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Non-Technical Summary

After a severe crisis that hit many advanced economies starting in 2007, various measures were adopted to stimulate economic activity. In many countries, monetary authorities stepped in, considerably reducing their nominal interest rates. In some of them, these rates reached the effective lower bound of zero or near zero. Precluded from continuing stimulating their economies through monetary policy, these countries promoted significant fiscal stimuli.

A larger stimulative effect of fiscal policy, when monetary policy is bounded by the nominal interest rate zero lower bound, has been vastly debated in the literature since then. However, in general, the focus of the discussion was on monetary and fiscal authorities's actions during the crisis. This work contributes to the discussion of the impacts of fiscal stimuli in an economy with zero nominal interest rate, but considering agents' expectations about future actions from monetary and fiscal authorities. It uses a simple model in which expectations with respect to future levels of output and inflation are key elements for the determination of their present levels as well. This allows one to show that not only policies implemented during the crisis are important to stimulate the economy, but also the correct signaling regarding future policies is crucial to quantify the effects of the implemented stimuli.

The discussion proposed presents how these effects primarily depend on what agents expect the authorities will do after the dissipation of the shock that led to the crisis and how these policies are coordinated. It shows that, to amplify the stimulative effect of fiscal policy, it is critical that monetary policy continue to be accommodative for a few periods after the crisis is over, that the fiscal authority be able to implement the proposed stimulus quickly and that its duration be correctly signaled. Lastly, the paper discusses the implications of various policies' combinations in terms of welfare and point that this evaluation must consider not only the effects of policies adopted during the crisis, but also of those expected to be adopted in the future.

Sumário Não Técnico

Após a grave crise que atingiu as principais economias avançadas a partir de 2007, várias medidas de estímulo da atividade econômica foram adotadas. Em muitos países, as respectivas autoridades monetárias agiram reduzindo consideravelmente suas taxas nominais de juros. Em alguns deles, essas taxas chegaram ao limite inferior efetivo de zero ou próximo de zero. Impedidos de continuar estimulando suas economias através da política monetária, esses países promoveram significativos incentivos fiscais.

O maior efeito estimulativo da política fiscal, quando a política monetária está restrita à taxa nominal de juros zero, tem sido vastamente debatido na literatura desde então. No entanto, o foco da discussão, em geral, tem sido nas ações das autoridades monetária e fiscal durante os momentos de crise. Esse artigo contribui para a discussão dos impactos de estímulos fiscais em uma economia com taxa nominal de juros zero, mas levando em consideração as expectativas dos agentes sobre ações futuras das autoridades monetária e fiscal. Usa-se um modelo simples no qual expectativas em relação aos níveis futuros de produto e inflação são elementos chave para determinação dessas variáveis também no presente. Com isso é possível mostrar que não apenas políticas implementadas durante a crise são importantes para estimular a economia, mas também a correta sinalização em relação às políticas futuras é fundamental para quantificar o efeito dos estímulos implementados.

A discussão aqui proposta apresenta como esses efeitos dependem primordialmente do que os agentes esperam que as autoridades farão tão logo o choque que levou à crise se dissipe e de como essas políticas são coordenadas. Mostra-se que, para ampliar o efeito estimulativo da política fiscal, é fundamental que a política monetária continue sendo acomodatícia por alguns períodos após a crise, que a autoridade fiscal seja capaz de implementar o estímulo rapidamente e que sua duração seja corretamente sinalizada. Por fim, o artigo discute as implicações em termos de bem-estar de várias combinações de políticas e aponta que essa avaliação deve levar em consideração não apenas os efeitos das políticas adotadas durante a crise, mas também daquelas que se espera sejam implementadas em períodos futuros.

Fiscal Stimulus at the Zero Lower Bound: the role of expectations and policy coordination*

Cyntia Freitas Azevedo†

Abstract

In this article, we discuss the role expectations regarding future policies play in determining the depth of a crisis when the economy hits the zero lower bound on nominal interest rates. We show that when analyzing the impact of a fiscal stimulus during a zero interest rate episode, there is more than just short-run output multipliers. We extend the analysis in Eggertsson (2011) by allowing for a transitional state in which the zero lower bound is no longer binding, but policies can be expected to credibly deviate from their steady-state values. The main result of the paper is that, to have larger positive effects on output, monetary and fiscal policies should last longer than the duration of the shock and be coordinated. This coordination is required not only during the crisis but also in the commitment to future policy actions. It is fundamental to the fiscal authority to be able to respond quickly to the shock, minimizing implementation delays and correctly signaling the duration of the stimulus. We also show that a thoughtful evaluation of a fiscal stimulus in terms of the implied welfare losses should account not only for the effects of policies on short-run output and inflation, but also for the present discounted value of output and inflation in future periods as well.

Keywords: Fiscal Policy, Fiscal Multipliers, Monetary-Fiscal Coordination, Zero Bound Constraint.

JEL codes: E58, E61, E62.

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1 Introduction

The significant amount of stimulus provided by fiscal authorities since the burst of the 2007-2008 financial crisis brought the discussion about the impact of these policies to the center of the economics debate.[1] Knowledge about the magnitude of the multipliers, which express how much output changes in response to a given fiscal measure, is relevant to guide the choice among the available policy instruments (taxes, transfers or spending). This knowledge is a crucial input for deciding the size and the duration of the stimulus that will be implemented. For the conduct of monetary policy, one must also understand the impact of fiscal policies on output and inflation and evaluate how monetary authority's actions might contribute to amplifying or undermining this impact.

There is a general understanding that fiscal multipliers cannot be established as an absolute number under all circumstances and across countries. In fact, the discussion about them has converged to the characterization of their determinants. Corsetti et al. (2012) point out that the specificities of each country, and the economic environment, largely affect the impact fiscal measures have on the real economy. Analyzing the characteristics of DSGE models that are relevant to fiscal multipliers, Coenen et al. (2012) observe that, besides depending on the structural parameters of the model, a series of factors are essential to explain such remarkably dispersed estimates. Among the most relevant, they highlight the duration of the stimulus, the degree of monetary policy accommodation, the fiscal instruments used and their operating channels.

The way monetary policy responds to the implementation of fiscal policies is a critical aspect that one should consider when studying the multipliers. Following an economic stimulus through increases in government spending, a higher degree of monetary accommodation implies larger multipliers. The extreme case is when, during a deep recession, monetary policy reaches its limit, with the nominal interest rate hitting the zero lower bound. In this circumstance, as shown by Christiano et al. (2011), fiscal policy ends up being more stimulative than in normal times. Eggertsson (2011b) studies the effects of a fiscal stimulus at zero interest rates using a standard New Keynesian model. His results point out that the impact of government spending increases is more substantial in this environment, while cuts in labor or capital taxes can be recessive.

In general, the debate about the interaction between fiscal and monetary policies is based on the analysis of their actions during the crisis. Nonetheless, there has been some discussion regarding the role played by expectations about future fiscal and monetary policy measures. Using a DSGE model in which policy rules may evolve according to a probability distribution, Davig and Leeper (2011) show that the multipliers dependent not only on the monetary-fiscal

[1] According to the IMF Fiscal Monitor, in the US, government spending as a fraction of GDP increased by 2.1% on average in the period 2006-2014, reaching its historical peak in 2009, when the Congress passed the American Recovery and Reinvestment Act. In the same period, gross debt to GDP ratio increased by 41.3%. Most advanced economies observed these increases and the peak in 2009.

regime during the period the stimulus is provided, but also on the regime expected in future periods. Woodford (2011) points that "careful signaling about the likely direction of future policy is likely to be as important as current actions." Eggertsson and Woodford (2004) analyzed the optimal conduct of monetary and fiscal policies in this environment. They highlight the role played by expectations regarding future policy in mitigating the distortions created by the zero lower bound.

I extend the work of Eggertsson (2011b) to study the impact on economic activity of agents' expectations regarding policies when the crisis is over and the zero lower bound is still binding. Eggertsson (2011b) studies fiscal policy in a two-state model where monetary and fiscal policies revert to their pre-crisis values when the shock goes back to the steady state. This paper extends his framework by allowing for a transitional state in which the shock that led to the zero lower bound is no longer affecting the economy, but policies can be expected to deviate credibly from their steady-state values.

This paper revisits the main results of optimal policies under a liquidity trap which supports that stimulus should stay in place for a longer horizon than the duration of the shock. Indeed, optimal monetary policy under commitment, when there are adverse shocks to the natural rate of interest, requires keeping the interest rate at the zero-lower bound, even when the shock is no longer causing it to bind, to generate inflationary expectations at the exit. I ask whether fiscal policy can directly help to boost the economy by also committing to keep the stimulus once the crisis is over. The inclusion of a transitional state allows analyzing several combinations of practical interest in which fiscal policy is expansionary for a longer horizon, while the monetary authority may or may not be accommodative, even when the zero lower bound is no longer binding.

On the fiscal side, I discuss the effect of stimuli that are expected to be temporary or permanent, or that might be carried out with a delay. On the monetary side, besides examining the effect of keeping the interest rate at zero for a few periods after the crisis is over, I also analyze an optimally chosen rate for this intermediate state.

I use the model setup in Eggertsson (2011b), which gives rise to an aggregate demand/supply sort of analysis. Although simple, this model allows obtaining useful analytical results which highlight that considering only short-run multipliers can be misleading and that expectations regarding future policies play a crucial role in determining the levels of output and inflation during the crisis state. This work contributes by extending the discussion about government spending multipliers, but also regarding the impact of stimulus implemented through income taxes.

An additional contribution is the derivation of a welfare loss function in present discounted value terms that allows ranking the various combinations of policies adopted in each state. Thus, I can discuss the impact of policies not only regarding their effects on output and inflation during the crisis but also considering the entire future.

The main result of the paper is that monetary and fiscal policies, to be more expansionary, should last longer than the duration of the shock and be coordinated in this expansionary move. This coordination is required not only during the crisis but also regarding the commitment to future policies. It is crucial to the fiscal authority to be able to respond quickly to the shock, minimizing implementation delays and correctly signaling the duration of the stimulus.

The most successful combinations of policies are those in which the fiscal stimulus is carried out with minimum delay during the crisis and is expected to be temporary. Keeping this policy combination when the shock is no longer causing the zero lower bound to bind and agents expect its association with an accommodative monetary policy, there is an overall improvement of welfare. These policies can create expectations of higher inflation when the crisis is over and, even if small decreases in output are expected, with impacts during the crisis positive enough to generate lower welfare losses.

It is interesting to point out that, as shown in Eggertsson (2011b), an increase in income taxes stimulates the economy when the zero lower bound is binding, although its impact is smaller than if providing the stimulus through increases in government spending. Nonetheless, in contrast to what we observe with government spending, keeping the stimulus after the crisis is over can improve the economy, even without monetary policy accommodation. Hence, a mistake in the time of ending the stimulus provided through increases in income taxes is not as harmful as the one in the timing of reverting an increase in government spending. This difference is also valid for the implementation of policies. Delaying the increase in taxes until the shock is no longer causing the zero lower bound to bind can still reduce the depth of the crisis. In this case, it is better than not implementing any policy at all. In contrast, if the stimulus is carried out through increases in government spending that are expected to occur only when the crisis is over, without any accommodation from monetary policy, it is better not to implement any policy at all.

I will discuss these various policy combinations in the analysis that follows. The next section situates this work on the recent literature about fiscal multipliers, especially considering their values when the zero lower bound for the nominal interest rate is binding. Section 3 introduces the setup of the model and discusses the available policy responses to the crisis under the extension of a 2-state to a 3-state economy, allowing for the occurrence of a transitional state. Section 4 derives the analytical solutions for output and inflation in each state and the welfare loss function in present discounted value terms. It also presents the results for the optimal nominal interest rate in the transitional state. Section 5 performs an analysis of the multipliers in a calibrated model, while Section 6 discusses the impact of different policies' combinations on output and inflation in each state and the associated welfare losses. Section 7 concludes the paper.

2 Related Literature

The discussion about the size of fiscal multipliers has been intense and led to a vast literature on the topic. I summarize here a few papers more closely related to the analysis presented in this work.[2] The focus is on the debate regarding the response of output and inflation to fiscal stimuli during a crisis caused by a shock that makes the nominal interest rate hit the zero lower bound. It is known that a combination of many factors is responsible for determining the impact of fiscal measures in this environment. Ramey and Zubairy (2014) point to theoretical evidence that characteristics such as the persistence of spending changes, how they are financed, how monetary policy reacts and the tightness of labor markets can significantly affect the magnitude of multipliers.

In a structural vector autoregression (VAR) analysis with data from groups of both advanced and emerging countries, Ilzetzki et al. (2013) obtain that the degree of openness of the economy, the exchange rate regime and the monetary policy reaction are the most influential factors explaining the broad dispersion of estimates for the multipliers across countries. Their estimations suggest larger multipliers in more closed economies (fewer leakages of aggregate demand towards imports), in open economies with fixed exchange rate regimes and when the monetary policy is more accommodative.

The extreme case of accommodative monetary policy is when the nominal interest rate hits the zero lower bound and the monetary authority can no longer stimulate the economy by decreasing it. In such an environment, with the economy facing deep reductions in both output and inflation, there is plenty of evidence that fiscal stimulus is effective in reducing the depth of the recession.

Christiano et al. (2011) show that spending multipliers are more than three times larger when the nominal interest rate is zero, compared with the situation of positive interest rates. They emphasize that the larger the fraction of government spending that occurs while the nominal interest rate is zero, the larger the value of the multiplier. Woodford (2011) also analyzes variations in government spending. In his setup, the multiplier at the zero lower bound is monotonically increasing in the expected duration of the crisis.

Eggertsson (2011b) studies this issue in a setup similar to Woodford's, but with a richer set of fiscal instruments. He shows that if private agents expect government spending to be higher in all states in which the lower bound is binding, the expected contraction is reduced, giving an incentive for them to spend more. Government's commitment to keeping higher spending as long as the zero lower bound is still binding is critical for this result. A surprising result that labor tax cuts can be recessive follows. In normal times, reductions in labor taxes would make workers willing to increase labor supply, leading to a more abundant supply of goods and, thus, downward pressures on prices. In response, the monetary authority reduces the nominal interest rate and output goes up. When the zero lower bound binds, the same tax cuts still

[2] For an overview of this discussion see Hall (2009) and Spilimbergo et al. (2009).

create deflationary pressures, making agents expect lower prices in all states while it is still binding. But the central bank cannot react by further decreasing the nominal interest rate. So the real interest rate goes up, implying a downward movement of output.[3]

All these authors highlight the importance of implementation timing for fiscal measures to have maximum effect in fighting the recession. The expectation that the stimulus might continue after the crisis is over, while monetary policy goes back to its usual stance, can undermine the positive effect obtained during the collapse. Woodford (2011) studies the possible occurrence of a transitional state generated by the continuation of higher government spending after the recession is over. This work corroborates his results. In the case of continuous fiscal expansion, the multiplier ends up being smaller than if the policy was perfectly timed to end as soon as the crisis is over. The longer agents expect the spending stimulus to last, the smaller is the multiplier. Ultimately, it can even become negative. Eggertsson (2011b) finds similar results.

In the extreme case, if the increase in government spending is expected to be permanent, meaning that it will be above its steady-state level in the long run, it will imply higher output and lower inflation in the long term. The expectation of lower inflation in the future undermines the effect of higher government spending in the short run and, for some calibrations, it can be contractionary even during the crisis. This effecy explains the contradictory results obtained by Cogan et al. (2010). Using a model similar to Christiano et al. (2011), but assuming government spending to increase permanently during the crisis, they obtain significantly smaller multipliers.

In the opposite direction, Corsetti et al. (2010) discuss the need of many countries to face significant retrenchment in government spending and analyze the effects of such anticipated spending reversals. They show that the beneficial effect of public expenditure is quite sensitive to when the reversal starts. An early and intense reversal may lower fiscal multipliers and extend the zero lower bound episode.

Eggertsson (2011b) also analyzes how expectations about future monetary policy can affect the macroeconomic aggregates during the crisis. He assumes the central bank credibly commits to a higher inflation target in the future. Expectations of higher inflation in the future decrease real interest rate in the crisis state, stimulating spending during the recession. This policy, however, requires a high degree of credibility for the central bank to make such trustworthy announcements.

Eggertsson and Woodford (2004) point that in a liquidity trap, this type of commitment would be optimal. In their setup, they show that keeping the nominal interest rate at the lower bound for a few quarters after the natural rate of interest has returned to its normal level, will cause the economy to display an output boom and an increase in inflation. The authors highlight that "a credible commitment to behave in this way, after the zero bound has ceased to bind, dramatically reduces the price and output declines that occur during the period when

[3]This is an application of the *Paradox of Toil* which says that, once the nominal interest rate hits the zero lower bound, if everybody tries to work more, there will be less work in the aggregate (see Eggertsson (2010)).

the central bank is constrained by the zero bound."

Denes et al. (2013) extend the work of Eggertsson (2011b) by explicitly considering the government budget constraint to analyze the consequences to debt dynamics of policies adopted when the zero lower bound is binding. They show that austerity measures (cutting government spending and/or increasing taxes) may increase, rather than decrease the short-run deficit. They also examine how the deficit created during the crisis affects expectations and, consequently, short-run demand. These expectations refer to how long-run taxes and spending will be adjusted to bring down debt to its pre-crisis level. However, a fundamental assumption in their work is that monetary policy sets the nominal interest rate so that inflation is zero when the zero lower bound is not binding.

The studies mentioned above use a setup similar to the one adopted in the present study, analyzing the effects of expectations about future policies. Nevertheless, they do not put together an extensive analysis of possible policy combinations like the one shown here. They investigate forward guidance considering monetary policy or government spending in isolation but do not study the combination of both policies when the crisis is over, besides only examining government spending as the fiscal instrument. Moreover, the setup adopted here also enables us to compare policy combinations regarding welfare losses, which has not been discussed so far in the literature.

3 The Model

I follow the setup of a simple New Keynesian model presented in Eggertsson (2011b). His paper provides the detailed microfoundations of the non-linear model. The model is linearized around the long-run steady state and summarized through an (IS) equation, an aggregate supply (AS) relation and the definition of a monetary policy rule (MP). These three equations are the starting point for our analysis.

The (IS) equation comes from the households' optimal decisions, using the aggregate resource constraint to substitute out for consumption. Its linearized version is given by

$$(IS) \quad \hat{Y}_t = E_t\hat{Y}_{t+1} - \sigma\left(i_t - E_t\pi_{t+1} - r_t^e\right) + \left(\hat{G}_t - E_t\hat{G}_{t+1}\right) \tag{3.1}$$

where i_t is the one-period risk-free nominal interest rate, π_{t+1} is the inflation rate, $\hat{Y}_t \equiv \log\left(\frac{Y_t}{\overline{Y}}\right)$ is the deviation of output from its steady-state value and $\hat{G}_t \equiv \left(\frac{G_t - \overline{G}}{\overline{Y}}\right)$ the deviation of government spending (G_t) from its steady-state value $(\overline{G})$, as a fraction of steady-state output $\left(\overline{Y}\right)$. r_t^e is an exogenous shock defined as $r_t^e \equiv \log\beta^{-1} + E_t\left(\hat{\xi}_t - \hat{\xi}_{t+1}\right)$, where $\hat{\xi}_t \equiv \log\xi_t/\bar{\xi}$ is the deviation of a consumer's preference shock from its steady state. $\beta > 0$ is the discount factor and $\sigma > 0$ is the intertemporal elasticity of substitution of private expenditure.

A critical assumption is that government spending is not a perfect substitute for private consumption. As pointed out by Eggertsson (2011b), this is the type of expenditure which is effective in increasing demand (infrastructure or military spending are examples). If they were substitutes, cuts in private spending would offset an increase in government spending, and aggregate spending would not change.

The aggregate supply relation (AS) is derived from the firms' optimal decisions and is given by

$$(AS) \quad \pi_t = \kappa \hat{Y}_t + \kappa\psi \left(\chi^I \hat{\tau}_t^I - \sigma^{-1} \hat{G}_t \right) + \beta E_t \pi_{t+1} \tag{3.2}$$

where $\hat{\tau}_t^I = \tau_t^I - \bar{\tau}^I$ is the percentage increase in income tax rate[4,5], while $\chi^I > 0$, $\kappa > 0$, $\omega > 0$ and $\psi > 0$ are model parameters.[6]

Finally, the monetary policy rule is as follows:

$$(MP) \quad i_t = \max \left(0, r_t^e + \phi_\pi \pi_t + \phi_y \hat{Y}_t \right) \tag{3.3}$$

where it is assumed that $\phi_\pi > 1$ and $\phi_y > 0$. This specification rules out negative values for the nominal interest rate.

The (IS) equation, together with the monetary policy rule, defines the aggregate demand (AD) relation in this model. Equilibrium will be the solution of the system formed by the aggregate demand and supply equations. Eggertsson (2011b) shows that, given a path for $\left\{ \hat{G}_t, \hat{\tau}_t^I \right\}$ determined by fiscal policy, and an exogenous path for $\{r_t^e\}$, an equilibrium is a collection of stochastic processes for output, inflation and the nominal interest rate $\left\{ \hat{Y}_t, \pi_t, i_t \right\}$ that solve the system of equations (3.1) – (3.3).

Another critical assumption is that Ricardian equivalence holds. This means that lump-sum transfers in period t or in future periods offset temporary variations in either $\hat{\tau}_t^I$ or $\hat{G}_t$. In the absence of this hypothesis, Denes et al. (2013) point out that how a fiscal expansion is financed, through adjustments in future taxes or spending, can have significant effects on short-run demand. This implies that a plan about how short-run budget deficits or surpluses will be met in the future should complement a given government stimulus provided when the zero lower bound is binding. Although important, I will not discuss the impact of the fiscal stimulus on debt generated by the absence of lump-sum taxes closing the government budget constraint. This would significantly increase the number of policy combinations to be analyzed

[4] We follow Denes et al. (2013) who model taxes $\left(\tau_t^I\right)$ levied on income from both labour and the households' claims on firms profits. They explain that if only wages were taxed, there would be a disproportionate drop in tax revenues in a recession. This would exaggerate the results and rely too much in the complete wage flexibility in the model. With this more conservative assumption, income tax is proportional to output. Any drop in real wages will be reflected by an increase in profits, and taxing wages and profits at the same rate means we abstract from this redistribution aspect of the model.

[5] Regarding other types of taxes discussed in Eggertsson (2011b), he shows that the effects of sales taxes represent a negative scale of those caused by variations in $\hat{G}_t$. So it would be redundant to analyze them here. For the sake of space, the analysis of taxes levied on capital income is left for future work.

[6] These parameters depend on the baseline parameters of the model as follows: $\chi^I \equiv \frac{1}{1-\bar{\tau}^I}$, $\omega \equiv \frac{\bar{v}_{hh}\bar{H}}{\bar{v}_h}$, where $\bar{v}_h$ and $\bar{v}_{hh}$ are, respectively, the first and second derivatives of the utility function with respect to labor, and $\bar{H}$ is the level of labor in steady state. $\psi \equiv \frac{1}{\sigma^{-1}+\omega}$ and $\kappa \equiv \frac{(1-\alpha)(1-\alpha\beta)}{\alpha} \frac{\sigma^{-1}+\omega}{1+\omega\theta}$, where $0 < \alpha < 1$ is the fraction of firms whose prices stay unchanged in each period and $\theta > 1$ is the elasticity of substitution among differentiated goods.

and negatively affect the tractability of the paper.

