. equity return ratio decreases from 2.51 at the 1-year horizon to 1.89 at the 25-year horizon. On the other end, the least volatile U. S. GDP growth is almost not affected. Intuitively, because of uncertainty about the mean return parameter, an investor considering different horizons formulate different point forecasts.

[18] Although the RMSE gain of the small-sample efficient estimator over the arithmetic mean is of *36%* for Jacquier et al. (2005) chosen *H/T=25/60*, mean $\mu = 0.10$ and volatility $\sigma = 0.20$, the small-sample efficient estimator bias amounts to *-34%* of the unbiased expected future portfolio in *H=25* periods. In their notation, the bias formula is: $[E(C)/E(V_H)] = exp\{0.5\sigma^2 H[k + (H/T) - 1]\}$ where C is the estimator. For $k = 1 - 3(H/T)$ of the small-sample efficient estimator, we get to a bias of $[E(C)/E(V_H)] = exp\{-\sigma^2(H^2/T)\}$. With the Brazilian parameters and *H/T = 25/50*, the small-sample efficient estimator bias amounts to *-95%* of the unbiased expected future portfolio in *H=25* periods.

Table III - Unbiased mean annual real returns for different horizons in Brazil and U.S. - 1968-2019

	"Arithmetic"	Horizon (*H* in years)				
		1	5	10	20	25
	Brazil					
Equity return	0.2078	0.2050	0.1938	0.1800	0.1528	0.1395
Short-term interest	0.0112	0.0112	0.0110	0.0107	0.0102	0.0100
GDP growth	0.0388	0.0388	0.0387	0.0386	0.0384	0.0384
	U.S.					
Equity return	0.0820	0.0816	0.0803	0.0787	0.0754	0.0738
Short-term interest	0.0169	0.0169	0.0168	0.0167	0.0166	0.0165
GDP growth	0.0284	0.0284	0.0284	0.0283	0.0283	0.0283

Notes : Annually compounded real rates expressed in % per year. The "Arithmetic" is *{exp[Geometric Mean + .5*(Std.Dev.^2)]-1}* . For horizon *H* , the unbiased mean annual real return is *{exp[Geometric Mean + .5*(Std.Dev.^2)*(1-(H/52)]-1}* . Computed using *208* quarterly observations.

Analogously, one could ask about the size of expected cumulated wealth. We provide this information in Table IV. The investment of *$1* in the Brazilian stock market is expected to return *$26.16* after 25 years, while *$1* in the U.S. stock market for 25 years is expected to return *$5.93*.

Table IV - Unbiased mean terminal wealth (in multiples of initial) for different horizons in Brazil and U.S. - 1968-2019

	"Arithmetic"	Horizon (*H* in years)				
		1	5	10	20	25
		Brazil				
Equity return	1.21	1.20	2.42	5.23	17.19	26.16
Short-term interest	1.01	1.01	1.06	1.11	1.23	1.28
GDP growth	1.04	1.04	1.21	1.46	2.13	2.56
		U.S.				
Equity return	1.08	1.08	1.47	2.13	4.28	5.93
Short-term interest	1.02	1.02	1.09	1.18	1.39	1.50
GDP growth	1.03	1.03	1.15	1.32	1.75	2.01

Note : Terminal wealth after *H* years investment (V_H) of *$1* . "Arithmetic" for *H=1* is *exp[Geometric Mean + .5*(Std.Dev.^2)]* . For horizon *H* , the unbiased mean terminal wealth V_H = *exp{[Geometric Mean + .5*(Std.Dev.^2)*(1-(H/52)]*H}* . Computed using *208* quarterly observations.

We warn that, besides the positive sample mean bias, corrected in Tables III and IV above, investors should be aware of the asymmetric effect of volatility on the percentiles of the lognormal distribution. Fama and French (2018a) present convincing simulations that high volatility implies nontrivial probabilities of negative realized premiums even for 10- and 20-year periods. And such negative realizations really happened in the past fifty-two years histories of Brazil and the U.S., as indicated in Table II. Because of the lognormal positive skewness, Hughson et al. (2006) even argue the median return is a better statistic than the mean (which is too optimistic) for those interested in forecasting future cumulative returns. [19]

Although these authors' perspectives are enlightening of relevant aspects of risk, their approaches do not prescribe a normative optimal allocation, which we hope for, to compare with observed allocations. If so, we choose to judge variance and lognormality through a risk-averse utility function in the next section.

3. Risk Aversion and Optimal Allocation

The very different equity returns histories raise the question: are national investors similar in nature? Precisely, is it possible to reconcile such different equity returns processes with similar risk preferences?

Merton's (1969) optimal lifetime-portfolio selection under uncertainty prescribes different allocations in equities according to the expected premium-volatility trade-off, for a given aversion to risk. Instead, we input the historical premium-volatility trade-offs and observed national allocations to stocks into Merton's (1969) optimal formula, with the hope of obtaining similar implied risk aversions in both countries.

[19] Kan and Zhou (2009) get to the point of combining Hughson et al. (2006) warning with Jacquier et al. (2005) bias correction to the lognormal distribution, deriving an unbiased median estimator equal to $e^{\left(\hat{r}+\frac{1}{2}\sigma^2\left(-\frac{H}{T}\right)\right)H}$, where the penalty of a high variance is sizeable. In section 3, we choose to penalize the variance through a concave (risk-averse) utility function.

Following Jacquier et al. (2005), we adapt Merton's (1969) problem to the context in which we have to estimate the expected equity return ($\bar{r_e}$). [20] The investor maximizes the expectations of her utility of final wealth in H periods from today, given the dataset $\mathcal{D}$:

$$E[U(V_H)|\mathcal{D}] = E\left[\frac{V_H^{1-\gamma}}{1-\gamma}|\mathcal{D}\right] = E\left[\frac{1}{1-\gamma}exp\{(1-\gamma)ln(V_H)\}|\mathcal{D}\right], \tag{13}$$

subject to:

$$V_{t+1} = \left[1 + \bar{r}_f + w\left(r_{e,t+1} - \bar{r}_f\right)\right]V_t\,. \tag{14}$$

The portfolio value H periods into the future is log-normal with parameters:

$$ln(V_H) \sim N(\alpha_H, \sigma_H^2) \equiv N\left(\begin{Bmatrix}\left[\bar{r}_f + w\left(\bar{r}_e - \bar{r}_f\right)\right] \\ -\frac{1}{2}w^2\sigma_{e-f}^2\end{Bmatrix}H,\ w^2\sigma_{e-f}^2 H\right) \tag{15}$$

If we knew $\bar{r}_e$ for sure, the optimal allocation would be constant and independent of the horizon H: $w^* = \frac{\bar{r}_e - \bar{r}_f + \frac{1}{2}\sigma_e^2}{\sigma_e^2\gamma}$. However, because we do not know $\bar{r}_e$, which we need to estimate with the dataset $\mathcal{D}$, the optimal allocation is

$$w^* = \frac{\hat{r}_e + \frac{1}{2}\sigma_{e-f}^2 - \bar{r}_f}{\sigma_{e-f}^2\left[\gamma + \frac{H}{T}(\gamma - 1)\right]}\,. \tag{16}$$

[20] In Appendix B, we develop a version where we have to estimate the expected short-term interest rate ($\bar{r_f}$). There is no critical qualitative difference and quantitative differences are small.

In the above equation, the $\frac{H}{T}(\gamma - 1)$ term comes from the sample variation of the estimated average $\hat{r}_e$, which amplifies the variance of $ln(V_H)$ by $\left(1 + \frac{H}{T}\right)$. Note that for $\gamma > 1$, the weight w^* decreases with the investment horizon. The reasoning is, as estimation risk increases with the horizon H, equity allocations decrease proportionally to the risk aversion. Here, it is not just that an investor formulates different point forecasts for different horizons, as illustrated in Table III. But investors with different risk aversions react differently to the horizon imprecision.

Table V - Optimal weights allocated to equity for different horizons in Brazil and U.S.