Although quite simple, the model has the advantage of providing closed-form solutions and allowing the discussion of the effects of policies during the zero lower bound episode. In what follows I analyze a set of possible short-run allocations in response to the exogenous shock r_t^e. Depending on the strength of the shock and the policy responses to it, two scenarios are possible. I call "normal times" the scenario where the shock is not large enough to make the nominal interest rate hit the zero lower bound. The other scenario, called the "crisis state," occurs when the shock causes the nominal interest rate implied by rule (3.3) to reach the zero lower bound.[7] The fall in output and inflation caused by the shock varies a lot, depending not only on the policy actions taken during the crisis state (short run) but also on expectations about the implementation of policy actions after the crisis is over. I analyze how possible combinations of present and future policies impact crisis-state allocations.

First, I summarize the 2-state economy case presented by Eggertsson (2011b). Then I show the setup of the 3-state economy that includes a transitional state. The next section displays the derivation of the allocations in each state.

3.1 Revisiting the Literature - Two-state economy

The analysis in Eggertsson (2011b) starts by assuming that at time $t = T_0$ the economy is hit by a shock that causes r_t^e to go to $r_S^e < 0$. The return of the shock to the steady state at time $t = T_{exit}$ is exogenously given. In his basic setup, there are two possible states for this economy. The short run, or crisis state, ($t \in [T_0, T_{exit})$), characterized by the period in which r_t^e goes below its steady-state value ($r_t^e = r_S^e$). And the long term ($t > T_{exit}$), when r_t^e returns to the steady state ($r_t^e = \bar{r}$). Woodford (2011) interprets this shock as a severe disruption in financial intermediation that causes a spike in credit spreads, thus decreasing r_t^e. Eggertsson (2011b) explains it as a preference shock that lowers ξ_t because suddenly everyone wants to save more, so the real interest rate must decline for output to stay constant. I assume that the evolution of this shock is independent of either monetary or fiscal policy actions. Therefore, measures to stimulate the economy do not revert the shock but can attenuate the recession it generates.

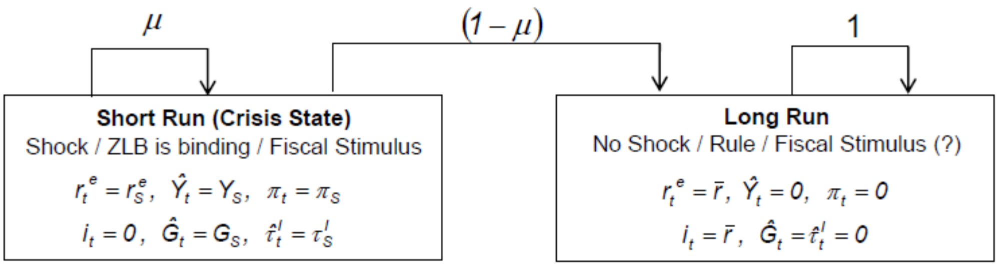
If the shock is large enough, it can make the zero lower bound on the nominal interest rate a binding constraint in policy rule (3.3). In this scenario, the monetary authority can no longer reduce the nominal interest rate to stimulate the economy further, so the government can only rely on a combination of the available fiscal instruments $\left(\hat{G}_t, \hat{\tau}_t^I\right)$ to mitigate the recession caused by the fall in r_t^e.

Assuming that the shock causes the nominal interest rate to go to zero ($i_t = 0$), Figure 3.1 illustrates the 2-state economy with a crisis state. In the short run, the shock causes output

[7] As in Eggertsson and Woodford (2004) and Woodford (2011), Eggertsson (2011b) assumes that even when the shock is large enough to cause the zero bound to bind, local approximations to both the model structural relations and the welfare objective are still accurate.

and inflation to go below their steady-state values (Y_S, π_S). This allocation depends on the size of the shock (r^e_S) and the fiscal measures adopted $\left(G_S, \tau^I_S\right)$. In each period $t \in [T_0, T_{exit})$, after the shock hits the economy, there is an exogenous probability μ that the shock will still be at r^e_S in period $t+1$. With probability $(1-\mu)$ it will return to $\bar{r}$ in period $t+1$ and the economy goes back to the steady state. I assume that once back to the steady state, r_t stays at $\bar{r}$ thereafter. Eggertsson (2011b) (Proposition 2 - Page 70) shows that, in the long run, with $\hat{G}_t = \hat{\tau}^I_t = 0$, there is a locally unique bounded solution to the system $(3.1)-(3.3)$ such that $\hat{Y}_t = \pi_t = 0$ and $i_t = \bar{r}$, $\forall t > T_{exit}$.

Figure 3.1: Diagram I - Benchmark Case



Eggertsson (2011b) highlights as a general principle in this class of models that, when the zero lower bound is binding, successful policy actions are those which do not aim at increasing aggregate supply, but increasing the aggregate level of spending in the economy. This happens because, with zero nominal interest rates, output is demand determined. This does not mean that aggregate supply is irrelevant since it is crucial to pin down expectations about future inflation. It is important to note that in this environment, incentives to aggregate supply are counterproductive since they can create deflationary expectations. He precisely points out that "policy should not be aimed at increasing the supply of goods when the problem is that there are not enough buyers."

3.2 The Three-State Economy

A critical assumption of significant part of the analysis in Eggertsson (2011b) is that the implemented stimulus policy is perfectly correlated with the shock. This means that government spending and/or income taxes will deviate from their steady-state values as long as $r^e_t = r^e_S$i.e., while the crisis lasts. Once r^e_t goes back to $\bar{r}$, fiscal instruments also return to their long-run levels $\left(\hat{G}_t = \hat{\tau}^I_t = 0, \forall t > T_{exit}\right)$.

I extend the analysis in Eggertsson (2011b) by allowing the government to credibly commit to implementing a combination of monetary and fiscal policies, even after the recession is over. During the crisis, expectations about future values of output and inflation play a crucial role in determining the depth of the collapse. Eggertsson and Woodford (2004) point out that it makes a crucial difference if the government can commit to creating inflation in the future. They show that it is optimal to maintain a loose monetary policy for a few quarters after the crisis is over. Assuming VAT taxes, they also show it is optimal to increase them during the crisis and reduce them when the zero lower bound is no longer binding.

Expectations regarding future actions from both monetary and fiscal authorities affect the levels of output and inflation during the crisis and the associated welfare losses. It is important to note from equations (3.1) and (3.2) that, besides the direct impact of future government spending on short-run aggregate demand through the term $E_t\hat{G}_{t+1}$, any future policy actions that affect output and inflation after the crisis is over, will also have effects on short-run aggregate demand and supply relations through the terms $E_t\hat{Y}_{t+1}$ and $E_t\pi_{t+1}$.

Our goal is to understand these effects and show how the impact of stimulus actions during the crisis depends on expectations about future interactions between these policies. On the monetary policy side, I want to model the expectation that the central bank might keep the interest rate at a fixed value i_M (which can be zero or an optimally chosen value) for a few periods[8], even after r_t^e returns to $\bar{r}$. On the fiscal policy side, I want to analyze the impact of the termination timing of the stimulus provided during the short run and of implementation delays. I examine these policies adopted in coordination, but also cases where each authority acts separately.

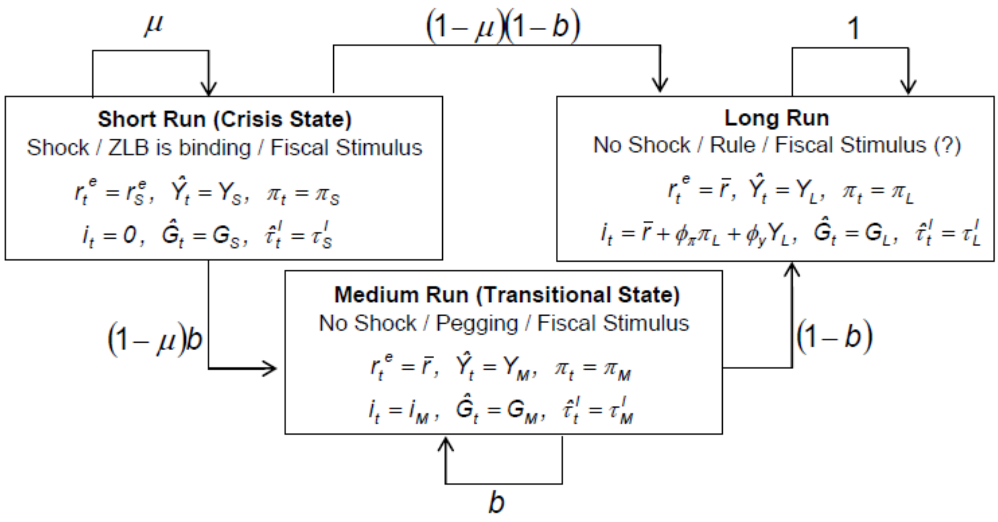
Diagram 3.2 illustrates the setup of this 3-state economy. As before, the idea is that at time $t = T_0$, the economy is hit by a shock $r_t^e = r_S^e$, the zero lower bound binds ($i_t = 0$), the fiscal authority uses a combination of its instruments $\left(G_S, \tau_S^I\right)$ to undermine the crisis' effects, but output and inflation still stay below their steady-state levels (Y_S, π_S). This state lasts until an uncertain period $t = T_{exit}$, when r_t^e returns to $\bar{r}$. In every period of the recession $t \in [T_0, T_{exit})$, there is probability μ that the economy stays in the crisis state in the following period. With probability $(1-\mu)\,b$ it goes to a transitional state, while with probability $(1-\mu)\,(1-b)$ it jumps straight to the long run.[9]

A transitional state occurs because agents associate some probability that after exiting the crisis state, the authorities might still deviate from their steady-state policies. They might expect that the monetary authority will keep an accommodative monetary policy for a few periods after the crisis is over, maintaining nominal interest at $i_M = 0$ or keeping it at an optimally chosen level i_M^*. Agents can also expect the fiscal stimulus implemented during the crisis not to be perfectly correlated with it, lasting for a few periods after the zero lower bound is no longer binding $\left(G_M, \tau_M^I\right)$. Expectations may be that only one or both these policies will be used in the transitional state. This creates a medium-run allocation, with output and inflation deviating from their steady-state levels (Y_M, π_M), that lasts until an exogenously given period $t = T_M$. In each period $t \in [T_{exit}, T_M)$, there is probability b that the economy stays in the

[8] We will show that this cannot be a permanent policy as it would lead to solution indeterminacy.

[9] Throughout the analysis, both probabilities μ and b are assumed independent from monetary and fiscal policy actions and from each other.

Figure 3.2: Diagram II - Transitional State Case



transitional state in $t+1$, while with probability $(1-b)$ it goes to the long run.

If the fiscal stimulus is temporary (during the crisis state or lasting until the transitional state only), in the long run, monetary and fiscal policies go back to their steady-state plans $(i_L = \bar{r}, G_L = \tau_L^I = 0)$, returning output and inflation to their steady-state levels as well $(Y_L = \pi_L = 0)$. A case in which fiscal measures implemented during the crisis are expected to be permanent $(G_L > 0$ and/or $\tau_L^I > 0)$ will also be analyzed. In such scenario, long-run output, inflation and the nominal interest rate are allowed to deviate from their steady-state levels (Y_L, π_L, i_L).[10] Once the economy goes to the long run, I assume that it stays at this level thereafter.[11] Eggertsson (2011b) analyzes the case of permanent fiscal policies as well. But his setup does not allow for the discussion of this policy associated with a temporary accommodative monetary policy after the crisis is over.

The proposal is to analyze a way the monetary authority has to create expectations of higher inflation in the future, by assuming that the central bank can credibly commit to keeping the nominal interest rate fixed for a few periods, even after r_t^e returns to its steady-state value at $\bar{r}$. I study the possibility that the fiscal authority can also signal that it will keep using a combination of its policy instruments when the zero lower bound is no longer binding, which might be associated with a future accommodative monetary policy or not.[12]

[10] We will not assume different levels for the fiscal instruments in each state. When used in all states, we will have $G_S = G_M = G_L$ or $\tau_S^I = \tau_M^I = \tau_L^I$. We keep the subscripts for the respective states to allow identifying where the impacts on the levels of output and inflation are coming from.

[11] One should note that the long-run equilibrium is different from the initial steady-state level around which the model was loglinearized around. We assume that this linearization is still valid at the long-run equilibrium. We are aware that this is a very strong assumption, but it significantly simplifies the analysis.

[12] Eggertsson (2011b) models the promise of future inflation with a monetary policy rule committed to a higher inflation target. He shows that this not only increases inflation expectations for all periods after r_t^e returns to $\bar{r}$ but also reduces the drop in short-run inflation in all periods when the zero lower bound still binds. He does not analyze an association of this policy with future fiscal actions though.

Table 3.1: Summary of Possible Policy Choices

	Monetary Policy	Fiscal Policy
Short run ($r_t^e = r_S^e$)	*ZLB is binding* ($i_S = 0$)	(i) *None* (ii) $\hat{G}_S$ (iii) $\hat{\tau}_S^I$
Medium run ($r_t^e = \bar{r}$)	(i) $i_M = 0$ (ii) $i_M = i_M^*$ (iii) $i_M = \bar{r} + \phi_\pi \pi_M + \phi_y Y_M$	(i) *None* (ii) $\hat{G}_M$ (iii) $\hat{\tau}_M^I$
Long run ($r_t^e = \bar{r}$)	$i_L = \bar{r} + \phi_\pi \pi_L + \phi_y Y_L$	(i) *None* (ii) $\hat{G}_L$ (iii) $\hat{\tau}_L^I$

Table 3.1 summarizes the policy choices available for monetary and fiscal authorities in each state. In the short run, the monetary authority cannot stimulate the economy further, since the zero lower bound is binding, leaving the fiscal authority as the only one able to provide stimulus. It will pick one of its instruments, either using government spending or income taxes. I assume that once chosen in the short run, the same instrument is used throughout the states (this rules out, for example, using government spending in the short run and income taxes in the medium run or vice-versa).

In the medium run, the fiscal authority can choose to return the instrument used in the short term to its steady state, or it can keep the stimulus for a few periods after the crisis is over. On the other hand, the monetary authority will choose from three possible policies: (i) keep the nominal interest rate at $i_M = 0$ for a few periods after the crisis is over; (ii) keep the nominal interest rate at an optimally chosen level i_M^* for a few periods after the crisis is over; (iii) return to policy rule (3.3) as soon as the crisis is over. I assume that if the fiscal stimulus provided during the crisis is not expected to be permanent, but it is extended to the transitional state associated with an accommodative monetary policy, than these policies are perfectly correlated, i.e., they go to the long run at the same time $t = T_M$.

In the long run, monetary policy returns to policy rule (3.3). If the fiscal stimulus provided during the crisis is expected to be permanent, it determines a new long-run allocation. I will also discuss the impact of implementation delays in fiscal policy, in an extreme case in which the stimulus is only carried out after the crisis is already over.

The next section presents solutions for output and inflation in each state, under different hypothesis regarding expectations about monetary and fiscal policies, after the crisis is over.

4 Solution Allocations

The solution for this type of model is obtained backward, getting the solution for the long-run allocation first. Then this will be used to compute expectations in the transitional state and obtain the medium-run solution. Finally, I use both medium- and long-run allocations to compute expectations in the crisis state and get the short-run solutions. Given the linearity of the model, to get the most general result, I derive these solutions assuming that both fiscal policy instruments $\left\{\hat{G}_t, \hat{\tau}_t^I\right\}$ are active in every state. According to each hypothesis made in the analysis that follows, one or the other instrument is muted.

The short- and medium-run allocations depend on the monetary policy adopted in the transitional state. Proposition 1 presents the results for those cases where the monetary authority either keeps the nominal interest rate at $i_M = 0$, $\forall t \in [T_{exit}, T_M)$, or it is set at an optimal level $i_M = i_M^*$, $\forall t \in [T_{exit}, T_M)$. Section 4.1 discusses the choice of this optimal level. Proposition 2 outlines the results if the monetary authority returns to rule (3.3) as soon as the crisis is over.

Proposition 1 *Assume that the nominal interest rate zero lower bound is binding in the short run* ($t \in [T_0, T_{exit})$) *and that the following conditions hold:*

$$
\begin{array}{ll}
(C1) & (1-\mu)(1-\beta\mu) - \mu\kappa\sigma > 0 \\
(C2) & (1-b)(1-\beta b) - b\kappa\sigma > 0 \\
(C3) & \phi_\pi + \dfrac{(1-\beta)}{\kappa}\phi_y > 1 \\
(C4) & r_S^e < -\Theta_{G_S} G_S - \Theta_{\tau_S^I}\tau_S^I - \Theta_{\bar{r}}(\bar{r} - i_M) - \Theta_{G_M} G_M - \Theta_{\tau_M^I}\tau_M^I - \Theta_{G_L} G_L - \Theta_{\tau_L^I}\tau_L^I
\end{array}
$$

If there is a transitional state generated by the monetary authority keeping the nominal interest rate at $i_M = 0$, or at an optimally chosen level $i_M = i_M^*$, after the crisis is over ($\forall t \in [T_{exit}, T_M)$), *solutions for output, inflation and the nominal interest rate in each state can be obtained backward as follows:*

(*i*) *In the long run* ($\forall t > T_M$), *with $r_t^e = \bar{r}$, there is a locally unique bounded solution such that*

$$Y_L = \Omega_{Y_L,G_L} G_L + \Omega_{Y_L,\tau_L^I}\tau_L^I \tag{4.1}$$
$$\pi_L = \Omega_{\pi_L,G_L} G_L + \Omega_{\pi_L,\tau_L^I}\tau_L^I \tag{4.2}$$
$$i_L = \bar{r} + \phi_\pi \pi_L + \phi_y Y_L \tag{4.3}$$

(*ii*) *There is a locally unique bounded medium-run solution* ($\forall t \in [T_{exit}, T_M)$), *with $r_t^e = \bar{r}$, such that*

$$Y_M = \Omega_{Y_M,i_M}(\bar{r} - i_M) + \Omega_{Y_M,G_M} G_M + \Omega_{Y_M,\tau_M^I}\tau_M^I + \Omega_{Y_M,G_L} G_L + \Omega_{Y_M,\tau_L^I}\tau_L^I \tag{4.4}$$

$$\pi_M = \Omega_{\pi_M,i_M}(\bar{r} - i_M) + \Omega_{\pi_M,G_M} G_M + \Omega_{\pi_M,\tau_M^I}\tau_M^I + \Omega_{\pi_M,G_L} G_L + \Omega_{\pi_M,\tau_L^I}\tau_L^I \tag{4.5}$$

$$i_M = \begin{cases} 0 \\ i_M^* \end{cases} \tag{4.6}$$

(*iii*) *In the short run* ($\forall t \in [T_0, T_{exit})$), *with $r_t^e = r_S^e$, there is a locally unique bounded solution, such that*

$$
\begin{aligned}
Y_S = \quad & \Omega_{Y_S,r_S^e} r_S^e && + \quad \Omega_{Y_S,G_S} G_S && + \quad \Omega_{Y_S,\tau_S^I} \tau_S^I && + \\
& \Omega_{Y_S,i_M} (\bar{r} - i_M) && + \quad \Omega_{Y_S,G_M} G_M && + \quad \Omega_{Y_S,\tau_M^I} \tau_M^I && + \\
& && \quad\;\; \Omega_{Y_S,G_L} G_L && + \quad \Omega_{Y_S,\tau_L^I} \tau_L^I &&
\end{aligned} \tag{4.7}
$$

$$
\begin{aligned}
\pi_S = \quad & \Omega_{\pi_S,r_S^e} r_S^e && + \quad \Omega_{\pi_S,G_S} G_S && + \quad \Omega_{\pi_S,\tau_S^I} \tau_S^I && + \\
& \Omega_{\pi_S,i_M} (\bar{r} - i_M) && + \quad \Omega_{\pi_S,G_M} G_M && + \quad \Omega_{\pi_S,\tau_M^I} \tau_M^I && + \\
& && \quad\;\; \Omega_{\pi_S,G_L} G_L && + \quad \Omega_{\pi_S,\tau_L^I} \tau_L^I &&
\end{aligned} \tag{4.8}
$$

$$
i_S = 0 \tag{4.9}
$$

where the analytical expressions for the coefficients $\Omega_{i,j}$*,* $i \in \{Y_S, \pi_S, Y_M, \pi_M, Y_L, \pi_L\}$ *and* $j \in \{r_s^e, i_M, G_S, \tau_S^I, G_M, \tau_M^I, G_L, \tau_L^I\}$*, and* Θ_k*,* $k \in \{i_M, G_S, \tau_S^I, G_M, \tau_M^I, G_L, \tau_L^I\}$ *are defined in the Appendix and depend on the structural parameters.*

Proof. See Appendix B. ■

The proof of Proposition 1 is long and its details are presented in Appendix B. Nevertheless, there are a few aspects to be highlighted. The proof is divided into three parts. The first part discusses determinacy of the solution in each state and obtains conditions $(C1)-(C3)$, which are shown to be necessary for a determinate solution. Given the parameters β, σ and κ, Condition $(C1)$ imposes an upper bound on the duration of the crisis state (μ), while $(C2)$ imposes an upper bound on the length of the transitional state (b). Condition $(C3)$ is the regular determinacy condition for a New Keynesian model with monetary policy given by a rule like (3.3), and it is satisfied given that I assume $\phi_\pi > 1$ and $\phi_y > 0$. The second part of the proof derives the analytical expressions for the solutions in each state (equations $(4.1)-(4.9)$). Finally, the third part derives the restriction on the size of the shock r_S^e and the stimulus policies to guarantee that the zero lower bound is binding in the short run (Condition $(C4)$). Note that if no fiscal instrument is used, $\hat{G}_t = \hat{\tau}_t^I = 0$, $\forall t$, and monetary policy goes back to rule (3.3) right after the crisis is over (implying $i_M = \bar{r}$), this condition states that $r_S^e < 0$. The condition derived here nests the one derived by Eggertsson (2011b).[13] It is interesting to point out that this condition depends not only on monetary and fiscal policies adopted during the crisis but on those expected to be adopted in future states as well.

One should also note that the results in Proposition 1 are presented in the most summarized form possible. Table A.1 in Appendix A summarizes the analytical expressions for medium- and long-run solutions coefficients ($\Omega's$) in equations $(4.1)-(4.2)$ and $(4.4)-(4.5)$, respectively. Table A.2 presents the short-run coefficients in equations $(4.7)-(4.8)$.

The values associated with each coefficient will be discussed in the next section.[14] But it

[13] Assuming $b = 0$ makes $\Theta_{\bar{r}} = \Theta_{G_M} = \Theta_{\tau_M^I} = 0$, shutting down the medium run, while $G_L = \tau_L^I = 0$ shuts down the long run.

[14] Tables A.1 and A.2 also include the coefficients for the responses to transitional state fiscal instruments, if

is important to underline that the solution derived in this proposition nests the benchmark case, presented by Eggertsson (2011b). Given the linearity of the model, it is easy to see that, if we assume there is no transitional state ($b = 0$), the economy jumps straight to the steady state, right after the crisis is over.[15] Besides, if the fiscal stimulus is expected to be temporary ($G_L = \tau_L^I = 0$), long-run output and inflation are given by $Y_L = \pi_L = 0$, while the short-run solution obtained from equations (4.7) and (4.8) are reduced to

$$\begin{aligned} Y_S = &\quad \Omega_{Y_S,r_S^e} r_S^e \quad + \quad \Omega_{Y_S,G_S} G_S \quad + \quad \Omega_{Y_S,\tau_S^I} \tau_S^I \\ \pi_S = &\quad \Omega_{\pi_S,r_S^e} r_S^e \quad + \quad \Omega_{\pi_S,G_S} G_S \quad + \quad \Omega_{\pi_S,\tau_S^I} \tau_S^I \end{aligned}$$

where the multipliers $\Omega_{i,j}$, $i \in \{Y_S, \pi_S\}$ and $j \in \{r_s^e, G_S, \tau_S^I\}$ are precisely those derived by Eggertsson (2011b).