	Mean	Std. Dev.	γ	With known parameters	Horizon (H in years)				
					1	5	10	20	25
				Brazil					
Market index	0.0677	0.4921	**2**	0.372	0.368	0.355	0.339	0.312	0.300
Short-term interest	0.0086	0.0719	**4**	0.186	0.183	0.173	0.163	0.144	0.137
			5	0.149	0.147	0.138	0.129	0.114	0.107
			6	0.124	0.122	0.115	0.107	0.094	0.089
			8	0.093	0.091	0.086	0.080	0.070	0.065
				U.S.					
Market index	0.0630	0.1776	**2**	0.996	0.986	0.950	0.908	0.835	0.803
Short-term interest	0.0159	0.0409	**4**	0.498	0.491	0.464	0.435	0.386	0.366
			5	0.398	0.392	0.370	0.345	0.305	0.288
			6	0.332	0.327	0.307	0.286	0.251	0.237
			8	0.249	0.245	0.230	0.213	0.186	0.175

Note: Mean and Std.Dev. of continuously compounded rates from 1968:Q1 to 2019:Q4 (*208* quarterly observations).
The proportion of the wealth allocated to equities is given by Equation (16).

Table V indicates the proportion allocated in equities for different risk aversions and horizons. For the same horizon (the H in a column) and risk aversion (the γ in a row), the percentage of the wealth allocated in equities is lower in Brazil than in the U.S., what is rationalized by the much higher Brazilian equity volatility.

From the Financial Accounts of the United States, we find the average participation of stocks in the total financial wealth of U.S. households is *0.323* between 1997 and 2016. [21] From Afonso (2014), which uses proprietary data from the Brazilian Revenue Service, we calculate the average participation of stocks in the total financial wealth of Brazilian households to be close to *0.124* between 2005 and 2012. [22]

Strikingly, given the parsimony of Merton's (1969) model, those two allocations can result from a risk aversion $\gamma = 5$ and investment horizons between 10 to 20 years in each country. Small perturbations to the equity premium, or the volatility, make $\gamma = 4$ or $\gamma = 6$ also possible. [23] Although we are not aware of data for the average duration of household equity investments in

[21] The Board of Governors of the Federal Reserve System publishes The Financial Accounts of the United States and makes them available at FRED Economic Data. We compute the average participation of stocks in the total financial assets of Households and Nonprofits as the ratio of the sum of corporate equities and mutual fund shares to the difference between total financial assets and the liability in credit market instruments, i.e., (HNOCEAQ027S+HNOMFAA027N)/(HNOTFAA027N-TCMILBSHNO).

[22] From Afonso (2014), we compute the average participation of stocks in the total financial assets of Households as the ratio of the sum of equities and equity funds to the difference between total household wealth and fixed assets, i.e., (Equity + Equity Fund)/(Total Household Wealth - Fixed Assets).

[23] For an example, see the Appendix exercise when we estimate the short-term interest rate r_f and the covariance between equity premium and interest rate is negative (though small).

each country, these bounds sound plausible, given the “planning horizon” figures in the Survey of Consumers Finance by the Federal Reserve Board. [24]

4. Conclusions

In this paper, we tell the history of stock market returns in Brazil during 1968-2019. Besides the documentation of the historical Brazilian long-term equity returns and premium, we assess them in comparison with the U.S. data and through the lens of Merton’s (1969) model, providing insights of asset allocation in emerging economies.

Through various descriptive statistics of the sample returns, including higher moments and the Aumann and Serrano (2008) riskiness index, we indicate that the most striking difference between the Brazilian and the U.S. stock market is the enormous variance of the former.

Following Jacquier et al. (2003), we assess the relative biases of arithmetic and geometric methods in these two countries with very different volatilities and compute an unbiased expected return estimator that penalizes longer-horizon returns for higher volatility due to the increasing imprecision of estimates. Because the Brazilian stock market is very volatile, its expected returns point estimates decrease considerably with the investment horizon.

Most interesting from an asset pricing research perspective, we show the difference between Brazilian and U.S. stock market returns can result from the demands of investors that handle risk similarly. In Merton’s (1969) optimal long-term allocation model, we show the much higher Brazilian

[24] The Survey of Consumers Finance by the Federal Reserve Board asks survey respondents about their most important saving and planning horizons. As described in Hong and Hanna (2014), *65.7%* of the respondents report planning horizons longer than one year, and *14.3%* report horizons longer than ten years.

equity-premium volatility discourages heavier investments in stocks, despite expected returns being higher in Brazil than in the U.S. With similar risk aversions, Brazilians should invest less in stocks than should North Americans.

In sum, our results are consistent with an equilibrium of emerging financial markets where the demand for equities is low despite stocks issued at a high cost of equity, a consequence of higher perceived risks. National investors can be modeled alike, despite the differences in macroeconomic environments.

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Appendix A

In this appendix, we assess the robustness of the findings in Table I, presenting our computations at different frequencies and samples, discussing the effect of expected inflation as the deflator to build real variables on the equity premium and its components, and examining the time series characteristics of our annual data.

Table A.I shows the arithmetic equity premium and its components measured at quarterly and monthly frequencies. We convert magnitudes for the mean and the standard deviation to annual terms to compare the figures at higher frequencies with the results at the annual frequency. We can see less volatility (smaller standard deviations) at quarterly and monthly frequencies. The mean equity returns and equity premium have somewhat smaller magnitudes at higher frequencies. North American figures are more stable across frequencies than their Brazilian counterparts. Generally, the qualitative results of Table I remains valid, i.e., very high and volatile equity premium for Brazil in contrast to a more stable and smaller equity premium for the U.S.

Note we have already considered the quarterly frequency for geometric returns to increase the data points in the sample to characterize more accurately the statistical distributions for the investigated variables, especially their higher moments, which are inputs to the computation of the Aumann-Serrano riskness index.

Table A. II displays the equity premium and its components in subsamples. Each period comprises 17 years, except the last one, which includes 18 years of data. For Brazil, equity and risk-free returns, and the equity premium differ substantially across subsamples. In the most recent subsamples, the equity premium has a much smaller mean and is much less volatile compared to the previous subsamples. This stability coincides with a more stable macroeconomic outlook due to a period of anchored inflation, a consequence of the inflation-targeting regime adopted in 1999. The figures for the U.S are somewhat more stable. However, we can see important differences across subsamples. For instance, the risk-free return becomes negative in the last subsample due to the effect of the great financial crisis on the nominal interest rate, kept at its zero lower bound. Figures A1 and A2 show the equity premium in a twenty-year-rolling window. These figures corroborate the instability of the equity premium across subsamples, especially for Brazil.

Table A.I: Returns 1968-2019 - Data Frequencies

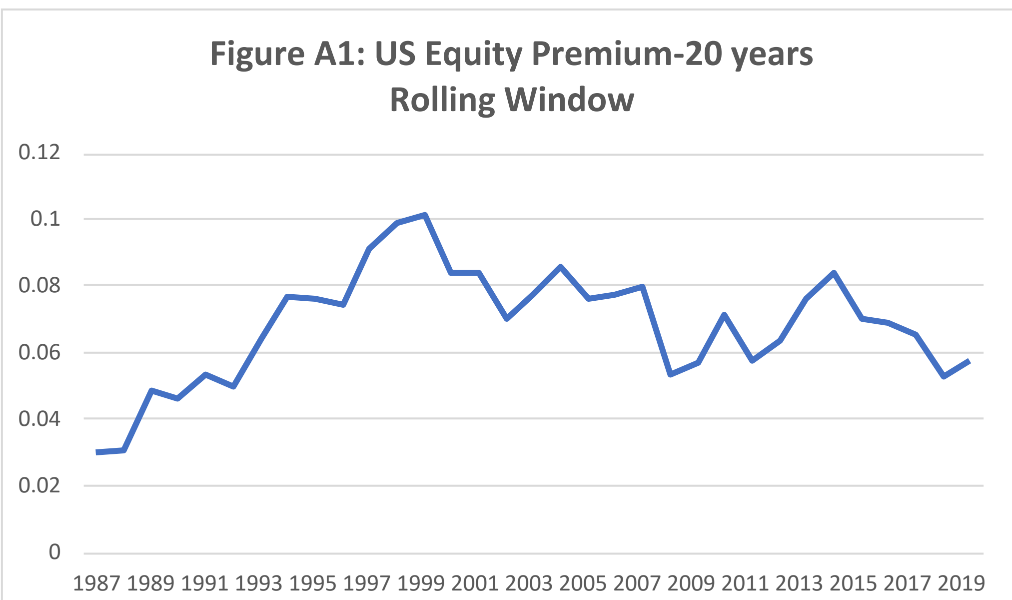
	Brazil		
	Annual	Quarterly	Monthly
Short-term interest - Mean	0.0115	0.0112	0.0107
Short-term interest - Std. Dev.	0.0749	0.0711	0.0642
Equity - Mean	0.2127	0.1749	0.1810
Equity - Std. Dev.	0.6656	0.5172	0.5129
Equity Premium - Mean	0.2012	0.1624	0.1686
Equity Premium - Std. Dev.	0.6704	0.5268	0.5160
	USA		
	Annual	Quarterly	Monthly
Short-term interest - Mean	0.0174	0.0174	0.0168
Short-term interest - Std. Dev.	0.0534	0.0426	0.0319
Equity - Mean	0.0817	0.0815	0.0785
Equity - Std. Dev.	0.1811	0.1760	0.1582
Equity Premium - Mean	0.0642	0.0633	0.0607
Equity Premium - Std. Dev.	0.1696	0.1709	0.1551