Proposition 2 presents the solution allocations for the crisis and transitional states if the monetary authority goes back to rule (3.3) as soon as the crisis is over, but fiscal stimulus continues in the transitional state. In this case, I assume that fiscal policy is temporary, since the previous proposition already nests the case of a permanent stimulus (assume $b = 0$ and fiscal instruments different from zero in the long run).

Proposition 2 *Assume that the nominal interest rate zero lower bound is binding in the short run* ($t \in [T_0, T_{exit})$) *and that the following conditions hold:*

$$\begin{array}{ll} (C1) & (1-\mu)(1-\beta\mu) - \mu\kappa\sigma > 0 \\ (C2) & (1-b)(1-\beta b) - b\kappa\sigma > 0 \\ (C3) & \phi_\pi + \dfrac{(1-\beta)}{\kappa}\phi_y > 1 \\ (C4') & r_S^e < -\Theta_{G_S} G_S - \Theta_{\tau_S^I} \tau_S^I - \Theta_{G_M}^T G_M - \Theta_{\tau_M^I}^T \tau_M^I \end{array}$$

If there is a transitional state generated by the fiscal authority keeping the stimulus provided in the short run for a few periods after the crisis is over ($\forall t \in [T_{exit}, T_M)$)*, while monetary policy returns to rule* (3.3) *as soon as* r_t^e *returns to* $\bar{r}$*, the solutions in each state can be obtained backward as follows:*

(i) In the long run, there is a locally unique bounded solution ($\forall t > T_M$)*, with* $r_t^e = \bar{r}$*, such that* $i_L = \bar{r}$ *and* $Y_L = \pi_L = 0$*.*

(ii) There is a locally unique bounded medium-run solution ($\forall t \in [T_{exit}, T_M)$)*, with* $r_t^e = \bar{r}$*, such*

the monetary policy goes back to the Taylor rule, as soon as the crisis is over (Ω_{*,G_M}^T and $\Omega_{*,\tau_M^I}^T$). Proposition 2 discusses this.

[15] It is important to note that making $G_M = \tau_M^I = 0$ is not enough to rule out the transitional state. The isolated action of the monetary authority keeping the nominal interest rate at $i_M = 0$ or i_M^* for a few periods after the crisis is over can still generate it. Eliminating fiscal instruments in the medium run would still leave the effect of the term ($\bar{r} - i_M$). As can be seen in Table A.2, for $i \in \{Y_S, \pi_S\}$, making $b = 0$ implies that $\Omega_{i,G_M} = \Omega_{i,\tau_M^I} = \Omega_{i,i_M} = 0$.

that

$$
\begin{aligned}
Y_M^T &= \Omega^T_{Y_M,G_M} G_M + \Omega^T_{Y_M,\tau^I_M} \tau^I_M & (4.10)\\
\pi_M^T &= \Omega^T_{\pi_M,G_M} G_M + \Omega^T_{\pi_M,\tau^I_M} \tau^I_M & (4.11)\\
i_M^T &= \bar{r} + \phi_\pi \pi_M^T + \phi_y Y_M^T & (4.12)
\end{aligned}
$$

(iii) *In the short run* $(\forall t \in [T_0, T_{exit}))$, *with* $r_t^e = r_S^e$, *there is a locally unique bounded solution, such that*

$$
\begin{aligned}
Y_S^T = {} & \Omega_{Y_S,r_S^e} r_S^e + \Omega_{Y_S,G_S} G_S + \Omega_{Y_S,\tau_S^I} \tau_S^I + \Omega^T_{Y_S,G_M} G_M + \Omega^T_{Y_S,\tau^I_M} \tau^I_M & (4.13)\\
\pi_S^T = {} & \Omega_{\pi_S,r_S^e} r_S^e + \Omega_{\pi_S,G_S} G_S + \Omega_{\pi_S,\tau_S^I} \tau_S^I + \Omega^T_{\pi_S,G_M} G_M + \Omega^T_{\pi_S,\tau^I_M} \tau^I_M & (4.14)\\
i_S = {} & 0 & (4.15)
\end{aligned}
$$

where the analytical expressions for the coefficients $\Omega_{i,j}$, $i \in \{Y_S, \pi_S\}$ *and* $j \in \{r_S^e, G_S, \tau_S^I\}$, *and* Θ_k, $k \in \{G_S, \tau_S^I\}$, *are the same as those defined in Proposition 1. The expressions for the coefficients* $\Omega^T_{m,n}$, $m \in \{Y_S, \pi_S, Y_M, \pi_M\}$ *and* $n \in \{G_M, \tau^I_M\}$, *and* Θ^T_h, $k \in \{G_M, \tau^I_M\}$, *are described in the Appendix and depend on the structural parameters.*

Proof. See Appendix B. ■

Following the same rationale for the proof of Proposition 1, the proof of Proposition 2 is also divided into three parts. The first part shows that Conditions $(C1)$ – $(C3)$ are also necessary for determinacy of the solutions. The second part derives the analytical expressions for the solutions in each state (equations $(4.10) - (4.15)$). Finally, the third part derives the condition that guarantees that the zero lower bound is binding in the short run $(C4')$.

Proposition 2 allows us to analyze a temporary continuation of the fiscal stimulus with the response of monetary policy through rule (3.3). Woodford (2011) analyzes this case, but he only deals with government spending and does not discuss the association with an accommodative monetary policy in the transitional state.

In what follows, given the results from Propositions 1 and 2, we analyze the properties of the coefficients $\left(\Omega' s\right)$ and the impact of different combinations of policies adopted during the crisis, and expected to be adopted in the future, on output and inflation in each state. We also examine the implications regarding welfare losses. But first we need to discuss how these losses are computed and how, based on the derived welfare loss function, the monetary authority would set the transitional-state nominal interest rate optimally.

4.1 Welfare Losses and Optimal Transitional State Monetary Policy

One of the policies the monetary authority may choose to implement in the transitional state is to set the nominal interest rate at an optimal level. To compute this optimal level, we need to solve the problem of minimizing the economy's welfare losses, restricted to output and inflation solution allocations in each state and the non-negativity condition for the nominal interest rate. The following proposition shows how the loss function is obtained and how it can be expressed in present discounted terms, given the probability structure of the model.

Proposition 3 *The welfare loss function in this model is given by*

$$L_t \equiv \sum_{t=0}^{\infty} \beta^t \left\{ \pi_t^2 + \lambda_y \left(\hat{Y}_t - \Gamma \hat{G}_t \right)^2 + \lambda_g \hat{G}_t^2 \right\} \tag{4.16}$$

where $\Gamma \equiv \dfrac{\sigma^{-1}\gamma}{\sigma^{-1}+\omega}$, $\lambda_y \equiv \dfrac{\kappa}{\theta}$ *and* $\lambda_g \equiv \lambda_y \Gamma (1-\gamma-\Gamma)$. *Given the probability structure of the model, this function can be expressed in present discounted terms as*

$$L^{PDV} = \left\{ \begin{array}{l} \dfrac{1}{1-\beta\mu} \left(\pi_S^2 + \lambda_y (Y_S - \Gamma G_S)^2 + \lambda_g G_S^2 \right) + \\ \dfrac{\beta(1-\mu) b}{(1-\beta\mu)(1-\beta b)} \left(\pi_M^2 + \lambda_y (Y_M - \Gamma G_M)^2 + \lambda_g G_M^2 \right) + \\ \dfrac{\beta(1-\mu)(1-b)}{(1-\beta)(1-\beta\mu)(1-\beta b)} \left(\pi_L^2 + \lambda_y (Y_L - \Gamma G_L)^2 + \lambda_g G_L^2 \right) \end{array} \right\} \tag{4.17}$$

Proof. See Appendix *B*. ■

Note that the welfare loss function derived here is analogous to that presented by Woodford (2011). He uses this function to discuss the optimal level of government spending during the crisis. However, since in his setup there is no transitional state and fiscal stimulus is temporary, the economy goes to the steady state ($Y_L = \pi_L = 0$) after the crisis. Thus, the present discounted value of his loss function corresponds only to the first line in equation (4.17). I will not discuss the optimal level for fiscal instruments here. They will be set at a given level throughout the analysis. I use the present discounted value of the welfare loss function to obtain the optimal transitional-state interest rate below and to compare the welfare implications of different policies' combinations in Section 6.

The solutions for output and inflation obtained in Proposition 1 form the restrictions of the minimization problem solved to find the optimal medium-run interest rate. The proposition below presents the result for this problem, which gives the optimal level for the nominal interest rate in the transitional state.

Proposition 4 *Assuming Conditions* $(C1)-(C4)$ *hold, if the monetary authority keeps the nominal interest rate fixed at an optimal level* i_M^* *in the transitional state, it picks this level by* *solving the following minimization problem*

$$\min_{\{i_M\}} L^{PDV}$$
$$s.t. \quad Y_S, \pi_S, Y_M, \pi_M, Y_L, \pi_L$$
$$i_M \geq 0$$

where L^{PDV} is defined in Proposition 3 and the levels of output and inflation in each state are provided by Proposition 1. The solution to this problem yields an optimal nominal interest rate given by

$$i_M^* = \begin{cases} i_M^{opt} & , \text{ if } i_M^{opt} > 0 \\ 0 & , \text{ otherwise} \end{cases}$$

$$i_M^{opt} = \bar{r} + \frac{1}{\Omega^*_{i_M,\bar{r}}} \left\{ \Omega^*_{i_M,r_S^e} r_S^e + \Omega^*_{i_M,G_S} G_S + \Omega^*_{i_M,\tau_S^I} \tau_S^I + \Omega^*_{i_M,G_M} G_M + \Omega^*_{i_M,\tau_M^I} \tau_M^I \right\} \tag{4.18}$$

where the analytical expressions for the coefficients $\Omega^*_{i_M,j}$, $j \in \{\bar{r}, r_S^e, G_S, \tau_S^I, G_M, \tau_M^I\}$ are defined in the appendix and depend on the structural parameters and the coefficients $\left(\Omega' s\right)$ from Proposition 1.

Proof. See Appendix B. ■

It is worth highlighting that the optimal nominal interest rate depends on the size of the shock (r_S^e), the expected duration of the financial disturbance (μ) and the transitional state (b). It also depends on fiscal policies implemented during the crisis and transitional states. The long-run allocation does not depend on the transitional-state monetary policy, so i_M^{opt} is not affected by the fact that fiscal policy is expected to be temporary or permanent.

5 Analysis of Fiscal Policy Multipliers

To discuss the behavior of multipliers in this section, and the solution allocations for output and inflation in each state in the next, I set the model's parameters[16] according to the values used by Eggertsson (2011b).

Given the results in the previous section, I first look at the isolated impact of each policy instrument on output and inflation in each state. The relations derived in Propositions 1 and 2 are linear functions of government spending and income taxes in each state (G_S, τ_S^I, G_M, τ_M^I, G_L and τ_L^I), the shock (r_s^e) and the gap between steady-state interest rate and medium-run interest rate ($\bar{r} - i_M$). The coefficients that appear in equations $(4.1) - (4.15)$ represent output and inflation multipliers ($\Omega' s$), where some of them depend on the monetary policy adopted in the transitional state.

[16] For a quartely model we assume: $\sigma^{-1} = 1.1599$; $\beta = 0.9970$; $\omega = 1.5692$; $\alpha = 0.7747$; $\theta = 12.7721$; $\phi_\pi = 1.5$; $\phi_y = 0.125$; $\bar{\tau}^I = 0.2$; $r_S^e = -0.0104$; $\mu = 0.903$. Probability μ is set to the maximum value that satisfies Condition $(C1)$, which corresponds to an average crisis duration of 10 quarters. The analogous Condition $(C2)$ establishes the same maximum value for the probability b, thus throughout the analysis we allow it to vary in the interval $[0, 0.9]$.

Table 5.1 presents these coefficients' values under the assumed parameters, as a function of probability b.[17] First I look at the direct impact of the shock (r_s^e) on crisis-state output and inflation ($\Omega_{Y_S,r_S^e} = 28.8$ and $\Omega_{\pi_S,r_S^e} = 2.5$, respectively). These coefficients do not depend on the probability associated with the transitional state (b), but they are increasing in the duration of the shock (determined by probability μ) as illustrated in Figure 5.1.

Table 5.1: Solution Coefficients in Each State a Function of Probability b

	Y_S	π_S	Y_M	π_M	Y_L	π_L
Ω_{*,r_S^e}	28.8	2.5				
Ω_{*,G_S}	2.3	0.2				
Ω_{*,τ_S^I}	1.0	0.1				
Ω_{*,i_M}	$[0.0, 123.2]$	$[0.0, 12.4]$	$[0.9, 24.5]$	$[0.0, 2.1]$		
Ω_{*,G_M}	$[0.0, 6.5]$	$[0.0, 0.7]$	$[1.0, 2.1]$	$[0.0, 0.1]$		
Ω^T_{*,G_M}	$[0.0, -1.4]^\dagger$	$[0.0, -0.1]^\dagger$	$[0.9, 0.5]$	$[0.004, 0.003]$		
Ω_{*,τ_M^I}	$[0.0, 5.2]$	$[0.0, 0.5]$	$[0.0, 0.9]$	$[0.0, 0.1]$		
Ω^T_{*,τ_M^I}	$[0.0, 0.6]$	$[0.0, 0.1]$	$[0.0, -0.1]^\dagger$	$[0.0, 0.03]$		
Ω_{*,G_L}	$[-4.7, -23.3]^\dagger$	$[-0.5, -2.4]^\dagger$	$[-0.7, -4.1]^\dagger$	$[-0.1, -0.4]^\dagger$	0.4	−0.1
Ω_{*,τ_L^I}	$[1.6, 9.4]$	$[0.2, 1.0]$	$[-0.3, 1.3]$	$[0.1, 0.2]$	−0.4	0.1

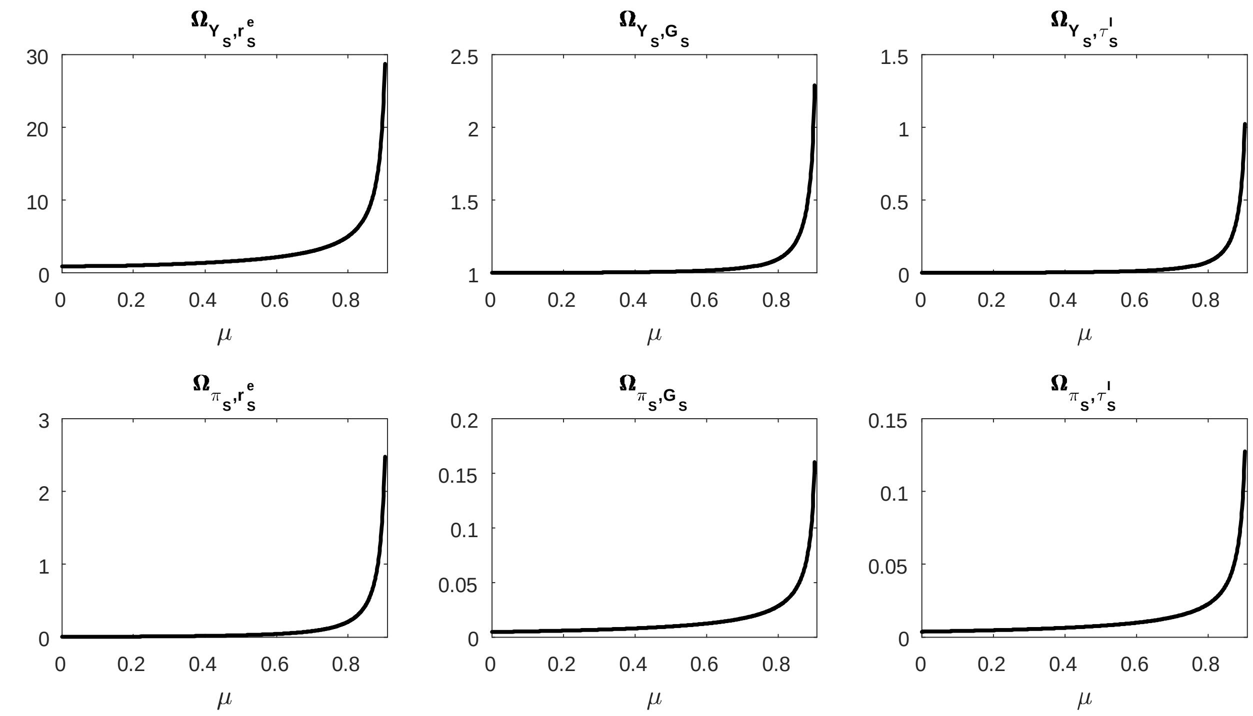
Note: The intervals marked with † are decreasing in b. The others are increasing in b. The * symbol represents the variables in each column.

As discussed in Section 4, if agents expect a continuation of the fiscal stimulus and/or an accommodative monetary policy after the crisis is over, this generates a transitional state that also affects the short-run allocations. If there is a positive probability associated with the monetary authority deciding to keep nominal interest rate at zero, or to choose an optimal rate, after r_s^e returns to $\bar{r}$, the term $(\bar{r} - i_M)$ impacts not only on medium-run output and inflation ($\Omega_{Y_M,i_M} = [0.9, 24.5]$ and $\Omega_{\pi_M,i_M} = [0.0, 2.1]$, respectively), but also on their short-run levels ($\Omega_{Y_S,i_M} = [0.0, 123.2]$ and $\Omega_{\pi_S,i_M} = [0.0, 12.4]$, respectively). Its effects on these allocations are positive and increasing in the expected duration of the transitional state. This means that monetary stimulus provided after the crisis is over can play a significant role in attenuating the depth of the recession caused by the shock. This happens because it increases output and generates inflation in the medium run, producing a positive impact on short-run output and inflation through expectations.

The coefficients on fiscal policy instruments reflect their isolated impact on each state's

[17] Tables A.1 and A.2 in the Appendix present their respective analytical expressions, while Figures A.1-A.3 illustrate their behavior as a function of b.

Figure 5.1: Short-run coefficients as a function of probability μ



allocation. These are the multipliers which most of the literature focuses the discussion on. In what follows, I examine the multipliers for each fiscal instrument separately.

5.1 Government Spending Multipliers

The impact of fiscal stimuli provided during the crisis (which are independent of b, but highly dependent on μ) replicates the multipliers found in Eggertsson (2011b). He shows that, when the zero lower bound is binding, an increase in government spending has a significant impact on crisis-state output ($\Omega_{Y_S,G_S} = 2.3$) and inflation ($\Omega_{\pi_S,G_S} = 0.2$), attenuating the depth of the recession. In fact, providing the same stimulus in normal times[18], for exactly the duration of the crisis (determined by the probability μ), would produce a smaller impact on output and inflation ($\Omega^N_{Y_S,G_S} = 0.5$ and $\Omega^N_{\pi_S,G_S} = 0.003$, respectively).[19]

As Woodford (2011) points out, the short-run government spending multiplier is the highest when it is needed the most, when fiscal stimulus becomes more urgent since interest rate cuts alone can no longer stimulate aggregate demand. He also observes that, if government spending is increased for just one period during the crisis,[20,21] the output multiplier would be

[18] When the zero lower bound is not binding and policy rule (3.3) determines the nominal interest rate.
[19] To obtain the multipliers in normal times we use the coefficients from equations (4.10) and (4.11) assuming $b = \mu$.
[20] To obtain the 1-quarter multiplier, consider $\mu = 0$ in Ω_{Y_S,G_S} and Ω_{π_S,G_S}.
[21] If government spending increases for one period only in normal times, the multipliers would be $\Omega^{N_1}_{Y_S,G_S} = 0.90$

$\Omega^1_{Y_S,G_S} = 1$. This means that from the 2.3 increase in output caused by the fiscal stimulus, 1 comes from the increase in government spending in the first quarter, while 1.3 results from the expected spending increase in the following quarters. The same happens to inflation, which would respond with 0.005 of the rise in the first quarter and a significant part, 0.155, coming from the increase in government spending in future periods when the zero lower bound is still binding.

One important observation regarding the multipliers obtained by Eggertsson (2011b) is that they are highly dependent on the duration of the crisis (μ). Figure 5.1 shows the responses of crisis-state output and inflation (Y_S, π_S) to the short-run fiscal instruments $\left(G_S, \tau^I_S\right)$ as a function of probability μ. We see the significant increase in these coefficients towards the end of the interval. Therefore, one should keep in mind that we are working with the highest values of these coefficients once we assume $\mu = 0.903$, which implies an average crisis duration of 10 quarters[22]. It is worth highlighting that this is a limitation of the framework used here, since the maximum value allowed for the probability μ, due to determinacy requirements as discussed in Propositions 1 and 2, does not admit a longer crisis spell.

I now turn to the analysis of how expectations regarding future policies affect short-run output and inflation (The coefficients discussed are those presented in Table 5.1). The impact of agents' expectations that the fiscal authority keeps the stimulus when the zero lower bound is no longer binding depends on how the monetary authority accommodates it. The association of higher government spending, with the nominal interest rate kept at $i_M = 0$, or optimally chosen (i^*_M), after the crisis is over, generates an impact that is positive and increasing in b on both transitional-state output ($\Omega_{Y_M,G_M} \in [1.0, 2.1]$) and inflation ($\Omega_{\pi_M,G_M} \in [0, 0.1]$)[23]. It also has positive effects on short-run output ($\Omega_{Y_S,G_M} \in [0, 6.5]$) and inflation ($\Omega_{\pi_S,G_M} \in [0, 0.7]$), which are significantly increasing in the probability b as well. It is interesting to note that for high values of b, these multipliers get even more substantial than those observed for the short-run stimulus (Ω_{Y_S,G_S} and Ω_{π_S,G_S}).[24]

On the other hand, if the monetary authority responds to the continuation of the government spending stimulus through policy rule (3.3), the impacts on medium-run output ($\Omega^T_{Y_M,G_M} \in$

and $\Omega^{N_1}_{\pi_S,G_S} = 0.004$.

[22] Probably, back in $2010 - 2011$, predicting an average expectation of 2.5 years of interest rates at the zero lower bound seemed very pessimistic. However, looking for how long the Fed Funds rate was maintained near zero, it might not have been so bad.

[23] Note that these multipliers accurately replicate the behavior of short-run multipliers varying μ. The only difference is that we assume the maximum value for $b = 0.90$, slightly smaller than $\mu = 0.903$. Making $b = \mu$ gets the top of the interval equal to the short-run multipliers. We set $\mu = 0.903$ to make the result comparable to those obtained by Eggertsson (2011b). It is just a matter of convenience that we do not extend the interval for b until the value of μ.

[24] This happens if $b \geq 0.87$ for output and $b \geq 0.85$ for inflation.

$[0.9, 0.5]^{\dagger}$) and inflation ($\Omega^T_{\pi_M,G_M} \in [0.004, 0.003]^{\dagger}$) are still positive[25], but decreasing in b and smaller than those observed with an accommodative monetary policy. In this case, the short-run impact is negative and decreasing in b ($\Omega^T_{Y_S,G_M} \in [0, -1.4]^{\dagger}$ and $\Omega^T_{\pi_S,G_M} \in [0, -0.1]^{\dagger}$ for output and inflation, respectively). This happens because output increases less than one-for-one with the increase in government spending in the transitional state, accompanied by a small rise in inflation. Therefore, the stimulus generated in the medium run is small compared to the cost of higher government spending. This is in line with the result obtained by Woodford (2011). His multiplier corresponds to $\left(\Omega_{Y_S,G_S} + \Omega^T_{Y_S,G_M}\right) \in [2.3, 0.9]^{\dagger}$ for $b \in [0, 0.9]$. He gets a negative total multiplier with a transitional-state probability higher than 0.91, which is not allowed here.

We also conclude that if the government spending stimulus provided during the crisis is expected to be permanent, it decreases short-run output and inflation. In this case, long-run output increases much less than one-for-one with the fiscal stimulus ($\Omega_{Y_L,G_L} = 0.4$), while inflation decreases ($\Omega_{\pi_L,G_L} = -0.1$). Besides, if there is a transitional state generated by an accommodative monetary policy, the impacts on medium-run output ($\Omega_{Y_M,G_L} \in [-0.7, -4.1]^{\dagger}$) and inflation ($\Omega_{\pi_M,G_L} \in [-0.1, -0.4]^{\dagger}$) are negative and decreasing in b. This is also the case for short-run output ($\Omega_{Y_S,G_L} \in [-4.7, -23.3]^{\dagger}$) and inflation ($\Omega_{\pi_S,G_L} \in [-0.5, -2.4]^{\dagger}$). We will see that the low increase in output and the deflation generated in the long run are determinant for the recrudescence of the crisis. As pointed out by Woodford (2011) and Eggertsson (2011b), this helps to understand the low values for the spending multipliers found by Cogan et al. (2010).

Note that with a permanent fiscal stimulus, a transitional state will be generated only if the monetary authority decides to keep the nominal interest rate at $i_M = 0$ or i^*_M, when the zero lower bound is no longer binding. In the case the monetary authority goes back to following policy rule (3.3) as soon as the crisis is over, with a permanent fiscal stimulus, there is no transitional state, and the economy jumps straight to the long run. To get short-run output and inflation, we just need to assume $b = 0$[26] in Proposition 1. In this case, the short-run impact of a permanent fiscal stimulus is at the lower bound of the intervals ($\Omega_{Y_S,G_L} = -4.7$ and $\Omega_{\pi_S,G_L} = -0.5$ for output and inflation, respectively).

It is essencial to observe that I assume the monetary policy reaction to long-run inflation and output deviations from the steady state does not change to offset fiscal stimulus expected to be permanent. It would be natural to imagine that a new long-run level for the fiscal instruments would imply a new stance for monetary policy as well. This is why the results differ from those obtained by Denes et al. (2013), which analyze the effects of long-run policies,

[25] The marker † highlights the intervals which are decreasing in b.

[26] This will make the coefficients Ω_{Y_S,i_M}, Ω_{π_S,i_M}, Ω_{Y_S,G_M}, Ω_{π_S,G_M}, Ω_{Y_S,τ^I_L}, Ω_{π_S,τ^I_L} equal to zero.

but with monetary policy targeting zero inflation when not at the zero lower bound. If the interest rate adjusts to guarantee that there is no inflation or deflation in the long run, the effect of permanent policies come only from their impact on long-run output.

5.2 Income Tax Multipliers

In general, one would expect that an increase in income taxes would reduce output and increase inflation[27]. However, Eggertsson (2011b) shows the surprising result that, when the zero lower bound is binding, an increase in income taxes has a positive effect on short-run output[28] $\left(\Omega_{Y_S,\tau_S^I} = 1.0\right)$, in contrast with the negative multiplier observed in normal times $\left(\Omega^N_{Y_S,\tau_S^I} = -0.1\right)$. The impact on inflation $\left(\Omega_{\pi_S,\tau_S^I} = 0.1\right)$ is also more substantial than in normal times[29] $\left(\Omega^N_{\pi_S,\tau_S^I} = 0.03\right)$. As seen in Figure 5.1, these multipliers are highly dependent on the value of probability μ which is set at the upper end of the feasible interval.

Eggertsson (2011b) explains that the output multiplier is now positive because, when the zero lower bound is binding, the short-run demand curve becomes upward slopping.[30] This implies that an increase in taxes in the short run creates inflationary pressures, increasing inflation expectations in every state where the zero lower bound is still binding. This reduces the real interest rate, making current consumption cheaper and raising demand. It is important to point out that this result is particular to the model setup assumed. Eggertsson (2011b) observes that income taxes may have a direct effect on demand if one assumes that a fraction of workers and/or firms are liquidity constrained, for example.

I show that if income taxes are kept at a higher level after the crisis is over, associated with an accommodative monetary policy, this generates a transitional state where the economy is stimulated, with positive output and inflation multipliers ($\Omega_{Y_M,\tau_M^I} \in [0.0, 0.9]$ and $\Omega_{\pi_M,\tau_M^I} \in [0.0, 0.1]$, respectively), which are increasing in b. Again, these are equivalent to the short-run multipliers if we were varying μ. This policy also has a positive impact on crisis-state output ($\Omega_{Y_S,\tau_M^I} \in [0.0, 5.2]$) and inflation ($\Omega_{\pi_S,\tau_M^I} \in [0.0, 0.5]$). These short-run multipliers are increasing in b, and they get larger than the effect of the short-run increase in taxes for higher levels of this probability ($b \geq 0.84$ for output and $b \geq 0.85$ for inflation).

On the other hand, if the monetary authority goes back to following policy rule (3.3) as

[27] The values for the coefficients discussed in this section are those presented in Table 5.1

[28] This multiplier $\left(\Omega_{Y_S,\tau_S^I} = \frac{\mu\kappa\sigma\psi\chi^I}{(1-\mu)(1-\beta\mu)-\mu\kappa\sigma}\right)$ is always positive and increasing on the crisis' duration (μ).

[29] Note that again we compute the multipliers in normal times assuming the expected stimulus duration to be the same of the crisis (determined by the probability μ). If the stimulus was provided for just one period, its impact on output and inflation would be approximately zero during normal times or the crisis state.

[30] This can be seen in the derivation of the short-run solution in the proof of Proposition 1. (See Equation (B.17))

soon as the crisis is over, the continuation of higher income taxes decreases output ($\Omega^T_{Y_M,\tau^I_M} \in [0.0, -0.1]^\dagger$) and causes inflation to increase less ($\Omega^T_{\pi_M,\tau^I_M} \in [0.0, 0.03]$) in the transitional state. The effects on short-run output and inflation are still positive ($\Omega_{Y_S,\tau^I_M} \in [0.0, 0.6]$ and $\Omega_{\pi_S,\tau^I_M} \in [0.0, 0.1]$, respectively), but smaller than those obtained when monetary policy accommodates the fiscal stimulus in the medium run.

If the stimulus provided during the crisis through an increase in income taxes is expected to be permanent, in the long run, there is a decrease in output ($\Omega_{Y_L,\tau^I_L} = -0.4$) and an increase in inflation ($\Omega_{\pi_S,\tau^I_S} = 0.1$). With a transitional state generated by an accommodative monetary policy kept after the crisis is over, the effect on medium-run output is initially negative, but it increases with b, getting positive if this transitional state is expected to last longer ($\Omega_{Y_M,\tau^I_L} \in [-0.3, 1.3]$). The expression for this multiplier is given by

$$\Omega_{Y_M,\tau^I_L} \equiv \frac{(1-b)\left[(1-\beta b)\,\Omega_{Y_L,\tau^I_L} + \sigma\Omega_{\pi_L,\tau^I_L}\right]}{\Gamma_{b\sigma}}$$

which shows that the medium-run output multiplier of long-run income taxes is a combination of long-run output and inflation multipliers (the same happens for the medium-run inflation multiplier). Thus, we see that as b increases, it puts more weight on the positive long-run inflation multiplier[31] (Ω_{π_L,τ^I_L}) than on the negative long-run output multiplier (Ω_{Y_L,τ^I_L}). The impact on medium-run inflation is positive and increasing in b ($\Omega_{\pi_M,\tau^I_L} \in [0.1, 0.2]$). Finally, the impact of a permanent increase in taxes on crisis-state output and inflation is positive and increasing in b ($\Omega_{Y_S,\tau^I_L} \in [1.6, 9.4]$ and $\Omega_{\pi_S,\tau^I_L} \in [0.2, 1.0]$, respectively).

We also observe that the short-run multipliers of long-run income taxes are a combination of medium- and long-run output and inflation multipliers. This can be seen in the expressions derived in the proofs of Propositions 1 and 2 in the Appendix and summarized in Tables A.1 and A.2. For example, the impact of long-run income taxes on short-run output is given by the expression

$$\Omega_{Y_S,\tau^I_L} \equiv \frac{(1-\mu)}{\Gamma_{\mu\sigma}} \left\{ \begin{array}{c} b\left[(1-\beta\mu)\,\Omega_{Y_M,\tau^I_L} + \sigma\Omega_{\pi_M,\tau^I_L}\right] + \\ (1-b)\left[(1-\beta\mu)\,\Omega_{Y_L,\tau^I_L} + \sigma\Omega_{\pi_L,\tau^I_L}\right] \end{array} \right\}$$

where $\Gamma_{\mu\sigma} \equiv (1-\mu)(1-\beta\mu) - \mu\kappa\sigma$.

This shows that the short-run multiplier is a weighted average of the medium- and long-run multipliers. In this specific case, it combines a negative multiplier ($\Omega_{Y_L,\tau^I_L} = -0.4$), a multiplier that is negative for low values of b, but positive for higher values ($\Omega_{Y_M,\tau^I_L} \in [-0.3, 1.3]$) and

[31] Initially, for $b \in [0, 0.13]$, we have that $(1-\beta b) > \sigma$, but for $b \geq 0.14$, we obtain $(1-\beta b) < \sigma$.

positive multipliers ($\Omega_{\pi_M,\tau_L^I} \in [0.1, 0.2]$ and $\Omega_{\pi_L,\tau_L^I} = 0.1$). Since $\sigma > (1-\beta\mu)$, we see that inflation multipliers have a larger weight on the short-run multiplier. Hence, although the permanent policy decreases output in the long run, and might also decrease it in the transitional state as well, its effect on generating more inflation in the future represents an improvement upon the recession observed during the crisis. However, this does not necessarily mean the economy is better off in terms of welfare. This will be discussed in the next section.

As it was the case with government spending, I need to point out that with a permanent increase in taxes, there will be a transitional state only if monetary policy accommodates the increase in taxes ($i_M = 0$ or i_M^*) after the crisis is over. If the monetary authority returns to rule (3.3), as soon as r_t^e returns to $\bar{r}$, there will be no transitional state and the economy jumps straight to the long run. In this case, short-run output and inflation are obtained by assuming $b = 0$ in Proposition 1 with multipliers at the lower bound of the intervals ($\Omega_{Y_S,\tau_L^I} = 1.6$ and $\Omega_{\pi_S,\tau_L^I} = 0.2$ for output and inflation, respectively).

It is important to note that these multipliers should be carefully interpreted since it is the interplay between policies that will determine the levels of output and inflation in each state. This is the topic of next section.

6 Analysis of Different Policies' Combinations

I now discuss how the interaction of policies carried out during the crisis, their implementation timing and expectations regarding future policies affect the depth of the recession caused by the shock that makes the zero lower bound binding. I look at their effects on output and inflation in each state and discuss their implications in terms of welfare losses.

Before analyzing the various cases, I point out to a result that, associated with those obtained in Propositions 1 and 2, is helpful to understand the mechanisms behind the effects obtained for output and inflation in each state. In the derivation of solution allocations in the proofs of these propositions, output and inflation are expressed as functions of the shock (r_S^e), the gap between steady-state interest rate and medium-run interest rate ($\bar{r} - i_M$) and the fiscal instruments in each state of the economy (G_S, τ_S^I, G_M, τ_M^I, G_L and τ_L^I). This is convenient because it will enable us to discuss the impact of each instrument on the levels of output and inflation. However, an intermediate step in the proof of Proposition 1 allows us to express short-run output and inflation as functions of output, inflation and government spending in the transitional state and the long run.[32]

[32]The relations presented in Propositions 1 and 2 are obtained by plugging the medium- and long-run levels of

$$Y_S = \frac{(1-\beta\mu)\,\sigma}{\Gamma_{\mu\sigma}} r_S^e + \frac{[(1-\beta\mu)(1-\mu)-\mu\kappa\psi]}{\Gamma_{\mu\sigma}} G_S + \frac{\mu\kappa\sigma\psi}{\Gamma_{\mu\sigma}} \chi^I \tau_S^I + \frac{(1-\mu)}{\Gamma_{\mu\sigma}} \left\{ \begin{array}{l} b\,[(1-\beta\mu)(Y_M - G_M) + \sigma\pi_M] + \\ (1-b)\,[(1-\beta\mu)(Y_L - G_L) + \sigma\pi_L] \end{array} \right\} \tag{6.1}$$

$$\pi_S = \frac{\kappa\sigma}{\Gamma_{\mu\sigma}} r_S^e + \frac{(1-\mu)\left(1-\psi\sigma^{-1}\right)\kappa}{\Gamma_{\mu\sigma}} G_S + \frac{(1-\mu)\,\kappa\psi}{\Gamma_{\mu\sigma}} \chi^I \tau_S^I + \frac{(1-\mu)}{\Gamma_{\mu\sigma}} \left\{ \begin{array}{l} b\,[\kappa(Y_M - G_M) + [\beta(1-\mu)+\kappa\sigma]\,\pi_M] + \\ (1-b)\,[\kappa(Y_L - G_L) + [\beta(1-\mu)+\kappa\sigma]\,\pi_L] \end{array} \right\} \tag{6.2}$$

where $\Gamma_{\mu\sigma} \equiv (1-\beta\mu)(1-\mu) - \mu\kappa\sigma > 0$.

The first line in equations (6.1) and (6.2) represents our benchmark case, where short-run output and inflation are affected by the shock and the fiscal stimulus provided during the crisis (G_S or τ_S^I). The second line shows how short-run allocations are affected by medium- and long-run output, inflation and government spending. This is a result of the expectation terms $E_t\hat{Y}_{t+1}$, $E_t\pi_{t+1}$ and $E_t\hat{G}_{t+1}$ in the (IS) and (AS) equations, (3.1) and (3.2), respectively. These relations are a weighted average between what happens in the medium and long runs, with weights defined by the probability b. This means that the longer the transitional state is expected to last (higher b), the larger its impact on short-run allocations will be. I should also point out that future inflation is relatively more important than future output in determining short-run allocations.[33]

I will also make use of the equations for medium-run output and inflation from Proposition 1, before substituting for the long-run allocations, which are similar to (6.1) and (6.2):

$$Y_M = \frac{1}{\Gamma_{b\sigma}} \left\{ \begin{array}{l} (1-\beta b)\,\sigma(\bar{r} - i_M) + [(1-\beta b)(1-b) - b\kappa\psi]\,G_M + b\kappa\sigma\psi\chi^I\tau_M^I + \\ (1-b)\,[(1-\beta b)(Y_L - G_L) + \sigma\pi_L] \end{array} \right\} \tag{6.3}$$

$$\pi_M = \frac{(1-b)}{\Gamma_{b\sigma}} \left\{ \begin{array}{l} \frac{\kappa\sigma}{(1-b)}(\bar{r} - i_M) + \left(1-\psi\sigma^{-1}\right)\kappa G_M + \kappa\psi\chi^I\tau_M^I + \\ \kappa(Y_L - G_L) + [\beta(1-b)+\kappa\sigma]\,\pi_L \end{array} \right\} \tag{6.4}$$

where $\Gamma_{b\sigma} \equiv (1-\beta b)(1-b) - b\kappa\sigma > 0$.

In the transitional state, the gap between long- and medium-run interest rates $(\bar{r} - i_M)$ and medium-run fiscal stimulus (G_M or τ_M^I) determine output and inflation. But if the fiscal policy implemented is expected to be permanent, medium-run output and inflation will also be

output and inflation (Y_M, π_M, Y_L, π_L) in these intermediate step equations.

[33] Under the parameterization assumed, $\sigma = 0.86 > (1-\beta\mu) = 0.10$ and $[\beta(1-\mu)+\kappa\sigma] = 0.1041 > \kappa = 0.0086$.

impacted by the long-run gap between output and government spending ($Y_L - G_L$) and, even more, by long-run inflation.[34]

Table 6.1 summarizes the results for short and medium-run output and inflation and the associated welfare losses for all the cases analyzed in the paper, with stimulus provided either through government spending or income taxes. In those cases for which the solutions depend on the value of probability b (Cases $(C) - (J)$, (L), (M), (O) and (P)), the table presents the solution associated with the maximum value of this probability for which there is a unique determined solution ($b_{\max}$). I will discuss how these solutions vary with b.

Table 6.1: Short and Medium-Run Output and Inflation and Total Welfare Losses

	G Policy						τ^I Policy					
	Y_S	π_S	Y_M	π_M	Welfare Losses	$b_{\max}$	Y_S	π_S	Y_M	π_M	Welfare Losses	$b_{\max}$
(A)	−29.9	−9.9			9.9		−29.9	−9.9			9.9	
(B)	−18.5	−6.9			4.8		−24.8	−7.5			5.7	
(C)	−25.7	−9.2	2.4	0.1	8.5	0.90	−22.0	−6.1	−0.4	0.6	3.8	0.90
(D)	−4.5	−1.6	3.2	0.8	0.3	0.88	−7.1	−0.7	3.9	1.1	0.1	0.89
(E)	−2.9	−0.7	8.0	1.3	0.1	0.85	−5.4	0.1	3.8	1.5	0.1	0.87
(F)	−4.5	−1.6	3.2	0.8	0.3	0.88	−5.7	0.0	3.8	1.3	0.1	0.90
(G)	−2.9	−0.7	8.0	1.3	0.1	0.85	−6.7	0.1	2.7	1.7	0.2	0.90
(H)	−37.1	−12.1	2.4	0.1	14.8	0.90	−27.1	−8.5	−0.4	0.6	7.3	0.90
(I)	−3.9	0.4	9.9	2.0	0.3	0.88	−6.3	−0.7	4.5	1.8	0.3	0.88
(J)	−6.0	−0.2	8.7	2.2	0.4	0.90	−5.7	−0.2	3.9	2.1	0.4	0.90
(K)	−42.0	−16.0			38.1		−16.9	−2.9			15.5	
(L)	−65.2	−23.9	−2.7	−3.5	70.6	0.90	−6.3	1.0	1.2	2.6	14.8	0.77
(M)	−118.9	−40.1	−13.4	−6.9	177.9	0.90	−6.3	1.0	1.2	2.6	14.8	0.77
(N)	−53.5	−18.8			47.9		−22.0	−5.4			17.6	
(O)	−76.6	−26.5	−2.7	−3.5	83.7	0.90	−5.7	0.7	2.4	2.9	14.8	0.82
(P)	−122.4	−40.1	−11.8	−6.4	177.4	0.90	−5.7	0.7	2.4	2.9	14.8	0.82

(A) No fiscal or monetary policy
(B) Fiscal stimulus during the crisis state
(C) Fiscal stimulus kept in the medium run, $i_M = r_t^e + \phi_\pi \pi_t + \phi_y Y_t$
(D) Fiscal stimulus in the crisis state only, $i_M = 0$
(E) Fiscal stimulus kept in the medium run, $i_M = 0$
(F) Fiscal stimulus in the crisis state only, $i_M = i_M^*$
(G) Fiscal stimulus kept in the medium run, $i_M = i_M^*$
(H) Fiscal stimulus implemented after the crisis, $i_M = r_t^e + \phi_\pi \pi_t + \phi_y Y_t$
(I) Fiscal stimulus implemented after the crisis, $i_M = 0$
(J) Fiscal stimulus implemented after the crisis, $i_M = i_M^*$
(K) Permanent fiscal stimulus, $i_M = r_t^e + \phi_\pi \pi_t + \phi_y Y_t$
(L) Permanent fiscal stimulus, $i_M = 0$
(M) Permanent fiscal stimulus, $i_M = i_M^*$
(N) Permanent fiscal stimulus implemented after the crisis, $i_M = r_t^e + \phi_\pi \pi_t + \phi_y Y_t$
(O) Permanent fiscal stimulus implemented after the crisis, $i_M = 0$
(P) Permanent fiscal stimulus implemented after the crisis, $i_M = i_M^*$
Note: b_{max} represents the maximum value of probability b for which there is a determined solution in the respective case. The allocations presented in this table, that depend on the value of b, are computed using the respective b_{max}.

In the rest of this section, I first analyze all the cases assuming the implementation of fiscal stimulus during the crisis through increases in government spending. In what follows,

[34] The coefficient $[\beta(1-b) + \kappa\sigma] \in [0.99, 0.11]$ for $b \in [0, 0.90]$, while $\kappa = 0.0086$.

I discuss the stimulus carried out through increases in income taxes. I assume that once a fiscal instrument is chosen during the crisis, the government sticks to it in every following state. This rules out switching between spending and taxes which, although interesting and realistic, complicate unnecessarily the analysis. The list of possible combinations is not exhaustive but allows us to explore the mechanism and understand some appealing results.

6.1 Impact of Government Spending Stimulus

In this first part of the analysis, I assume that when a shock hits the economy, there is a 5% increase in government spending.[35] I will show that this stimulus is not sufficient to fully stabilize output and inflation during the crisis, but it is still large enough to reduce the fall in short-run output and inflation significantly. I will not pin down the optimal level of government spending increases in response to the crisis. However, Woodford (2011) observes that, in this context, *"it is not optimal to fully stabilize inflation and the output gap, despite the feasibility of doing so, because of the inefficient composition of expenditure that this would involve."* Since I am focusing the discussion on the comparison of different policies' combinations, the value assumed for the rise in government spending seems reasonable.

In what follows, it becomes clear how the impact of this policy on short-run output and inflation depends not only on the size of the stimulus provided while the zero lower bound is still binding but also on what agents expect regarding the interaction between monetary and fiscal policies once the crisis is over. Expectations about the timing of implementation and termination of the fiscal stimulus are also crucial in determining the depth of the recession.

To begin with, I assume that the increase in government spending is expected to be temporary (Cases $(A) - (J)$), but I will also discuss what happens if it is supposed to be permanent (Cases $(K) - (P)$). In all the cases, I assume the economy is hit by a shock that makes the zero lower bound binding in the short run. Case (A) represents the fall in output and inflation (30% and 10% respectively, according to the parameterization discussed above) that occurs if no stimulus policy is implemented in response to the crisis.

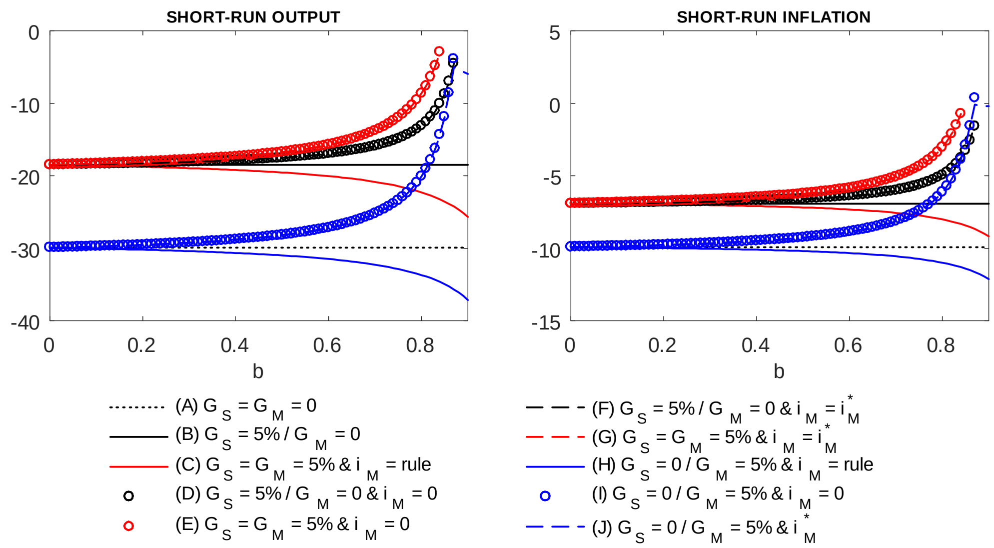
I consider (B) the benchmark case in our analysis. This is the case analyzed in Eggertsson (2011b), and that is much discussed in the literature. It assumes a perfect timing between the fiscal stimulus and the shock, meaning that the fiscal instrument returns to its pre-crisis level as soon as r_t^e returns to $\bar{r}$, while the monetary authority goes back to following rule (3.3) right away. This stimulus represents a significant improvement upon the recession caused by

[35] According to the IMF Fiscal Monitor, this was approximately the increase in government expenditure in the US in 2009.

the shock (Case (A)). The fall in output goes from 30% to 19%, while inflation decreases 7%, instead of 10%. Using the welfare loss function derived in Proposition 3, I obtain that the stimulus provided during the crisis takes the economy from losses of 9.9% in Case (A), to 4.8% in Case (B).