Note: Magnitudes in annualized terms

Table A.II: Returns Annual Subsamples

Brazil	1968-1984	1985-2001	2002-2019
Short-term interest - Mean	0.0101	0.0205	0.0042
Short-term interest - Std. Dev.	0.0853	0.0866	0.0526
Equity - Mean	0.2307	0.3096	0.1040
Equity - Std. Dev.	0.6011	0.9439	0.3583
Equity Premium - Mean	0.2206	0.2891	0.0998
Equity Premium - Std. Dev.	0.6042	0.9652	0.3299

USA	1968-1984	1985-2001	2002-2019
Short-term interest - Mean	0.0131	0.0491	-0.0084
Short-term interest - Std. Dev.	0.0603	0.0414	0.0422
Equity - Mean	0.0311	0.1402	0.0741
Equity - Std. Dev.	0.1914	0.1676	0.1773
Equity Premium - Mean	0.0181	0.0911	0.0825
Equity Premium - Std. Dev.	0.1664	0.1523	0.1876



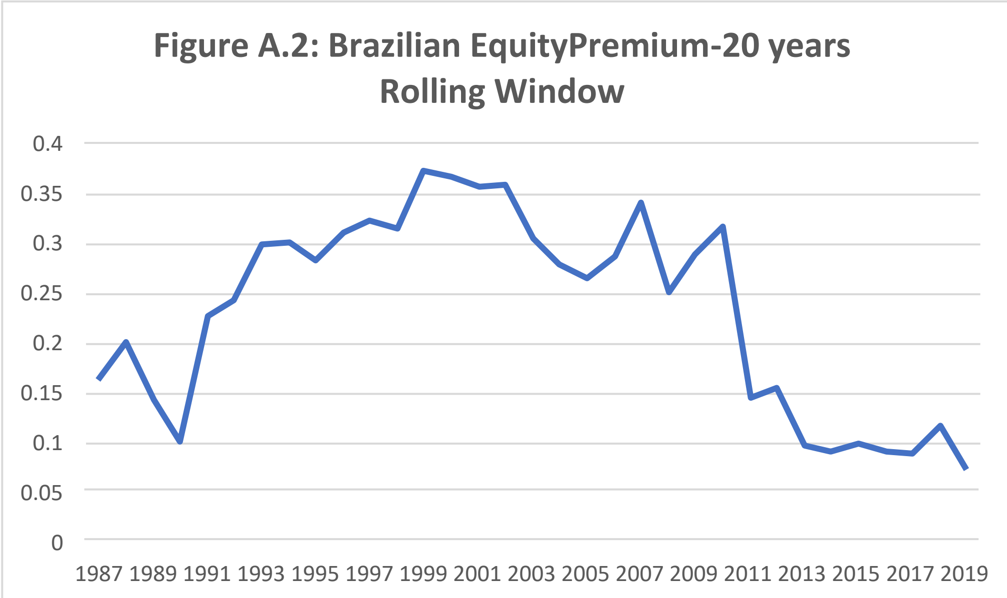


Table A.III: Nominal Returns deflated by Expected Inflation

Brazil (2001-2019)		
	Expected Inflation	Benchmark
Short-term interest - Mean	0.0201	0.0031
Short-term interest - Std. Dev.	0.0199	0.0514
Equity - Mean	0.1015	0.0883
Equity - Std. Dev.	0.3378	0.3548
Equity Premium - Mean	0.0814	0.0852
Equity Premium - Std. Dev.	0.3279	0.3268
USA (1982-2019)		
	Expected Inflation	Benchmark
Short-term interest - Mean	-0.0263	0.0250
Short-term interest - Std. Dev.	0.0103	0.0522
Equity - Mean	0.0972	0.1083
Equity - Std. Dev.	0.1603	0.1681
Equity Premium - Mean	0.1234	0.0833
Equity Premium - Std. Dev.	0.1604	0.1642

In Table A.III, we gauge the effect of using expected inflation as the deflator to build the real figures for the equity premium and its components. For the U.S., we employ the one-year expected inflation calculated by the Federal Bank of Cleveland.[25] For Brazil, we use the Focus Survey, sponsored by the Central bank of Brazil. The choices for expected inflation, however, are only compatible with smaller annual samples: 1982-2019 for the U.S. and 2001-2019 for Brazil. To compare with our benchmark computations in Table I, we restrict the baseline figures for returns and the equity premium in this table to coincide with the same period in which expected inflation measures are available.

For Brazil, employing expected inflation as the deflator to compute real variables affects more the real risk-free rate and real equity returns, but the equity premium remains stable across alternative deflators. For the U.S., choosing expected inflation affects the real risk-free rate, but does not substantially change the magnitudes for equity returns. Note that real risk-free rates are negative. Therefore, the North American equity premium changes, becoming higher compared to our benchmark calculations.

Finally, we discuss some time-series properties of our arithmetic returns annual data. For the sake of brevity, we do not present the details of the test results. We consider the Ljung-Box test for autocorrelation and the ARCH-LM test for conditional heteroscedasticity. If feasible, our goal is to specify a simple model for the investigated variables to compute alternative measures, based on the chosen specification, for the mean and the standard deviations of the equity premium and its components

[25] One can find more explanations about the methodology at https://www.clevelandfed.org/our-research/indicators-and-data/inflation-expectations.aspx

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To be clearer, suppose a specific variable r_t follows a first-order autoregressive process (AR(1) for short), with GARCH (1,1) process describing its conditional variance. Under this specification, the mean equation would be:

$$r_t = c + \rho r_{t-1} + u_t$$

The coefficients c and ρ characterizes the unconditional mean. The variable u_t is a stochastic disturbance and the equation for its conditional variance σ_t^2 is:

$$\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2$$

The coefficients ω, α and β characterize the unconditional variance for the time series studied.

If we could specify such model, alternative measures for the mean and standard deviation for the generic series r_t would be:

$$E(r_t) = \frac{c}{1-\rho} \text{ and } \sigma(r_t) = \sqrt{\frac{\omega}{1-\alpha-\beta}}.$$

With homoscedastic errors, we have $\sigma(r_t) = \sqrt{\frac{\delta^2}{1-\rho^2}}$ for a simple AR(1) process. In this expression, δ^2 is the variance of the error term u_t.

In our annual data concerning arithmetic returns, however, the tests results do not support the AR(1)-GARCH(1,1) specification. Concerning the ARCH-LM series, we are not able to detect conditional heteroscedasticity for any of the time series investigated, since the test statistics displayed p-values greater than 0.5, thus we cannot reject the null hypothesis of homoscedasticity. Concerning the Ljung-Box, we find evidence of autocorrelation only for the U.S. risk-free return. This result coincides with the evidence for quarterly frequency and geometric returns reported in the last row of Table II.

This result coincides with the evidence for quarterly frequency and geometric returns reported in the last row of Table II.

The best-fitting model for the U.S. risk-free interest rate is an AR(1) specification with homoscedastic errors, characterized by the following estimated coefficients: $c = 0.0178$, $\rho = 0.4164$ and $\delta^2 = 0.0023$ We calculate $E(r_t) = \frac{c}{1-\rho}$, which gives a point-estimate of 0.0305. Then, we use a Wald test to gauge the null hypothesis that $\frac{c}{1-\rho}$ equals the historical mean of 0.0174 for the U.S. risk-free interest rate. We find a t -statistics of 0.63 with a p-value of 0.53. Therefore, we cannot reject the null hypothesis. In addition, the point-estimate for the standard deviation is 0.0526 vary close to the figure displayed in Table I.