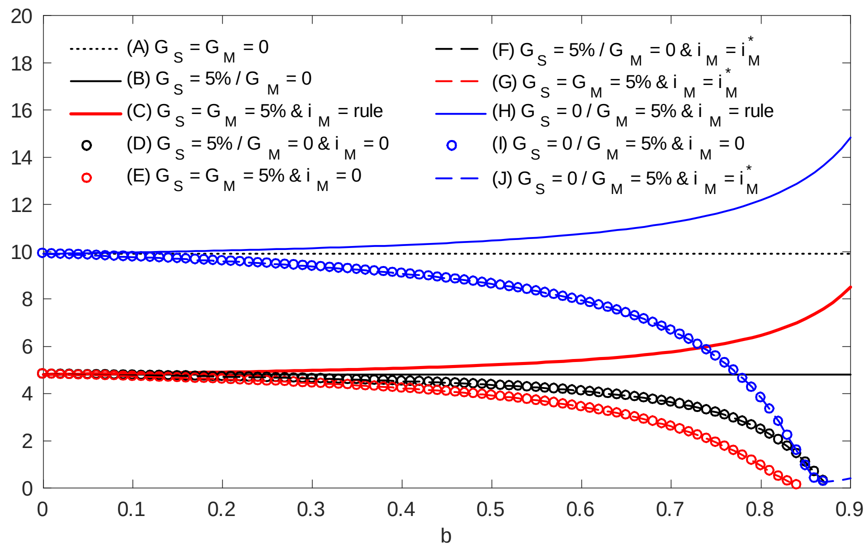
We note that the termination timing of this stimulus is crucial. If the rise in government spending is kept after the crisis is over, with monetary policy going back to rule (3.3) as soon as r_t^e returns to $\bar{r}$ (Case (C)), the decrease in short-run output and inflation will be larger than in Case (B). Figure 6.1 shows how the solutions depend on the probability associated with the transitional state (b). The longer it is expected to last i.e., the longer the government spending stimulus is kept after the crisis is over, deeper will be the fall in output and inflation during the crisis. This happens because an increase in government spending in the transitional state, without monetary policy accommodation, has a negative impact on crisis-state output and inflation. In this case, keeping the fiscal stimulus, when the zero lower bound is no longer binding, creates a transitional state with higher output and inflation. However, the medium-run output multiplier smaller than one $(\Omega^T_{Y_M,G_M} \in [0.90, 0.47])$ implies an increase in activity smaller than the increment in government spending. This makes the term $(Y_M - G_M)$ in (6.1) and (6.2) negative and large enough to cancel out the positive effect coming from higher medium-run inflation.

Figure 6.1: Short-Run Impact of Temporary Government Spending Stimulus



Although getting the timing of returning the fiscal instrument to its pre-crisis level wrong is harmful to the economy, it is still better than not providing any stimulus (Case (A)). One can also see in terms of welfare losses, which are more significant than in Case (B), but still smaller than in Case (A), even for the highest value of b (See Figure 6.2). It is important to observe that welfare losses depend not only on output and inflation in the crisis state but also on their medium- and long-run levels, as well as on the level of government spending in each state (see equation (4.17)).

Figure 6.2: Welfare Losses - Temporary Government Spending Stimulus



If the transitional state is created by agents' expectations that the monetary authority will keep the nominal interest rate at zero after the crisis is over, with government spending going back to its pre-crisis level as soon as r_t^e returns to $\bar{r}$ (Case (D)), the decrease in short-run output and inflation is reduced. This effect is stronger for a longer expected transitional state. Keeping the interest rate at $i_M = 0$ increases output and inflation in the transitional state, with a positive direct effect on short-run output and inflation as can be seen in equations (6.1) and (6.2). Note that, in this case, the sequence of black circles in Figure 6.1 stops before the highest value assumed for $b = 0.9$. This happens because for $b > 0.88$ the nominal interest rate implied by rule (3.3) in the short run is positive and the zero lower bound is no longer binding. In fact, for these values of b, Condition $(C4)$ is violated; thus the results from Proposition 1 do not apply. Then, I disregard these values.

We see that the solution in Case (D) coincides with the case in which medium-run interest rate is optimally chosen (Case (F)). This is the case since the optimal transitional-state interest rate problem in Proposition 4 gives precisely $i_M^* = 0$, until $b = 0.88$. But it also implies that Condition $(C4)$ is no longer satisfied for $b > 0.88$. Considering welfare, keeping an accommodative monetary policy, even when the zero lower bound is no longer binding, represents a significant improvement upon Case (B), where the authority goes back to following rule (3.3) as soon as the crisis is over, getting close to zero for higher values of b (see Figure 6.2).

In the cases where there is an association of the medium-run accommodative monetary policy with a continuation of the fiscal stimulus after the crisis is over (Cases (E) and (G)), short-run output and inflation decrease less than when the monetary authority provides the stimulus in isolation. In these cases, Condition $(C4)$ is no longer satisfied for $b > 0.85$. We see that with monetary accommodation of the fiscal stimulus in the transitional state, output and inflation increase significantly in the medium run compared to Case (C), where the monetary authority goes back to rule (3.3) as soon as the crisis is over. This happens because monetary accommodation causes the medium-run multipliers to increase ($\Omega_{Y_M,G_M} \in [1, 2.1]$ and $\Omega_{\pi_M,G_M} \in [0.0, 0.1]$), allowing output to increase more than one-for-one with the increase in government spending. Thus, the term $(Y_M - G_M)$ in (6.1) and (6.2) will be positive and increasing in b, which comes along with higher inflation (π_M) as well, undermining the decrease in output and inflation during the crisis. We see in Figure 6.2 that this would imply the lowest losses for the economy, getting close to zero for $b = 0.85$.

Regarding the implementation timing of the fiscal stimulus, I analyze an increase in government spending implemented when the zero lower bound is no longer binding. Woodford (2011) points out that, even if there is an implementation delay, as long as the eventual increase in government spending occurs during the financial disruption, there will be a positive effect on short-run output. If most of the stimulus is carried out in a post-crisis environment, without accommodation by the monetary authority, it will be ineffective, or even counter-productive.

I study the extreme case in which the rise in government spending only occurs after the crisis is over. This will also generate a transitional state where the impacts on output and inflation depend on the monetary authority's response. If it goes back to rule (3.3) as soon as r_t^e returns to $\bar{r}$ (Case (H)), medium-run output and inflation increase, but the multiplier on output is smaller than one and decreasing in b ($\Omega^T_{Y_M,G_M} \in [0.9, 0.5]$). This implies that the term $(Y_M - G_M)$ in short-run solution equations (6.1) and (6.2) is negative and significant enough to cancel the positive impact of higher inflation (π_M). In this case, we do not have the positive effect of the stimulus provided in the short run (Case (C)), thus the expectation of an implementation delay makes the economy even worse than if there was no stimulus at all (Case (A)). This is also reflected in welfare losses which grows the longer the transitional state is expected to last.

Nonetheless, if the fiscal stimulus is supposed to be carried out after the crisis is over, but agents also assume it to be associated with an accommodative monetary policy ($i_M = 0$ or i^*_M)

(Cases (I) and (J)),[36] transitional-state output and inflation increase more than in Case (H). In fact, the monetary accommodation makes the medium-run output multiplier larger than one ($\Omega_{Y_M,G_M} \in [1.0, 2.1]$), causing $(Y_M - G_M)$ to be positive and increasing in b. There is still no stimulus provided in the short run, but there is an improvement upon not providing any stimulus at all (Case (A)).

It is interesting to note that, for most values of b, delaying the increase in government spending for after the crisis is over, even with monetary policy accommodation, is worse than providing the stimulus only during the crisis. Nevertheless, when the transitional state is expected to be long (higher b), the decrease in short-run output ($b \geq 0.83$) and inflation ($b \geq 0.79$) can get smaller than in the benchmark (Case (B)). They are close to those observed when there is a transitional state generated by an accommodative policy with fiscal stimulus only during the crisis (Cases (D) and (F)). The key for this result is that with a more extended transitional state, medium-run inflation is expected to be high, which helps to reduce the depth of the recession. In terms of welfare losses, Figure 6.2 shows the improvement of Cases (I) and (J) upon having no stimulus at all (Case (A)). With a longer transitional state ($b \geq 0.79$), losses get smaller than in the benchmark case (Case (B)) and close to the cases with short-run fiscal stimulus and medium-run accommodative monetary policy.

Summarizing, the timing of implementation of a fiscal stimulus through increases in government spending, and expectations regarding its termination, are fundamental in determining the depth of the recession caused by a shock that makes the zero lower bound binding. It is crucial to the fiscal authority to be able to respond quickly when the shock hits the economy. But it is also essential for both monetary and fiscal authorities to coordinate their policies after the crisis is over and, above all, to be able to signal them correctly, so agents incorporate this into their expectations, helping to mitigate the decrease observed in output and inflation and the associated welfare losses. Besides, it is critical that agents understand that the policies implemented are supposed to be temporary. If they expect the increase in government spending to be permanent, the results are disastrous. I discuss this in the next subsection.

6.1.1 Permanent Government Spending Stimulus

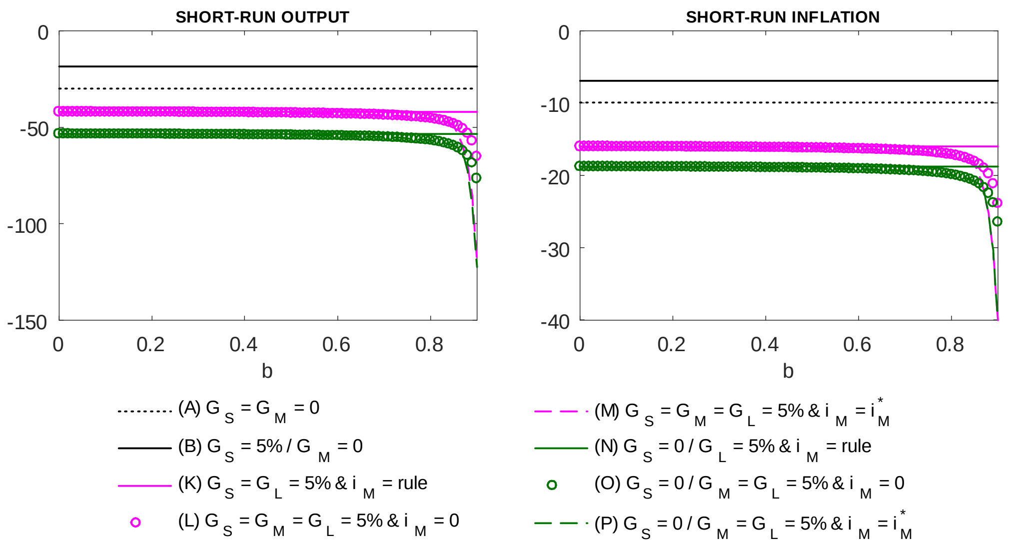
As we saw in the discussion so far, agents' expectation regarding future policies are crucial in determining the levels of output and inflation during the crisis. It is important to the fiscal

[36] Note that in Case (I), Condition ($C4$) is no longer satisfied for $b > 0.88$. However, for Case (J), it is satisfied until $b = 0.90$, with $i^*_M = 0$ until $b = 0.88$ and $i^*_M > 0$ in the higher end of the interval. This explains the inflection observed in short- and medium-run solutions and in welfare losses.

authority to make clear that the stimulus provided is supposed to be temporary, even if there is an implementation delay. Table 6.1 shows short-run output and inflation and the associated welfare losses for Cases (K) – (P) for which the fiscal stimulus is expected to be permanent. Cases (K) – (M) assume that the increase in government spending occurs during the crisis, while Cases (N) – (P) assume it is expected to be implemented after the crisis is over. They differ on expectations regarding the monetary authority's behavior in the transitional state.

Figure 6.3 shows what happens to short-run output and inflation in these cases. If the stimulus carried out during the crisis is expected to be permanent, with monetary policy going back to rule (3.3) as soon as r_t^e returns to $\bar{r}$ (Case (K)), crisis-state output and inflation decrease more (42% and 16%, respectively) than if no stimulus was provided at all (Case (A)). In the long run, output increases much less than one-for-one with government spending ($\Omega_{Y_L,G_L} = 0.4$), making the term $(Y_L - G_L)$ negative, while inflation decreases with a permanent increase in government spending ($\Omega_{\pi_L,G_L} = -0.1$). Hence, both relevant terms in (6.1) and (6.2) are negative, canceling the positive effect of the short-run stimulus and exacerbating the depth of the crisis. Comparing with the case without stimulus (Case (A)), this policy almost quadruples the losses in terms of welfare, as can be seen in Table 6.1.

Figure 6.3: Short-Run Impact of Permanent Government Spending Stimulus



Even if the monetary authority decides to temporarily keep an accommodative monetary policy ($i_M = 0$ or i_M^*), when the zero lower bound is no longer binding (Cases (L) and (M)), the expectation of a permanent increase in government spending dominates the effect on short-run output and inflation. In fact, if the transitional state created by an accommodative monetary policy is expected to last longer (higher b), output and inflation can fall even more than if it returns to rule (3.3) right after the crisis is over (Case (K)).

This happens because expectations of a sharp deflation, in the long run, have a large negative effect on transitional-state output and inflation (see equations (6.3) and (6.4)). Also, since output is expected to increase less than one-for-one with government spending, the term $(Y_L - G_L)$ is negative. The effect of this term, associated with long-run deflation, overcomes the effect of higher government spending accommodated by the monetary authority in the transitional state. It is bad for the economy to keep an accommodative policy after the crisis is over if low output growth and deflation are expected in the long run.[37]

This is also reflected in crisis-state output and inflation, as can be seen in Figure 6.3. With a small duration expected for the transitional state, the effects of long-run deflation and low output growth dominate the marked decrease in output and inflation during the crisis, overcoming the positive impact of the short-run stimulus. However, with a long transitional state under an accommodative policy, the decreases in medium-run output and inflation remarkably exacerbate the drop in short-run output and inflation. In both cases, the welfare losses are remarkable, especially when optimal medium-run interest rate gets positive.

Cases $(N)-(P)$ repeat this analysis assuming the fiscal stimulus is implemented when the crisis is over but is expected to be permanent. Since short-run allocations do not affect the following states, medium- and long-run output and inflation are the same as in Cases $(K)-(M)$. The only difference occurs when the medium-run interest rate is optimally chosen (Cases (M) and (P)), making medium-run output and inflation slightly different.[38] In the short run, the economy is worse off since there is no stimulus during the crisis to counteract the effect of the shock and further decreases in output and inflation are expected in future states. Consequently, these are the cases where welfare losses are more pronounced.

Summing up, it is crucial that the fiscal authority correctly signals that an increase in government spending provided in response to a shock that makes the zero lower bound binding is supposed to be temporary. If agents expect this increase to be permanent, it remarkably exacerbates the depth of the recession. It can get even worse if the monetary authority extends the accommodative monetary policy after r_t^e returns to $\bar{r}$ and/or if the fiscal measure is expected to be implemented with a delay.

[37] Note that for both Cases (L) and (M), Condition $(C4)$ is satisfied $\forall b \in [0, 0.90]$. In Case (M), for $b < 0.86$, $i_M^* = 0$. For higher values of b, i_M^* gets positive, but not large enough to imply the violation of Condition $(C4)$.

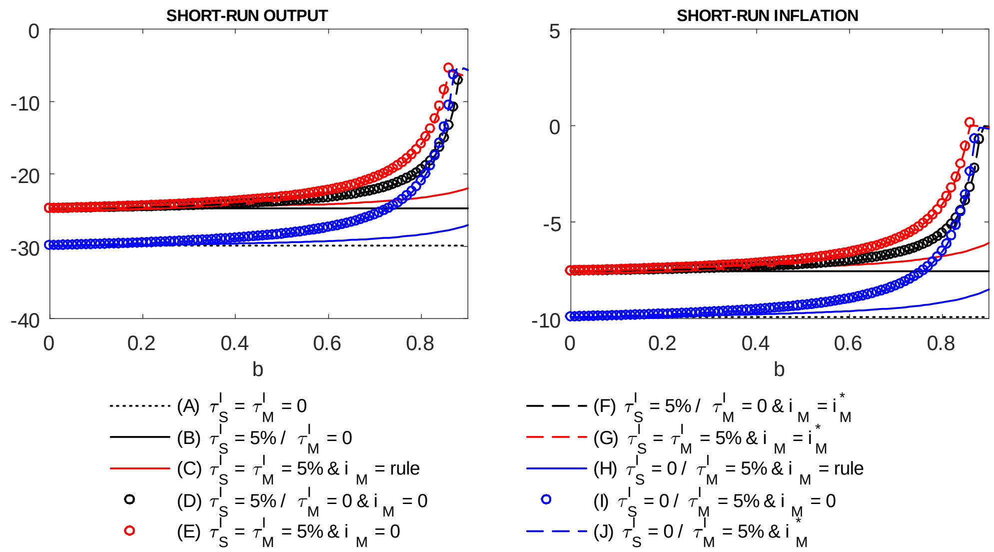
[38] It is important to note that, according to Proposition 4, i_M^* does not depend on long-run values of ouptut and inflation. It depends on the shock and the policy instruments used in the short and medium runs. That is why the solution is different with the implementation delay because instruments are not used in the short run. But note that since for most values of b, the restriction $i_M \geq 0$ is binding, medium-run allocations only differ in these cases for higher values of b, when this restriction is no longer binding.

6.2 Impact of Income Taxes Stimulus

I repeat the discussion using income taxes as the fiscal instrument to stimulate the economy, also assuming a 5% increase in taxes during the crisis.[39] Table 6.1 summarizes the results for short-run and medium-run output and inflation and the associated welfare losses for all the cases included in the analysis. In a nutshell, an increase in income taxes is stimulative during the crisis and it continues boosting the economy as long as monetary policy keeps being accommodative. An implementation delay is not as harmful as in the case of government spending stimulus, and it can still be beneficial if expected to be associated with an accommodative monetary policy. Finally, even if the stimulus is supposed to be permanent, it can mitigate the drop in short-run output and inflation, but the long-run impact is negative, implying more considerable welfare losses. I analyze first those cases in which the increase in income taxes is expected to be temporary. I later discuss the impact of an expansion expected to be permanent.

Figure 6.4 presents short-run output and inflation for Cases $(A) - (J)$. As discussed above regarding the multipliers, an increase in taxes during the crisis (Case (B)) has a positive effect, reducing the fall in output to 25% and inflation to 7.5%. We observe that this stimulus is not as powerful as the one provided by government spending, but it helps to reduce the depth of the recession. It also lowers welfare losses caused by the crisis from 9.9% to 5.7%.

Figure 6.4: Short-Run Impact of Temporary Income Tax Stimulus

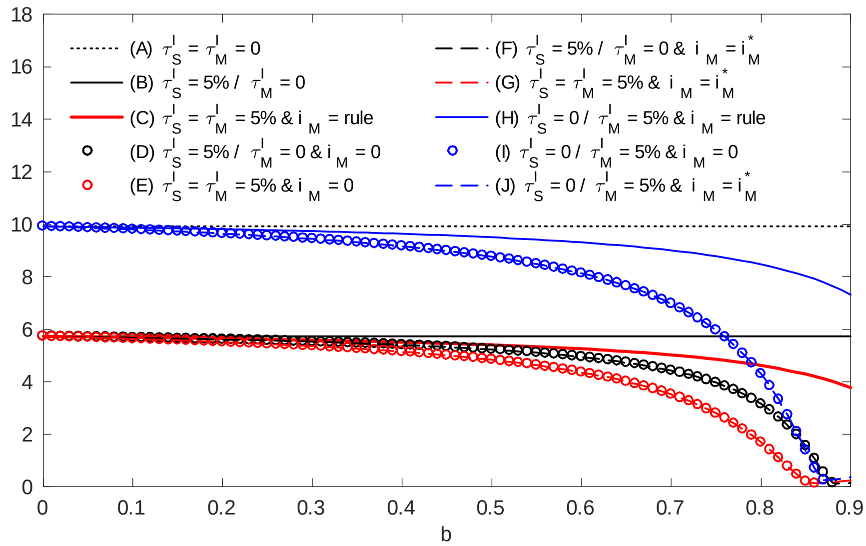


However, in contrast to government spending, if the fiscal authority temporarily keeps in-

[39] In fact, according to the Fiscal Monitor, 2009 data for the US shows there was a drop of 2% in government revenues as a percent of GDP. We assume a 5% increase in income taxes to make the results comparable to those obtained with government spending stimulus and discuss the impact of a positive short-run output multiplier for income taxes.

come taxes higher after the crisis is over (Case (C)), short-run output and inflation will decrease less than if the timing was perfect relative to the crisis. In this case, the recession is smaller the longer the transitional state is expected to last (higher b). To understand this result, I analyze the transitional state created by keeping income taxes higher after the crisis is over. Given that monetary policy returns to rule (3.3) as soon as r_t^e returns to $\bar{r}$, medium-run output decreases ($\Omega^T_{Y_M,\tau^I_M} \in [0.0, -0.1]^\dagger$), while inflation increases ($\Omega^T_{Y_M,\tau^I_M} \in [0.004, 0.003]$). Using equations (6.1) and (6.2), we see that medium-run inflation is more relevant than output in determining the short-run allocations. Therefore, even with an expected small decrease in activity after the crisis is over, the fact that this policy generates expectations of higher inflation when the zero lower bound is no longer binding stimulates the economy in the short run. This reduces welfare losses, compared with the case where income taxes return to their pre-crisis level as soon as r_t^e returns to $\bar{r}$ (Case (B)). The losses in Case (C) are smaller the longer the transitional state is expected to last (See Figure 6.5).

Figure 6.5: Welfare Losses - Temporary Income Tax Stimulus



Instead, if the fiscal stimulus is dropped as soon as the crisis is over, but monetary policy can stimulate the economy keeping the transitional-state nominal interest rate at $i_M = 0$ (Case (D)) or setting it optimally at i^*_M (Case (F)), the longer the transitional state generated by this policy is expected to last, smaller will be the fall in output and inflation. Again, for most values of b, the optimal medium-run interest rate is $i^*_M = 0$. It only becomes positive for $b > 0.89$, with Condition ($C4$) still satisfied. For Case (D), ($C4$) no longer holds for $b > 0.89$. In the transitional state, keeping the nominal interest rate at zero increases output and inflation through the positive impact of the gap between steady-state and medium-run nominal interest rate $(\bar{r} - i_M)$ (See equations (6.3) and (6.4)), which is increasing while $i_M^* = 0$. Once the optimal medium-run interest rate gets positive, transitional-state output and inflation slightly decrease, affecting the short-run allocations as well. Under the expectation of both output and inflation being higher after the crisis is over, the economy gets a boost in the short run, reducing the associated welfare losses even more.