In short for, all variables, except the U.S. risk-free interest rate, the best model specifies only a constant denoted by the letter c, with $\rho = 0$ and homoscedastic errors. In these cases, the historical mean and standard deviation coincides with the measures computed from the time series model associated with the investigated variables. Though we can describe the U.S. risk-free rate as an AR(1) process, we cannot reject the hypothesis that the alternative measure for the unconditional mean given by the ratio $\frac{c}{1-\rho}$ differs from the historical mean displayed in Table I. Moreover, the point-estimate for the standard deviation, which equals 0.0527, is very close in magnitude to 0.0534, the historical standard deviation also reported in Table I. These set of evidences support the historical mean and standard deviation as sensible measures of average returns and volatility for our annual data set.

Appendix B

When we estimate the expected equity return $\bar{r}_e$ and the short-term interest $\bar{r}_f$, the portfolio value H periods into the future is lognormal with parameters:

$$ln(V_H) \sim N(\alpha_H, \sigma_H^2) \equiv N\left(\begin{Bmatrix} \left[\bar{r}_f + w(\bar{r}_e - \bar{r}_f)\right] \\ -\frac{1}{2}\left[\sigma_f^2 + 2w\sigma_{f,e-f} + w^2\sigma_{e-f}^2\right] \end{Bmatrix} H, \left[\sigma_f^2 + 2w\sigma_{f,e-f} + w^2\sigma_{e-f}^2\right] H\right).$$

Because we concede there is a real short-term interest risk and do not know $\bar{r}_e$ and $\bar{r}_f$, which we need to estimate with the dataset $\mathcal{D}$, the optimal allocation is:

$$w^* = \frac{\left(\hat{r}_e - \hat{r}_f + \frac{1}{2}\sigma_{e-f}^2\right) - \sigma_{f,e-f}\left[\gamma + \frac{H}{T}(\gamma - 1)\right]}{\sigma_{e-f}^2\left[\gamma + \frac{H}{T}(\gamma - 1)\right]}. \qquad (B.1)$$

Note that (A.1) incorporates the sample variation of $\hat{r}_f$. Additionally, the covariance between short-term interest rate and equity premium $\sigma_{f,e-f} = \left(\sigma_{f,e} - \sigma_f^2\right)$ in the numerator takes advantage of the diversifying opportunities. Smaller $\sigma_{f,e}$ and greater σ_f^2 justify heavier allocations in equities. Regarding σ_{e-f}^2 in the denominator of Equation (A.1), recall that: $\sigma_{e-f}^2 = \left(\sigma_e^2 - 2\sigma_{f,e} + \sigma_f^2\right)$

Table B - Weights allocated to equity for different horizons according to Equation (B.1)

	Mean	Std. Dev.	Covar.	γ	With known parameters	Horizon (H in years)				
						1	5	10	20	25
					Brazil					
Market index	0.0677	0.4921		**2**	0.40	0.39	0.38	0.36	0.34	0.32
Short-term int.	0.0086	0.0719		**4**	0.21	0.21	0.20	0.19	0.17	0.16
Equity premium		0.5009		**5**	0.17	0.17	0.16	0.16	0.14	0.13
E.prem.xS.-t. int.			-6.9E-03	**6**	0.15	0.15	0.14	0.13	0.12	0.12
				8	0.12	0.12	0.11	0.11	0.10	0.09
					U.S.A.					
Market index	0.063	0.1776		**2**	1.04	1.03	0.99	0.95	0.87	0.84
Short-term int.	0.0159	0.0409		**4**	0.52	0.51	0.48	0.45	0.40	0.38
Equity premium		0.1730		**5**	0.42	0.41	0.39	0.36	0.32	0.30
E.prem.xS.-t. int.			-2.2E-05	**6**	0.35	0.34	0.32	0.30	0.26	0.25
				8	0.26	0.26	0.24	0.22	0.19	0.18

Note: Mean and Std. Dev. of continuously compounded rates from 1968:Q1 to 2019:Q4 (*208* quarterly observations). The proportion of the wealth allocated to equities is given by Equation (A.1).