The recession will be further mitigated if the government associates a medium-run accommodative monetary policy with a continuation of the fiscal stimulus (Cases (E) and (G)). Besides getting the effect of the positive interest rate gap $(\bar{r} - i_M)$, this policy extends the period with a positive output and inflation multiplier ($\Omega_{Y_M,\tau_M^I} \in [0, 0.9]$ and $\Omega_{Y_M,\tau_M^I} \in [0, 0.1]$, respectively), boosting the economy in the transitional state. Note that Condition $(C4)$ is no longer satisfied for $b > 0.87$ in Case (E), which is also the value of b for which optimal medium-run interest rate becomes positive. The stimulative effect of higher income taxes after the crisis is over is exacerbated by the accommodative monetary policy, further mitigating the decrease in short-run output and inflation and, consequently, attenuating the associated welfare losses.

I also analyze what happens if the effective rise in taxes is expected to be delayed, with its implementation only occurring when the zero lower bound is no longer binding (Cases $(H)-(J)$). In the case of taxes, this is even more likely than with government spending, which is usually able to respond quicker. Again, the impact depends on how the monetary authority responds to this policy. If it goes back to the rule (3.3) as soon as the crisis is over (Case (H)), it can still undermine the depth of the recession because, although it decreases output in the medium run, it is able to create expectations of higher inflation after r_t^e returns to $\bar{r}$. Even though the decrease of the short-run output and inflation is higher than if the policy was implemented during the crisis (Case (B)), it is still better than not implementing any policy at all (Case (A)). This contrasts with what happens under a government spending stimulus where the delayed policy, under a non-accommodative monetary policy, is worse than a total absence of fiscal policy. Delayed implementation of an increase in taxes can reduce the welfare losses compared to Case (A).

However, associating the delayed rise in income taxes with an accommodative monetary policy ($i_M = 0$ or i_M^*) (Cases (I) and (J)), medium-run output and inflation increase (Note that in Case (I), Condition $(C4)$ is no longer satisfied for $b > 0.88$). For low values of b, it is still worse than the timely implementation because it does not count with the stimulative effect of the policy in the short run. But, for $b \geq 0.74$, the fall in output is smaller than in the benchmark case. The same happens to inflation for $b \geq 0.77$. We observe that short-run allocations get close to those found in Cases $(D) - (G)$ for levels of b close to the top of the interval. The same happens to welfare losses, which are more significant than in the benchmark case but get smaller for $b \geq 0.77$.

To sum up, besides being stimulative during the crisis, an increase in income taxes that continues once the zero lower bound is no longer binding can boost the economy more, especially under an accommodative monetary policy. Even if there is an implementation delay, it is still better than not providing any policy at all, particularly under monetary policy accommodation with the transitional state expected to last longer. Therefore, independently of the fiscal instrument used to stimulate the economy, it is crucial for monetary and fiscal authorities to be able to coordinate and correctly signal their policies, not only during the crisis but in future periods as well. It is also crucial that agents comprehend that the policies carried out are supposed to be temporary. A permanent increase in income taxes can also have deceiving results as I will discuss next.

6.2.1 Permanent Income Taxes Stimulus

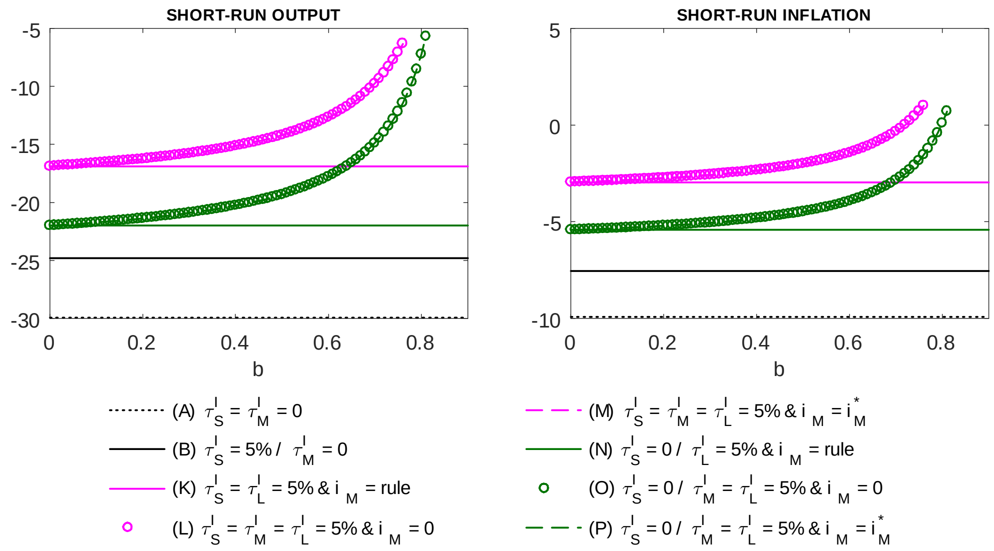
I examine the impact of an increase in income taxes which is expected to be permanent. While Cases $(K) - (M)$ assume the rise in income taxes to be carried out during the crisis, Cases $(N) - (P)$ suppose it only occurs when the zero lower bound is no longer binding. They also differ on the expected behavior of the monetary authority once the crisis is over.

Figure 6.6 presents the behavior of short-run output and inflation in each case. If the rise in income taxes that occurs in response to the crisis is expected to be permanent, with monetary policy expected to return to the rule (3.3) as soon as r_t^e returns to $\bar{r}$ (Case (K)), there is indeed a smaller decrease in short-run output and inflation, compared to the temporary benchmark Case (B). This occurs because inflation increases, even though long-run output decreases. Significant higher inflation in the future has a crucial stimulative impact on crisis-state allocations as can be seen from equations (6.1) and (6.2). However, welfare losses get dominated by long-run output depression and higher inflation. It remarkably increases to 15.5%, thus higher than if no policy was implemented 9.92% (Case (A)).

It is crucial to highlight that assuming the monetary authority keeps responding through rule (3.3), unmodified under permanent changes in fiscal instruments, is critical for the results obtained. If I assumed that it follows a strict zero inflation target, it would be allowed to adjust the nominal interest rate to avoid this spike in long-run inflation, although we would still have a decrease in long-run output. This would have a negative effect on short-run allocations.

In those cases where one associates the permanent rise in income taxes with an accommodative monetary policy in the transitional-state (Cases (L) and (M), with $i_M = 0$ or i_M^*, respectively), output and inflation indeed decrease less in the short run, compared to the case with non-accommodative response from the monetary authority (Case (K)). If this transitional state is expected to last longer, the decrease in short-run output and inflation get even smaller (note that for both Cases (L) and (M) Condition $(C4)$ does not hold for $b > 0.77$).

Figure 6.6: Short-Run Impact of Permanent Income Taxes Stimulus



If there is an implementation delay and the increase in taxes is only carried out when the zero lower bound is no longer binding (Cases $(N) - (P)$), medium- and long-run allocations are the same as in Cases $(K) - (M)$, since they do not depend on short-run allocations. The only difference is that with an accommodative monetary policy (Cases (O) and (P)), Condition $(C4)$ does not hold for $b > 0.82$. If the effective rise in taxes only occurs after the crisis is over, output and inflation do not count with the stimulative effect of higher income taxes in the short run. They only decrease less than in the benchmark Case (B) because of the expectation of higher inflation in the future. But we still get the highest welfare losses among all the cases since the short-run recession and deflation associated with the long-run recession and inflation take the economy far away from the stability that would imply no losses. It is interesting to observe that, if a transitional state generated by an accommodative monetary policy (Cases (O) and (P)) is expected to have a long duration, short-run output and inflation get larger than those observed with a permanent rise in taxes carried out during the crisis (Case (K)). This happens with $b \geq 0.65$ for output and $b \geq 0.71$ for inflation, reducing the associated welfare losses, which get close to those observed in Cases (L) and (M).

Summarizing, as remarked when government spending was the instrument being discussed, it is critical to the fiscal authority to signal that a response to the crisis through an increase in income taxes is supposed to be temporary. Although mitigating the drop in output and inflation during the crisis state, an expected permanent rise in taxes implies significant losses in terms of welfare, even with an accommodative monetary policy stance after the crisis is over. Losses are deeper if this policy is expected to be permanent and carried out with a delay.

7 Conclusion

This work discussed the crucial role expectations regarding future policies play in determining the depth of a crisis produced by a shock that causes the nominal interest rate to reach the zero lower bound. It showed that when analyzing the impact of a fiscal stimulus provided during the crisis, we need to go beyond looking at short-run multipliers. When the stimulus is expected to be implemented and terminated, and how it is expected to be accommodated by the monetary authority when the zero lower bound ceases to bind, is determinant to evaluate its effects on output and inflation during the crisis. It also showed that a thoughtful evaluation of a fiscal stimulus regarding the implied welfare losses should account not only for the effects of policies on short-run output and inflation but also for the present discounted value of future output and inflation as well.

In such a severe downturn, monetary and fiscal policies will be more expansionary if they last longer than the shock spell and coordinate their moves. This coordination is necessary not only during the crisis but also regarding the commitment to future policies. Another crucial aspect for stimulating the economy is that the fiscal authority needs to be able to respond quickly to the shock, minimizing implementation delays and correctly signaling the duration of the stimulus.

The most successful combinations of policies are those in which the fiscal stimulus is expected to be temporary and carried out with minimum delay once the crisis hits the economy. If kept when the shock is no longer causing the zero lower bound to bind, the fiscal policy only improves overall welfare if expected to be associated with an accommodative monetary policy. This combination of policies can create expectations of higher inflation when the crisis is over and, even if small decreases in output are expected, generate positive impacts on economic activity during the crisis, implying in lower welfare losses.

We also saw that an increase in income taxes stimulates the economy when the zero lower bound is binding. However, this instrument's impacts are smaller than those obtained with a stimulus provided through increases in government spending. Besides, keeping higher income taxes after the crisis is over, even without monetary policy accommodation, contributes to improving the economy, in contrast to what we observe with government spending. This means that a mistake in ending the stimulus implemented through increases in income taxes is not as harmful as the one in the timing of reverting an increase in government spending. This is also valid for the implementation timing of policies. If the stimulus is carried out through increases in government spending that are expected to occur only when the crisis is over, without any accommodation from monetary policy, it is better not to implement any policy at all. Delaying the increase in taxes until the shock is no longer causing the zero lower bound to bind can still reduce the depth of the crisis. In this case, it is even better than not implementing any policy at all.

A critical aspect of the analysis is that the trustworthy coordination between monetary and fiscal policy authorities is crucial in determining the outcome of expectations regarding future policies. It is important to highlight that I assumed agents to believe that both authorities can commit to the announced policies, once the zero lower bound is no longer binding. I did not discuss time consistency issues, but we understand that monetary and fiscal authorities may not have the incentive to stick to stimulative policies once the crisis is over, especially under such rare event which is not expected to occur in a considerable time horizon.

Another explicit limitation of the analysis presented is that it does not account for the effects of these policies on debt dynamics. It would be interesting to relax the Ricardian equivalence hypothesis and discuss how expected future fiscal policy adjustments, necessary to balance the government budget constraint, would impact the crisis-state allocations.

Other avenues for future research should include investigating how much the results depend on the structure of the model and the parameters adopted in the calibration, enriching the discussion of other fiscal policy instruments and improving the probability structure of the model to account for the possibility of the economy hitting the zero lower bound again in future periods, besides allowing for a longer crisis spell.

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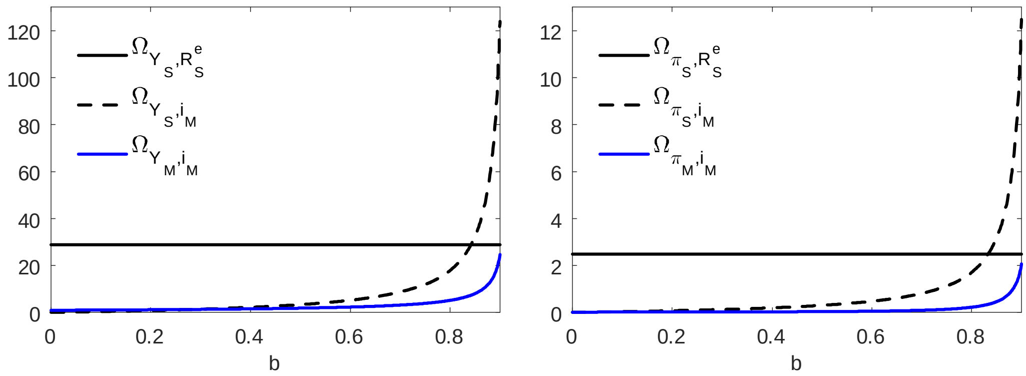
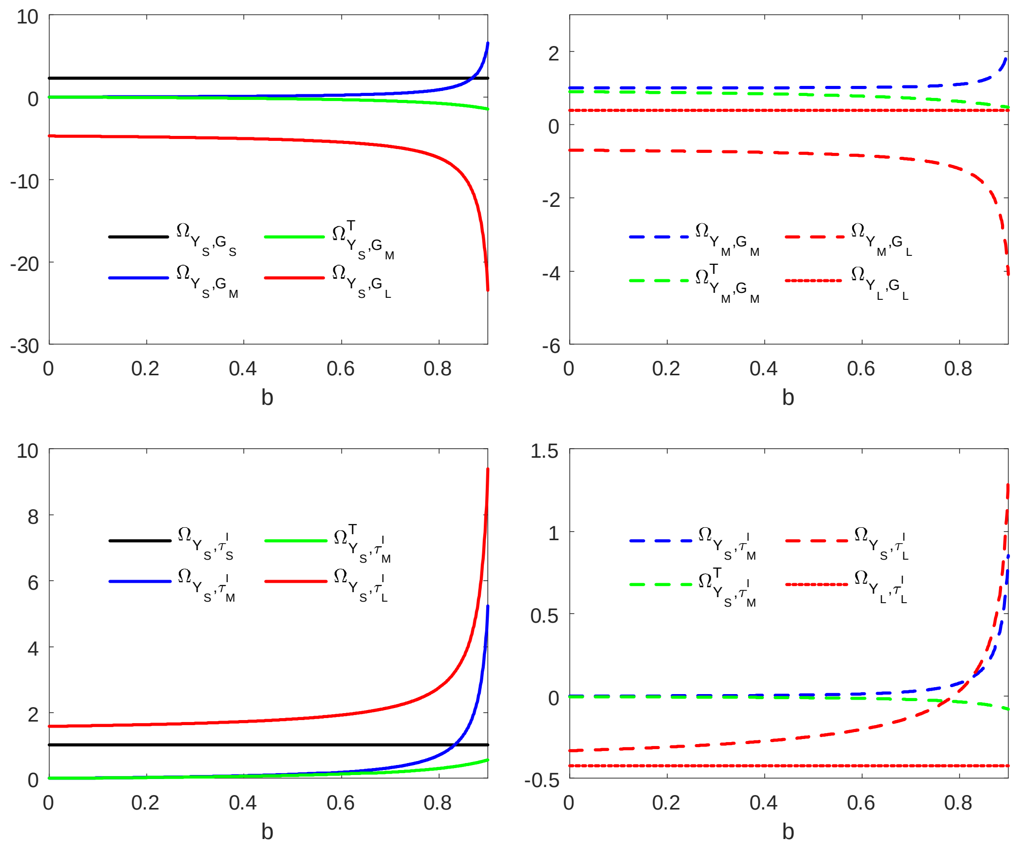
A Figures and Tables

Figure A.1: Shock and Interest Rate Coefficients as a Function of Probability b



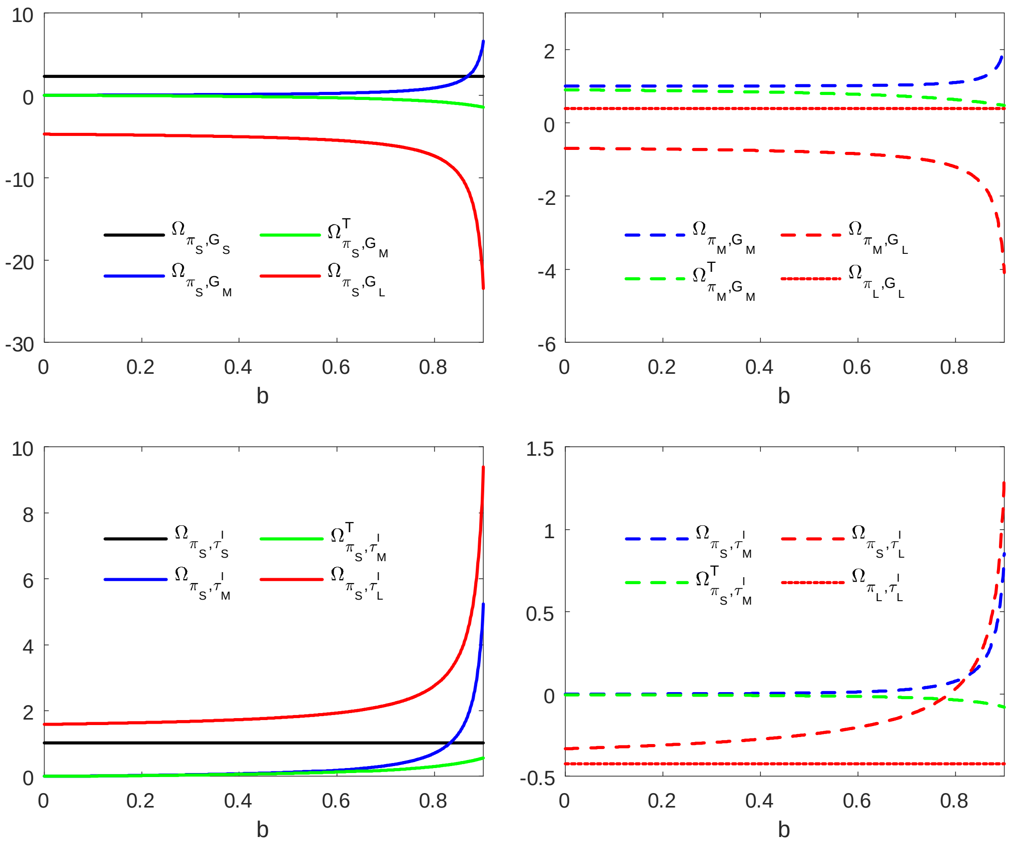
Note: This figure presents the behavior of the coefficients in response to the shock (r^e_S) and to the differential between long- and medium-run interest rates ($\bar{r} - i_M$) as a function of the probability associated with the transitional state (b). These terms appear in equations (4.4)-(4.5) and (4.7)-(4.8).

Figure A.2: Output Multipliers



Note: This figure presents the behavior of the coefficients that represent the responses of output to the fiscal stimulus (G or τ^I), in each state, according to Propositions 1 and 2, as a function of the probability associated with the transtional state (b).

Figure A.3: Inflation Multipliers



Note: This figure presents the behavior of the coefficients that represent the responses of inflation to the fiscal stimulus (G or τ^I), in each state, according to Propositions 1 and 2, as a function of the probability associated with the transtional state (b).

Table A.1: Analytical Expressions for Medium- and Long-Run Solution Coefficients - Propositions 1 and 2

	Y_M	π_M	Y_L	π_L
Ω_{*,i_M}	$\Gamma_{b\sigma}^{-1}(1-\beta b)\sigma$	$\Gamma_{b\sigma}^{-1}\kappa\sigma$	–	–
Ω_{*,G_M}	$\Gamma_{b\sigma}^{-1}[(1-b)(1-\beta b)-b\kappa\psi]$	$\Gamma_{b\sigma}^{-1}\kappa(1-b)\left(1-\psi\sigma^{-1}\right)$	–	–
Ω_{*,G_M}^{T}	$\Gamma_{b\phi}^{-1}[(1-b)(1-\beta b)+(\phi_\pi-b)\kappa\psi]$	$\Gamma_{b\phi}^{-1}\kappa\left[(1-b)\left(1-\psi\sigma^{-1}\right)-\psi\phi_y\right]$	–	–
Ω_{*,G_L}	$(1-b)\Gamma_{b\sigma}^{-1}\begin{bmatrix}(1-\beta b)(\Omega_{Y_L,G_L}-1)+\\ \sigma\Omega_{\pi_L,G_L}\end{bmatrix}$	$(1-b)\Gamma_{b\sigma}^{-1}\begin{bmatrix}\kappa(\Omega_{Y_L,G_L}-1)+\\ [\beta(1-b)+\kappa\sigma]\Omega_{\pi_L,G_L}\end{bmatrix}$	$\Gamma_{\phi}^{-1}(\phi_\pi-1)\kappa\psi\sigma^{-1}$	$-\Gamma_{\phi}^{-1}\phi_y\kappa\psi\sigma^{-1}$
Ω_{*,τ_M^I}	$b\Gamma_{b\sigma}^{-1}\kappa\sigma\psi\chi^I$	$(1-b)\Gamma_{b\sigma}^{-1}\kappa\psi\chi^I$	–	–
$\Omega_{*,\tau_M^I}^{T}$	$-\Gamma_{b\phi}^{-1}(\phi_\pi-b)\kappa\sigma\psi\chi^I$	$\Gamma_{b\phi}^{-1}\left(1-b+\sigma\phi_y\right)\kappa\psi\chi^I$	–	–
Ω_{*,τ_L^I}	$(1-b)\Gamma_{b\sigma}^{-1}\begin{bmatrix}(1-\beta b)\Omega_{Y_L,\tau_L^I}+\\ \sigma\Omega_{\pi_L,\tau_L^I}\end{bmatrix}$	$(1-b)\Gamma_{b\sigma}^{-1}\times\begin{bmatrix}\kappa\Omega_{Y_L,\tau_L^I}+\\ [\beta(1-b)+\kappa\sigma]\Omega_{\pi_L,\tau_L^I}\end{bmatrix}$	$-\Gamma_{\phi}^{-1}(\phi_\pi-1)\kappa\psi\chi^I$	$\Gamma_{\phi}^{-1}\phi_y\kappa\psi\chi^I$

Notes: $\Gamma_\phi \equiv (1-\beta)\phi_y+(\phi_\pi-1)\kappa$ and $\Gamma_{b\sigma} \equiv (1-\beta b)(1-b)-b\kappa\sigma$.

Table A.2: Analytical Expressions for Short-Run Solution Coefficients - Propositions 1 and 2

	Y_S	π_S
Ω_{*,r_S^e}	$\Gamma_{\mu\sigma}^{-1}(1-\beta\mu)\sigma$	$\Gamma_{\mu\sigma}^{-1}\kappa\sigma$
Ω_{*,i_M}	$b\Gamma_{\mu\sigma}^{-1}(1-\mu)\left[(1-\beta\mu)\Omega_{Y_M,i_M}+\sigma\Omega_{\pi_M,i_M}\right]$	$b\Gamma_{\mu\sigma}^{-1}(1-\mu)\left[\kappa\Omega_{Y_M,i_M}+\left[\beta(1-\mu)+\kappa\sigma\right]\Omega_{\pi_M,i_M}\right]$
Ω_{*,G_S}	$\Gamma_{\mu\sigma}^{-1}\left[(1-\beta\mu)(1-\mu)-\mu\kappa\psi\right]$	$\Gamma_{\mu\sigma}^{-1}(1-\mu)\left(1-\psi\sigma^{-1}\right)\kappa$
Ω_{*,G_M}	$b\Gamma_{\mu\sigma}^{-1}(1-\mu)\left[(1-\beta\mu)\left(\Omega_{Y_M,G_M}-1\right)+\sigma\Omega_{\pi_M,G_M}\right]$	$b\Gamma_{\mu\sigma}^{-1}(1-\mu)\left[\kappa\left(\Omega_{Y_M,G_M}-1\right)+\left[\beta(1-\mu)+\kappa\sigma\right]\Omega_{\pi_M,G_M}\right]$
Ω_{*,G_M}^T	$b\Gamma_{\mu\sigma}^{-1}(1-\mu)\left[(1-\beta\mu)\left(\Omega_{Y_M,G_M}^T-1\right)+\sigma\Omega_{\pi_M,G_M}^T\right]$	$b\Gamma_{\mu\sigma}^{-1}(1-\mu)\left[\kappa\left(\Omega_{Y_M,G_M}^T-1\right)+\left[\beta(1-\mu)+\kappa\sigma\right]\Omega_{\pi_M,G_M}^T\right]$
Ω_{*,G_L}	$\Gamma_{\mu\sigma}^{-1}(1-\mu)\left\{\begin{array}{l}(1-\beta\mu)\left[b\Omega_{Y_M,G_L}+(1-b)\left(\Omega_{Y_L,G_L}-1\right)\right]+\\ \sigma\left[b\Omega_{\pi_M,G_L}+(1-b)\Omega_{\pi_L,G_L}\right]\end{array}\right\}$	$\Gamma_{\mu\sigma}^{-1}(1-\mu)\left\{\begin{array}{l}\kappa\left[b\Omega_{Y_M,G_L}+(1-b)\left(\Omega_{Y_L,G_L}-1\right)\right]+\\ \left[\beta(1-\mu)+\kappa\sigma\right]\left[b\Omega_{\pi_M,G_L}+(1-b)\Omega_{\pi_L,G_L}\right]\end{array}\right\}$
Ω_{*,τ_S^I}	$\Gamma_{\mu\sigma}^{-1}\mu\kappa\sigma\psi\chi^I$	$\Gamma_{\mu\sigma}^{-1}(1-\mu)\kappa\psi\chi^I$
Ω_{*,τ_M^I}	$b\Gamma_{\mu\sigma}^{-1}(1-\mu)\left[(1-\beta\mu)\Omega_{Y_M,\tau_M^I}+\sigma\Omega_{\pi_M,\tau_M^I}\right]$	$b\Gamma_{\mu\sigma}^{-1}(1-\mu)\left[\kappa\Omega_{Y_M,\tau_M^I}\tau_M^I+\left[\beta(1-\mu)+\kappa\sigma\right]\Omega_{\pi_M,\tau_M^I}\right]$
$\Omega_{*,\tau_M^I}^T$	$b\Gamma_{\mu\sigma}^{-1}(1-\mu)\left[(1-\beta\mu)\Omega_{Y_M,\tau_M^I}^T+\sigma\Omega_{\pi_M,\tau_M^I}^T\right]$	$b\Gamma_{\mu\sigma}^{-1}(1-\mu)\left[\kappa\Omega_{Y_M,\tau_M^I}^T\tau_M^I+\left[\beta(1-\mu)+\kappa\sigma\right]\Omega_{\pi_M,\tau_M^I}^T\right]$
Ω_{*,τ_L^I}	$\Gamma_{\mu\sigma}^{-1}(1-\mu)\left\{\begin{array}{l}(1-\beta\mu)\left[b\Omega_{Y_M,\tau_L^I}+(1-b)\Omega_{Y_L,\tau_L^I}\right]+\\ \sigma\left[b\Omega_{\pi_M,\tau_L^I}+(1-b)\Omega_{\pi_L,\tau_L^I}\right]\end{array}\right\}$	$\Gamma_{\mu\sigma}^{-1}(1-\mu)\left\{\begin{array}{l}\kappa\left[b\Omega_{Y_M,\tau_L^I}+(1-b)\Omega_{Y_L,\tau_L^I}\right]+\\ \left[\beta(1-\mu)+\kappa\sigma\right]\left[b\Omega_{\pi_M,\tau_L^I}+(1-b)\Omega_{\pi_L,\tau_L^I}\right]\end{array}\right\}$

Note: $\Gamma_{\mu\sigma}\equiv(1-\beta\mu)(1-\mu)-\mu\kappa\sigma$

B Proofs of Propositions

Proposition 1 *Assume that the nominal interest rate zero lower bound is binding in the short run* ($t \in [T_0, T_{exit})$) *and that the following conditions hold:*

$$
\begin{array}{ll}
(C1) & (1-\mu)(1-\beta\mu) - \mu\kappa\sigma > 0 \\
(C2) & (1-b)(1-\beta b) - b\kappa\sigma > 0 \\
(C3) & \phi_\pi + \frac{(1-\beta)}{\kappa}\phi_y > 1 \\
(C4) & r_S^e < -\Theta_{G_S} G_S - \Theta_{\tau_S^I}\tau_S^I - \Theta_{\bar{r}}(\bar{r} - i_M) - \Theta_{G_M} G_M - \Theta_{\tau_M^I}\tau_M^I - \Theta_{G_L} G_L - \Theta_{\tau_L^I}\tau_L^I
\end{array}
$$

If there is a transitional state generated by the monetary authority keeping the nominal interest rate at $i_M = 0$*, or at an optimally chosen level* $i_M = i_M^*$*, after the crisis is over* ($\forall t \in [T_{exit}, T_M)$)*, solutions for output, inflation and the nominal interest rate in each state can be obtained backward as follows:*

(i) *In the long run* ($\forall t > T_M$)*, with* $r_t^e = \bar{r}$*, there is a locally unique bounded solution such that*

$$
\begin{aligned}
Y_L &= \Omega_{Y_L,G_L} G_L + \Omega_{Y_L,\tau_L^I}\tau_L^I && (4.1)\\
\pi_L &= \Omega_{\pi_L,G_L} G_L + \Omega_{\pi_L,\tau_L^I}\tau_L^I && (4.2)\\
i_L &= \bar{r} + \phi_\pi \pi_L + \phi_y Y_L && (4.3)
\end{aligned}
$$

(ii) *There is a locally unique bounded medium-run solution* ($\forall t \in [T_{exit}, T_M)$)*, with* $r_t^e = \bar{r}$*, such that*

$$
\begin{aligned}
Y_M = {} & \Omega_{Y_M,i_M}(\bar{r} - i_M) + \Omega_{Y_M,G_M} G_M + \Omega_{Y_M,\tau_M^I}\tau_M^I + \\
& \Omega_{Y_M,G_L} G_L + \Omega_{Y_M,\tau_L^I}\tau_L^I && (4.4)\\
\pi_M = {} & \Omega_{\pi_M,i_M}(\bar{r} - i_M) + \Omega_{\pi_M,G_M} G_M + \Omega_{\pi_M,\tau_M^I}\tau_M^I + \\
& \Omega_{\pi_M,G_L} G_L + \Omega_{\pi_M,\tau_L^I}\tau_L^I && (4.5)\\
i_M = {} & \begin{cases} 0 \\ i_M^* \end{cases} && (4.6)
\end{aligned}
$$

(iii) *In the short run* ($\forall t \in [T_0, T_{exit})$)*, with* $r_t^e = r_S^e$*, there is a locally unique bounded solution, such that*

$$
\begin{aligned}
Y_S = {} & \Omega_{Y_S,r_S^e} r_S^e + \Omega_{Y_S,G_S} G_S + \Omega_{Y_S,\tau_S^I}\tau_S^I + \\
& \Omega_{Y_S,i_M}(\bar{r} - i_M) + \Omega_{Y_S,G_M} G_M + \Omega_{Y_S,\tau_M^I}\tau_M^I + \\
& \Omega_{Y_S,G_L} G_L + \Omega_{Y_S,\tau_L^I}\tau_L^I && (4.7)\\
\pi_S = {} & \Omega_{\pi_S,r_S^e} r_S^e + \Omega_{\pi_S,G_S} G_S + \Omega_{\pi_S,\tau_S^I}\tau_S^I + \\
& \Omega_{\pi_S,i_M}(\bar{r} - i_M) + \Omega_{\pi_S,G_M} G_M + \Omega_{\pi_S,\tau_M^I}\tau_M^I + \\
& \Omega_{\pi_S,G_L} G_L + \Omega_{\pi_S,\tau_L^I}\tau_L^I && (4.8)\\
i_S = {} & 0 && (4.9)
\end{aligned}
$$

where the analytical expressions for the coefficients $\Omega_{i,j}$*,* $i \in \{Y_S, \pi_S, Y_M, \pi_M, Y_L, \pi_L\}$ *and* $j \in \{r_s^e, i_M, G_S, \tau_S^I, G_M, \tau_M^I, G_L, \tau_L^I\}$ *and* $\Theta_k, k \in \{i_M, G_S, \tau_S^I, G_M, \tau_M^I, G_L, \tau_L^I\}$*, are defined in the Appendix and depend on the structural parameters.*

The proof of this proposition is divided into three parts. First, we discuss determinacy of the solutions in each state. The second part is the derivation of analytical expressions for output and inflation allocations in each state. The last part is the derivation of condition ($C4$) that guarantees that the zero lower bound is binding in the short run.

Proof. (Proposition 1 - Part I - Determinacy) The derivation of the conditions for determinacy and existence of a solution in this model follows the generalization of the Taylor principle, presented by Davig and Leeper (2007) for a New Keynesian model with regime changes in monetary policy. The states in the economy are interpreted as different regimes, and it is possible to construct a probability transition matrix from the probability structure defined in the three-state economy setup presented in Section 3.2. One aspect that makes this matrix simpler is the assumption that once the economy moves forward to the transitional state, or to the long run, it does not go back to the previous states.

Here it is assumed that the first state ($s_t = 1$) is the short run (crisis state), where the shock hits the economy ($r_1^e = r_S^e$) and the zero lower bound is binding ($i_1 = 0$). The second state ($s_t = 2$) is the medium run (transitional state), where the shock is no longer active ($r_2^e = \bar{r}$), but the monetary authority keeps the nominal interest rate at a fixed value ($i_2 = i_M$) for a few periods. Finally, the third state ($s_t = 3$) is the long run ($r_3^e = \bar{r}$), where the monetary policy goes back to following the rule (3.3) $\left(i_3 = \bar{r} + \phi_\pi \pi_t + \phi_y Y_t\right)$.

The solutions are then defined by the set of equations formed, respectively, by the (IS) equation (3.1), the aggregate supply relation (3.2) and the monetary policy rule (3.3).

$$\hat{Y}_t = E_t\hat{Y}_{t+1} - \sigma\left(i_t - E_t\pi_{t+1} - r_t^e\right) + \left(\hat{G}_t - E_t\hat{G}_{t+1}\right) \tag{B.1}$$

$$\pi_t = \kappa\hat{Y}_t + \kappa\psi\left(\chi^I\hat{\tau}_t^I - \sigma^{-1}\hat{G}_t\right) + \beta E_t\pi_{t+1} \tag{B.2}$$

$$i_t = \begin{cases} 0, & s_t = 1\ (Short\ \ run) \\ i_M, & s_t = 2\ (Medium\ \ run) \\ \bar{r} + \phi_\pi\pi_t + \phi_y\hat{Y}_t, & s_t = 3\ (Long\ \ run) \end{cases} \tag{B.3}$$

According to the diagram in Figure (3.2), we can set up the probability transition matrix:

$$\Pi \equiv \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \begin{bmatrix} \mu & (1-\mu)\,b & (1-\mu)\,(1-b) \\ 0 & b & (1-b) \\ 0 & 0 & 1 \end{bmatrix} \tag{B.4}$$

The state-contingent expectations for output and inflation are given by

$$E_t\hat{Y}_{t+1} = E\left[Y_{t+1}|s_t = j, \Omega_t^{-s}\right] = p_{j1}E\left[Y_{1t+1}|\Omega_t^{-s}\right] + p_{j2}E\left[Y_{2t+1}|\Omega_t^{-s}\right] + p_{j3}E\left[Y_{3t+1}|\Omega_t^{-s}\right]$$
$$E_t\pi_{t+1} = E\left[\pi_{t+1}|s_t = j, \Omega_t^{-s}\right] = p_{j1}E\left[\pi_{1t+1}|\Omega_t^{-s}\right] + p_{j2}E\left[\pi_{2t+1}|\Omega_t^{-s}\right] + p_{j3}E\left[\pi_{3t+1}|\Omega_t^{-s}\right]$$

where $j \in \{1, 2, 3\}$ and Ω_t^{-s} denotes agents' information set at time t, not including the current regime and $\Omega_t = \Omega_t^{-s} \cup \{s_t\}$.

Expectations regarding government spending are given by:

$$E_t\hat{G}_{t+1} = E\left[G_{t+1}|s_t = j, \Omega_t^{-s}\right] = p_{j1}G_1 + p_{j2}G_2 + p_{j3}G_3$$

All expectations in $(B.1)$ and $(B.2)$ are formed conditional on Ω_t.

The (IS) and (AS) equations can be written as:

$$(IS_j) \quad Y_{jt} = p_{j1}E_tY_{1t+1} + p_{j2}E_tY_{2t+1} + p_{j3}E_tY_{3t+1} + \sigma\left(p_{j1}E_t\pi_{1t+1} + p_{j2}E_t\pi_{2t+1} + p_{j3}E_t\pi_{3t+1}\right) + \left(p_{j1}G_1 + p_{j2}G_2 + p_{j3}G_3\right) - G_j - \sigma i_{jt} + \sigma r_{jt}^e$$
$$(AS_j) \quad \pi_{jt} = \beta\left(p_{j1}E_t\pi_{1t+1} + p_{j2}E_t\pi_{2t+1} + p_{j3}E_t\pi_{3t+1}\right) + \kappa Y_{jt} + \kappa\psi\left(\chi^I\tau_j^I - \sigma^{-1}G_j\right)$$

Define state-contingent forecast errors for each state j:

$$\begin{aligned}\eta^{\pi}_{jt+1} &= \pi_{jt+1} - E_t\pi_{jt+1} \Rightarrow E_t\pi_{jt+1} = \pi_{jt+1} - \eta^{\pi}_{jt+1} \\ \eta^{Y}_{jt+1} &= Y_{jt+1} - E_tY_{jt+1} \Rightarrow E_tY_{jt+1} = Y_{jt+1} - \eta^{Y}_{jt+1}\end{aligned}$$

and use them to eliminate the conditional expectations and rewrite (IS_j) and (AS_j) as:

$$\begin{aligned}(IS_j) \quad Y_{jt} = \; & p_{j1}\left(Y_{1t+1} - \eta^{Y}_{1t+1}\right) + p_{j2}\left(Y_{2t+1} - \eta^{Y}_{2t+1}\right) + p_{j3}\left(Y_{3t+1} - \eta^{Y}_{3t+1}\right) + \\ & \sigma\left(p_{j1}\left(\pi_{1t+1} - \eta^{\pi}_{1t+1}\right) + p_{j2}\left(\pi_{2t+1} - \eta^{\pi}_{2t+1}\right) + p_{j3}\left(\pi_{3t+1} - \eta^{\pi}_{3t+1}\right)\right) + \\ & \left(p_{j1}G_1 + p_{j2}G_2 + p_{j3}G_3\right) - G_j - \sigma i_{jt} + \sigma r^{e}_{jt} \\ (AS_j) \quad \pi_{jt} = \; & \beta p_{j1}\left(\pi_{1t+1} - \eta^{\pi}_{1t+1}\right) + \beta p_{j2}\left(\pi_{2t+1} - \eta^{\pi}_{2t+1}\right) + \beta p_{j3}\left(\pi_{3t+1} - \eta^{\pi}_{3t+1}\right) + \\ & \kappa Y_{jt} + \kappa\psi\left(\chi^{I}\tau^{I}_{j} - \sigma^{-1}G_j\right)\end{aligned}$$

Hence, we can write the system of equations for each state, incorporating the monetary policy stance into the (IS) equation to get the aggregate demand relation. In the first state (short run) we have:

$$\begin{aligned}(AD_1) \quad & p_{11}Y_{1t+1} + p_{12}Y_{2t+1} + p_{13}Y_{3t+1} + \sigma\left(p_{11}\pi_{1t+1} + p_{12}\pi_{2t+1} + p_{13}\pi_{3t+1}\right) = Y_{1t} + \\ & \left(p_{11}\eta^{Y}_{1t+1} + p_{12}\eta^{Y}_{2t+1} + p_{13}\eta^{Y}_{3t+1}\right) + \sigma\left(p_{11}\eta^{\pi}_{1t+1} + p_{12}\eta^{\pi}_{2t+1} + p_{13}\eta^{\pi}_{3t+1}\right) \\ & -\sigma r^{e}_{S} - \left[\left(p_{11} - 1\right)G_1 + p_{12}G_2 + p_{13}G_3\right] \\ (AS_1) \quad & \beta\left(p_{11}\pi_{1t+1} + p_{12}\pi_{2t+1} + p_{13}\pi_{3t+1}\right) = \pi_{1t} - \kappa Y_{1t} + \\ & \beta\left(p_{11}\eta^{\pi}_{1t+1} + p_{12}\eta^{\pi}_{2t+1} + p_{13}\eta^{\pi}_{3t+1}\right) - \kappa\psi\left(\chi^{I}\tau^{I}_{1} - \sigma^{-1}G_1\right)\end{aligned}$$

In the second state (medium run) we have:

$$\begin{aligned}(AD_2) \quad & p_{21}Y_{1t+1} + p_{22}Y_{2t+1} + p_{23}Y_{3t+1} + \sigma\left(p_{21}\pi_{1t+1} + p_{22}\pi_{2t+1} + p_{23}\pi_{3t+1}\right) = Y_{2t} + \\ & \left(p_{21}\eta^{Y}_{1t+1} + p_{22}\eta^{Y}_{2t+1} + p_{23}\eta^{Y}_{3t+1}\right) + \sigma\left(p_{21}\eta^{\pi}_{1t+1} + p_{22}\eta^{\pi}_{2t+1} + p_{23}\eta^{\pi}_{3t+1}\right) \\ & -\sigma\left[\bar{r} - i_M\right] - \left[p_{21}G_1 + \left(p_{22} - 1\right)G_2 + p_{23}G_3\right] \\ (AS_2) \quad & \beta\left(p_{21}\pi_{1t+1} + p_{22}\pi_{2t+1} + p_{23}\pi_{3t+1}\right) = \pi_{2t} - \kappa Y_{2t} + \\ & \beta\left(p_{21}\eta^{\pi}_{1t+1} + p_{22}\eta^{\pi}_{2t+1} + p_{23}\eta^{\pi}_{3t+1}\right) - \kappa\psi\left(\chi^{I}\tau^{I}_{2} - \sigma^{-1}G_2\right)\end{aligned}$$

Finally, in the third state (long run) the system is given by:

$$\begin{aligned}(AD_3) \quad & p_{31}Y_{1t+1} + p_{32}Y_{2t+1} + p_{33}Y_{3t+1} + \sigma\left(p_{31}\pi_{1t+1} + p_{32}\pi_{2t+1} + p_{33}\pi_{3t+1}\right) = \\ & \sigma\phi_{\pi}\pi_{3t} + \left(1 + \sigma\phi_{y}\right)Y_{3t} + \left(p_{31}\eta^{Y}_{1t+1} + p_{32}\eta^{Y}_{2t+1} + p_{33}\eta^{Y}_{3t+1}\right) + \\ & \sigma\left(p_{31}\eta^{\pi}_{1t+1} + p_{32}\eta^{\pi}_{2t+1} + p_{33}\eta^{\pi}_{3t+1}\right) - \left[p_{31}G_1 + p_{32}G_2 + \left(p_{33} - 1\right)G_3\right] \\ (AS_3) \quad & \beta\left(p_{31}\pi_{1t+1} + p_{32}\pi_{2t+1} + p_{33}\pi_{3t+1}\right) = \pi_{3t} - \kappa Y_{3t} + \\ & \beta\left(p_{31}\eta^{\pi}_{1t+1} + p_{32}\eta^{\pi}_{2t+1} + p_{33}\eta^{\pi}_{3t+1}\right) - \kappa\psi\left(\chi^{I}\tau^{I}_{3} - \sigma^{-1}G_3\right)\end{aligned}$$

The complete system can be expressed in matrix form

$$AX_{t+1} = BX_t + A\eta_t + Ce_t$$

$$X_t \equiv \begin{bmatrix}\pi_{1t} \\ Y_{1t} \\ \pi_{2t} \\ Y_{2t} \\ \pi_{3t} \\ Y_{3t}\end{bmatrix} \qquad \eta_t \equiv \begin{bmatrix}\eta^{\pi}_{1t+1} \\ \eta^{Y}_{1t+1} \\ \eta^{\pi}_{2t+1} \\ \eta^{Y}_{2t+1} \\ \eta^{\pi}_{3t+1} \\ \eta^{Y}_{3t+1}\end{bmatrix} \qquad e_t \equiv \begin{bmatrix}r^{e}_{S} \\ i_M \\ \bar{r} \\ G_1 \\ \tau^{I}_{1} \\ G_2 \\ \tau^{I}_{2} \\ G_3 \\ \tau^{I}_{3}\end{bmatrix}$$

$$A = \Pi \otimes \begin{bmatrix} \beta & 0 \\ \sigma & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -\kappa & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\kappa & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\kappa \\ 0 & 0 & 0 & 0 & \sigma\phi_\pi & \left(1+\sigma\phi_y\right) \end{bmatrix}$$

$$C \equiv \begin{bmatrix} 0 & 0 & 0 & \kappa\psi\sigma^{-1} & -\kappa\psi\chi^I & 0 & 0 & 0 & 0 \\ -\sigma & 0 & 0 & (1-\mu) & 0 & -(1-\mu)b & 0 & -(1-\mu)(1-b) & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa\psi\sigma^{-1} & -\kappa\psi\chi^I & 0 & 0 \\ 0 & \sigma & \sigma & 0 & 0 & (1-b) & 0 & -(1-b) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \kappa\psi\sigma^{-1} & -\kappa\psi\chi^I \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since A is invertible, we can write the system as:

$$X_{t+1} = A^{-1}BX_t + \eta_t + A^{-1}Ce_t$$

where

$$A^{-1}B = \begin{bmatrix} \frac{1}{\beta\mu} & -\frac{\kappa}{\beta\mu} & -\frac{(1-\mu)}{\beta\mu} & \frac{\kappa(1-\mu)}{\beta\mu} & 0 & 0 \\ -\frac{\sigma}{\beta\mu} & \frac{\beta+\kappa\sigma}{\beta\mu} & \frac{\sigma(1-\mu)}{\beta\mu} & -\frac{(\beta+\kappa\sigma)(1-\mu)}{\beta\mu} & 0 & 0 \\ 0 & 0 & \frac{1}{\beta b} & -\frac{\kappa}{\beta b} & -\frac{(1-b)}{\beta b} & \frac{\kappa(1-b)}{\beta b} \\ 0 & 0 & -\frac{\sigma}{\beta b} & \frac{\beta+\kappa\sigma}{\beta b} & \frac{(1-\beta\phi_\pi)\sigma(1-b)}{\beta b} & -\frac{\left(\beta\left(1+\sigma\phi_y\right)+\kappa\sigma\right)(1-b)}{\beta b} \\ 0 & 0 & 0 & 0 & \frac{1}{\beta} & -\frac{\kappa}{\beta} \\ 0 & 0 & 0 & 0 & \frac{\sigma(\beta\phi_\pi-1)}{\beta} & \frac{\beta\left(1+\sigma\phi_y\right)+\kappa\sigma}{\beta} \end{bmatrix}$$

The rational-expectations solution is determined if and only if matrix $A^{-1}B$ has all its eigenvalues outside the unit circle. Since this matrix is block lower triangular, its eigenvalues are those defined by the block matrices in the diagonal.

$$Z_1 \equiv \begin{bmatrix} \frac{1}{\beta\mu} & -\frac{\kappa}{\beta\mu} \\ -\frac{\sigma}{\beta\mu} & \frac{\beta+\kappa\sigma}{\beta\mu} \end{bmatrix} \qquad Z_2 \equiv \begin{bmatrix} \frac{1}{\beta b} & -\frac{\kappa}{\beta b} \\ -\frac{\sigma}{\beta b} & \frac{\beta+\kappa\sigma}{\beta b} \end{bmatrix} \qquad Z_3 \equiv \begin{bmatrix} \frac{1}{\beta} & -\frac{\kappa}{\beta} \\ \frac{\sigma(\beta\phi_\pi-1)}{\beta} & \frac{\beta\left(1+\sigma\phi_y\right)+\kappa\sigma}{\beta} \end{bmatrix}$$

Hence, we can use the result from Proposition C.1 in Woodford (2003) (Pgs. 670-71) to verify the determinacy conditions. This proposition states that the eigenvalues of a 2×2 matrix Z are outside the unit circle if and only if one of the following two cases are satisfied:

Case I: $(a)\det(Z) > 1$ and $(b)\det(Z) - tr(Z) > -1$ and $(c)\det(Z) + tr(Z) > -1$;

Case II: $(d)\det(Z) - tr(Z) < -1$ and $(e)\det(Z) + tr(Z) < -1$

So, I verify the conditions under which one of these cases are satisfied for each matrix Z_i, reminding that all parameters $\left(\beta, \kappa, \sigma, \mu, b, \phi_\pi, \phi_y\right)$ are positive. Starting with Z_1:

$$tr(Z_1) = \frac{1}{\beta\mu} + \left(1 + \frac{\kappa\sigma}{\beta}\right)\frac{1}{\mu} \qquad det(Z_1) = \frac{1}{\beta\mu^2}$$

Both conditions in Case II are clearly not satisfied since all parameters are positive. So I check conditions in Case I:

$(a)\, det\,(Z_1) = \dfrac{1}{\beta\mu^2} > 1$

$(b)\, det\,(Z_1) - tr\,(Z_1) > -1$

$$\begin{aligned} det\,(Z_1) - tr\,(Z_1) &= \frac{1}{\beta\mu^2} - \frac{1}{\beta\mu} - \frac{1}{\mu} - \frac{\kappa\sigma}{\beta\mu} > -1 \\ &\Rightarrow \beta\mu^2 - \mu - \beta\mu - \mu\kappa\sigma + 1 > 0 \\ &\Rightarrow \beta\mu^2 - \beta\mu + 1 - \mu - \mu\kappa\sigma > 0 \\ &\Rightarrow -(1-\mu)\,\beta\mu + 1 - \mu - \mu\kappa\sigma > 0 \\ &\Rightarrow \boxed{(1-\mu)(1-\beta\mu) - \mu\kappa\sigma > 0} \qquad \text{(Condition } (C1)\text{)} \end{aligned}$$

$(c)\, det\,(Z_1) + tr\,(Z_1) > -1$

$$det\,(Z_1) + tr\,(Z_1) = \frac{\beta^{-1}}{\mu^2} + \frac{1 + \beta^{-1}(1+\kappa\sigma)}{\mu} > -1$$

Thus, matrix Z_1 has both eigenvalues with modulus $|\lambda| > 1$, if Condition $(C1)$ holds. Matrix Z_2 is analogous to Z_1. We just need to switch μ for b to get that its eigenvalues are outside the unit circle if the following condition holds:

$$\boxed{(1-b)(1-\beta b) - b\kappa\sigma > 0} \qquad \text{(Condition } (C2)\text{)}$$

For matrix Z_3 we have:

$$tr\,(Z_3) = \frac{1 + \beta\sigma\phi_y + \kappa\sigma + \beta}{\beta} \qquad det\,(Z_3) = \frac{\sigma\left(\phi_y + \kappa\phi_\pi\right) + 1}{\beta}$$

Again, since all parameters are positive, the conditions in Case II are clearly not satisfied. So I verify the conditions in Case I:

$(a)\, det\,(Z_3) = \dfrac{1 + \sigma\left(\phi_y + \kappa\phi_\pi\right)}{\beta} > 1$

$(b)\, det\,(Z_3) - tr\,(Z_3) > -1$

$$\begin{aligned} det\,(Z_3) - tr\,(Z_3) &= \frac{1 + \sigma\left(\phi_y + \kappa\phi_\pi\right) - 1 - \beta\sigma\phi_y - \kappa\sigma - \beta}{\beta} > -1 \\ &\Rightarrow \sigma\kappa\phi_\pi + \sigma\phi_y - \beta\sigma\phi_y - \kappa\sigma - \beta > -\beta \\ &\Rightarrow \sigma\kappa\phi_\pi + (1-\beta\sigma)\,\phi_y - \kappa\sigma > 0 \\ &\Rightarrow \boxed{\phi_\pi + \frac{(1-\beta)}{\kappa}\phi_y > 1} \qquad \text{(Condition } (C3)\text{)} \end{aligned}$$

which holds since we assume $\phi_\pi > 1$ and $\phi_y > 0$.

$(c)\, det\,(Z_3) + tr\,(Z_3) > -1$

$$det\,(Z_3) + tr\,(Z_3) = \frac{2 + \sigma\left(\phi_y + \kappa\phi_\pi\right) + \beta\sigma\phi_y + \kappa\sigma + \beta}{\beta} > -1$$

I should point out that Condition $(C1)$ is necessary for the short-run solution to be defined and it does not depend on the fiscal measures taken. However, if the fiscal policy implemented during the crisis state is expected to be permanent or expected to last for a few periods after the crisis is over, while the monetary policy is expected to go back to the rule right after the zero lower bound is no longer binding, than we need Condition ($C3$) to be satisfied as well. In the case that the transitional state is generated by the monetary authority keeping the nominal interest rate fixed for a few periods after the crisis is over (independently if the fiscal stimulus is also maintained or not), Condition ($C2$) also needs to be satisfied to guarantee that the solution is determinate. ■

Proof. (Proposition 1 - Part II - Output and Inflation Allocations in Each State) All solutions in this model are obtained by solving the system formed by the (IS) and the aggregate demand equations, besides the specifications for monetary and fiscal policies in each state.

$$(IS) \quad \hat{Y}_t = E_t\hat{Y}_{t+1} - \sigma\left(i_t - E_t\pi_{t+1} - r_t^e\right) + \left(\hat{G}_t - E_t\hat{G}_{t+1}\right) \tag{B.5}$$

$$(AS) \quad \pi_t = \kappa\hat{Y}_t + \kappa\psi\left(\chi^I\hat{\tau}_t^I - \sigma^{-1}\hat{G}_t\right) + \beta E_t\pi_{t+1} \tag{B.6}$$

Given the expectations terms in these equations, to obtain the allocations in each state, it is necessary to solve for them backward. One should first compute long-run output and inflation, then use them to get the medium-run allocations and, finally, use both of them to obtain output and inflation in the short run.[40]

Long-Run Solution Allocations

I assume that, in the long run, the shock that hits the economy during the crisis state has already faded away ($r_t = \bar{r}, \forall t > T_M$) and the zero lower bound is no longer binding. If the fiscal measure implemented during the crisis is permanent, the economy will deviate from the steady state in the long run, with the new allocation ($\pi_L \neq 0, Y_L \neq 0$) lasting forever. This implies that the expectation terms in ($B.5$) and ($B.6$) will be given by $E_t\hat{Y}_{t+1} = Y_L$ and $E_t\pi_{t+1} = \pi_L, \forall t > T_M$. Assuming that the monetary authority follows rule (3.3), the long-run aggregate demand is given by:

$$\begin{aligned} Y_L &= Y_L - \sigma\left(\bar{r} + \phi_\pi\pi_L + \phi_y Y_L - \pi_L - \bar{r}\right) + G_L - G_L \\ 0 &= -\sigma\phi_\pi\pi_L - \sigma\phi_y Y_L + \sigma\pi_L \\ Y_L &= -\frac{\phi_\pi - 1}{\phi_y}\pi_L \end{aligned} \tag{B.7}$$

While the aggregate supply is given by:

$$\begin{aligned} \pi_L &= \kappa Y_L + \kappa\psi\left(\chi^I\tau_L^I - \sigma^{-1}G_L\right) + \beta\pi_L \\ \pi_L &= \frac{\kappa}{(1-\beta)}Y_L + \frac{\kappa\psi}{(1-\beta)}\left(\chi^I\tau_L^I - \sigma^{-1}G_L\right) \end{aligned} \tag{B.8}$$

[40] Given the linearity of the model, in the derivation of the solution allocations we assume that both fiscal instruments are used to obtain the full expressions for output and inflation. To discuss the impact of each instrument in Section 6, we shut down the other according to the policy that we want to analyze.

Substituting the expression for the aggregate demand into the aggregate supply:

$$\pi_L = -\frac{(\phi_\pi - 1)\,\kappa}{(1-\beta)\,\phi_y}\pi_L + \frac{\kappa\psi}{(1-\beta)}\left(\chi^I \tau_L^I - \sigma^{-1} G_L\right)$$
$$\left(1 + \frac{\kappa\,(\phi_\pi - 1)}{(1-\beta)\,\phi_y}\right)\pi_L = \frac{\kappa\psi}{(1-\beta)}\left(\chi^I \tau_L^I - \sigma^{-1} G_L\right)$$
$$\pi_L = \frac{\kappa\psi\phi_y}{(1-\beta)\,\phi_y + \kappa\,(\phi_\pi - 1)}\left(\chi^I \tau_L^I - \sigma^{-1} G_L\right)$$

Substituting back into the aggregate demand:

$$Y_L = \frac{(\phi_\pi - 1)\,\kappa\psi}{(1-\beta)\,\phi_y + (\phi_\pi - 1)\,\kappa}\left(\sigma^{-1} G_L - \chi^I \tau_L^I\right)$$

Defining $\Gamma_\phi \equiv (1-\beta)\,\phi_y + (\phi_\pi - 1)\,\kappa$, the long-run solutions can be summarized as follows:

$$\boxed{Y_L = \Omega_{Y_L,G_L} G_L + \Omega_{Y_L,\tau_L^I}\tau_L^I} \tag{B.9}$$

where $\Omega_{Y_L,G_L} \equiv \frac{(\phi_\pi - 1)\,\kappa\psi\sigma^{-1}}{\Gamma_\phi}$ and $\Omega_{Y_L,\tau_L^I} \equiv -\frac{(\phi_\pi - 1)\,\kappa\psi\chi^I}{\Gamma_\phi}$.

$$\boxed{\pi_L = \Omega_{\pi_L,G_L} G_L + \Omega_{\pi_L,\tau_L^I}\tau_L^I} \tag{B.10}$$

where $\Omega_{\pi_L,G_L} \equiv -\frac{\phi_y\kappa\psi\sigma^{-1}}{\Gamma_\phi}$ and $\Omega_{\pi_L,\tau_L^I} \equiv \frac{\phi_y\kappa\psi\chi^I}{\Gamma_\phi}$.

Note that I assume the fiscal stimulus to be permanent to derive the analytical solutions for long-run output and inflation as a function of long-run government spending (G_L) and taxes $\left(\tau_L^I\right)$. However, if the stimulus is expected to be temporary, it is straightforward to set $G_L = 0$ and $\tau_L^I = 0$ to obtain $\pi_L = Y_L = 0$.

Medium-Run Solution Allocations (Transitional State)

The transitional-state allocations (π_M, Y_M) will also be obtained by solving the system formed by (IS) and aggregate supply equations, given monetary and fiscal policies. But first it is necessary to get the expectations regarding future values of inflation, output and government spending that enter into these equations. Recall that once the economy enters the medium run ($t \in [T_{exit}, T_M)$ and $b \neq 0$), in each following period there is probability b that it stays there, while with probability $(1-b)$ it goes to the long-run allocation (π_L, Y_L, G_L). Thus, expectations in the transitional state are given by:

$$\begin{aligned} E_t\hat{Y}_{t+1} &= bY_M + (1-b)\,Y_L \\ E_t\pi_{t+1} &= b\pi_M + (1-b)\,\pi_L \\ E_t\hat{G}_{t+1} &= bG_M + (1-b)\,G_L \end{aligned}$$

Once the economy exits the crisis and the zero lower bound is no longer binding, the monetary authority can keep stimulating it by holding the nominal interest rate at zero for a few periods ($i_M = 0$) or it can choose the optimal value ($i_M = i_M^*$) for the nominal interest rate to minimize the welfare losses implied by its policy. Hence i_M is kept in the aggregate demand equation to derive the medium-run solution as a function of it. It can later be set to zero or i_M^*, according to the hypothesis made.

Using the expectations obtained above, with the value for the nominal interest rate, the aggregate demand equation becomes:

$$\begin{aligned} Y_M &= Y_L + \frac{\sigma}{(1-b)}(\bar{r} - i_M) + \frac{\sigma b}{(1-b)}\pi_M - \sigma \pi_L + G_M - G_L \\ Y_M &= bY_M + (1-b) Y_L - \sigma (i_M - (b\pi_M + (1-b)\pi_L) - \bar{r}) + (G_M - (bG_M + (1-b) G_L)) \\ Y_M &= \frac{\sigma}{(1-b)}(\bar{r} - i_M) + \frac{b\sigma}{(1-b)}\pi_M + Y_L + \sigma\pi_L + G_M - G_L \end{aligned} \tag{B.11}$$

while the aggregate supply is given by

$$\begin{aligned} \pi_M &= \kappa Y_M + \kappa\psi\left(\chi^I \tau_M^I - \sigma^{-1} G_M\right) + \beta (b\pi_M + (1-b)\pi_L) \\ \pi_M &= \frac{\kappa}{(1-\beta b)} Y_M + \frac{\kappa\psi}{(1-\beta b)}\left(\chi^I \tau_M^I - \sigma^{-1} G_M\right) + \frac{\beta(1-b)}{(1-\beta b)}\pi_L \end{aligned} \tag{B.12}$$

Substituting $(B.11)$ into $(B.12)$:

$$\begin{aligned} \pi_M = {} & \frac{\kappa}{(1-\beta b)}\left\{\frac{\sigma}{(1-b)}(\bar{r} - i_M) + \frac{b\sigma}{(1-b)}\pi_M + Y_L + \sigma\pi_L + G_M - G_L\right\} + \\ & \frac{\kappa\psi}{(1-\beta b)}\left(\chi^I \tau_M^I - \sigma^{-1} G_M\right) + \frac{\beta(1-b)}{(1-\beta b)}\pi_L \end{aligned}$$

$$\pi_M = \frac{(1-b)}{(1-\beta b)(1-b) - b\kappa\sigma}\left\{\begin{array}{l} \frac{\kappa\sigma}{(1-b)}(\bar{r} - i_M) + \left(1 - \psi\sigma^{-1}\right)\kappa G_M + \kappa\psi\chi^I\tau_M^I + \\ \kappa(Y_L - G_L) + [\beta(1-b) + \kappa\sigma]\pi_L \end{array}\right\} \tag{B.13}$$

Substitute back into $(B.11)$:

$$\begin{aligned} Y_M = {} & \frac{\sigma}{(1-b)}(\bar{r} - i_M) + G_M + Y_L + \sigma\pi_L - G_L + \\ & \frac{b\sigma}{(1-\beta b)(1-b) - b\kappa\sigma}\left\{\begin{array}{l} \frac{\kappa\sigma}{(1-b)}(\bar{r} - i_M) + \kappa\psi\chi^I\tau_M^I + \left(1 - \psi\sigma^{-1}\right)\kappa G_M + \\ \kappa Y_L + [\beta(1-b) + \kappa\sigma]\pi_L - \kappa G_L \end{array}\right\} \end{aligned}$$

$$Y_M = \frac{1}{(1-\beta b)(1-b) - b\kappa\sigma}\left\{\begin{array}{l} (1-\beta b)\sigma(\bar{r} - i_M) + [(1-\beta b)(1-b) - b\kappa\psi] G_M + b\kappa\sigma\psi\chi^I\tau_M^I + \\ (1-\beta b)(1-b)(Y_L - G_L) + (1-b)\sigma\pi_L \end{array}\right\} \tag{B.14}$$

Given that we got the long-run allocations (Y_L, π_L) in equations $(B.9)$ and $(B.10)$, we can already use equations $(B.13)$ and $(B.14)$ to obtain the transitional-state solutions. However, I want to expresses them in terms of the fiscal instruments used in each state. So I define $\Gamma_{b\sigma} \equiv (1-\beta b)(1-b) - b\kappa\sigma$ and plug the long-run conditions $(B.9)$ and $(B.10)$ into the medium-run output equation $(B.14)$:

$$\begin{aligned} Y_M = {} & \frac{1}{\Gamma_{b\sigma}}\left\{(1-\beta b)\sigma(\bar{r} - i_M) + [(1-\beta b)(1-b) - b\kappa\psi] G_M + b\kappa\sigma\psi\chi^I\tau_M^I\right\} + \\ & \frac{(1-\beta b)(1-b)}{\Gamma_{b\sigma}}\left[\Omega_{Y_L,G_L} G_L + \Omega_{Y_L,\tau_L^I}\tau_L^I\right] + \frac{(1-b)\sigma}{\Gamma_{b\sigma}}\left[\Omega_{\pi_L,G_L} G_L + \Omega_{\pi_L,\tau_L^I}\tau_L^I\right] - \frac{(1-\beta b)(1-b)}{\Gamma_{b\sigma}} G_L \end{aligned}$$

$$\begin{aligned} Y_M = {} & \frac{1}{\Gamma_{b\sigma}}\left\{(1-\beta b)\sigma(\bar{r} - i_M) + [(1-\beta b)(1-b) - b\kappa\psi] G_M + b\kappa\sigma\psi\chi^I\tau_M^I\right\} + \\ & \frac{(1-b)}{\Gamma_{b\sigma}}\left[(1-\beta b)\left(\Omega_{Y_L,G_L} - 1\right) + \sigma\Omega_{\pi_L,G_L}\right] G_L + \frac{(1-b)\left[(1-\beta b)\Omega_{Y_L,\tau_L^I} + \sigma\Omega_{\pi_L,\tau_L^I}\right]}{\Gamma_{b\sigma}}\tau_L^I \end{aligned}$$

which can be summarized as:

$$\boxed{\begin{array}{lllllll} Y_M = & \Omega_{Y_M,i_M}\left(\overline{r}-i_M\right) & + & \Omega_{Y_M,G_M}G_M & + & \Omega_{Y_M,\tau_M^I}\tau_M^I & + \\ & & & \Omega_{Y_M,G_L}G_L & + & \Omega_{Y_M,\tau_L^I}\tau_L^I & \end{array}} \tag{B.15}$$

where $\Omega_{Y_M,i_M} \equiv \dfrac{(1-\beta b)\,\sigma}{\Gamma_{b\sigma}}$, $\Omega_{Y_M,G_M} \equiv \dfrac{[(1-\beta b)(1-b)-b\kappa\psi]}{\Gamma_{b\sigma}}$,

$\Omega_{Y_M,\tau_M^I} \equiv \dfrac{b\kappa\sigma\psi\chi^I}{\Gamma_{b\sigma}}$, $\Omega_{Y_M,G_L} \equiv \dfrac{(1-b)\left[(1-\beta b)\left(\Omega_{Y_L,G_L}-1\right)+\sigma\Omega_{\pi_L,G_L}\right]}{\Gamma_{b\sigma}}$ and

$$\Omega_{Y_M,\tau_L^I} \equiv \frac{(1-b)\left[(1-\beta b)\,\Omega_{Y_L,\tau_L^I}+\sigma\Omega_{\pi_L,\tau_L^I}\right]}{\Gamma_{b\sigma}}.$$

Analogously, plugging the long-run relations $(B.9)$ and $(B.10)$ into the medium-run inflation equation $(B.13)$:

$$\begin{aligned}\pi_M \;=\; & \frac{\kappa\sigma}{\Gamma_{b\sigma}}\left(\overline{r}-i_M\right)+\frac{(1-b)\left(1-\psi\sigma^{-1}\right)\kappa}{\Gamma_{b\sigma}}G_M+\frac{(1-b)\,\kappa\psi\chi^I}{\Gamma_{b\sigma}}\tau_M^I+ \\ & \frac{(1-b)}{\Gamma_{b\sigma}}\left\{\kappa\left[\Omega_{Y_L,G_L}G_L+\Omega_{Y_L,\tau_L^I}\tau_L^I\right]+\left[\beta(1-b)+\kappa\sigma\right]\left[\Omega_{\pi_L,G_L}G_L+\Omega_{\pi_L,\tau_L^I}\tau_L^I\right]-\kappa G_L\right\}\end{aligned}$$

$$\begin{aligned}\pi_M \;=\; & \frac{\kappa\sigma}{\Gamma_{b\sigma}}\left(\overline{r}-i_M\right)+\frac{(1-b)\left(1-\psi\sigma^{-1}\right)\kappa}{\Gamma_{b\sigma}}G_M+\frac{(1-b)\,\kappa\psi\chi^I}{\Gamma_{b\sigma}}\tau_M^I+ \\ & \frac{(1-b)\left[\kappa\Omega_{Y_L,G_L}+\left[\beta(1-b)+\kappa\sigma\right]\Omega_{\pi_L,G_L}-\kappa\right]}{\Gamma_{b\sigma}}G_L+\frac{(1-b)\left[\kappa\Omega_{Y_L,\tau_L^I}+\left[\beta(1-b)+\kappa\sigma\right]\Omega_{\pi_L,\tau_L^I}\right]}{\Gamma_{b\sigma}}\tau_L^I\end{aligned}$$

which can be summarized as:

$$\boxed{\begin{array}{lllllll} \pi_M = & \Omega_{\pi_M,i_M}\left(\overline{r}-i_M\right) & + & \Omega_{\pi_M,G_M}G_M & + & \Omega_{\pi_M,\tau_M^I}\tau_M^I & + \\ & & & \Omega_{\pi_M,G_L}G_L & + & \Omega_{\pi_M,\tau_L^I}\tau_L^I & \end{array}} \tag{B.16}$$

where $\Omega_{\pi_M,i_M} \equiv \dfrac{\kappa\sigma}{\Gamma_{b\sigma}}$, $\Omega_{\pi_M,G_M} \equiv \dfrac{(1-b)\left(1-\psi\sigma^{-1}\right)\kappa}{\Gamma_{b\sigma}}$, $\Omega_{\pi_M,\tau_M^I} \equiv \dfrac{(1-b)\,\kappa\psi\chi^I}{\Gamma_{b\sigma}}$,

$\Omega_{\pi_M,G_L} \equiv \dfrac{(1-b)\left[\kappa\left(\Omega_{Y_L,G_L}-1\right)+\left[\beta(1-b)+\kappa\sigma\right]\Omega_{\pi_L,G_L}\right]}{\Gamma_{b\sigma}}$ and

$$\Omega_{\pi_M,\tau_L^I} \equiv \frac{(1-b)\left[\kappa\Omega_{Y_L,\tau_L^I}+\left[\beta(1-b)+\kappa\sigma\right]\Omega_{\pi_L,\tau_L^I}\right]}{\Gamma_{b\sigma}}.$$

Short-Run Solution Allocations (Crisis State)

The last step is to solve for the short-run allocations (π_S, Y_S) that characterize the crisis state. First, it is necessary to obtain expectations regarding future values of inflation, output and government spending that enter into the aggregate demand and supply equations. Recall that once the economy enters the crisis state at $t = T_0$, in each following period, there is a probability μ that it stays there, while with probability $(1-\mu)\,b$ it goes to the transitional state (π_M, Y_M, G_M) and with probability $(1-\mu)(1-b)$ it goes straight to the long-run (π_L, Y_L, G_L). Thus expectations are given by:

$$E_t\hat{Y}_{t+1} = \mu Y_S + (1-\mu)\, bY_M + (1-\mu)(1-b)\, Y_L$$
$$E_t\pi_{t+1} = \mu\pi_S + (1-\mu)\, b\pi_M + (1-\mu)(1-b)\,\pi_L$$
$$E_t\hat{G}_{t+1} = \mu G_S + (1-\mu)\, bG_M + (1-\mu)(1-b)\, G_L$$

The crisis state is characterized by a shock ($r_t^e = r_S^e$) that makes the zero lower bound binding for the nominal interest rate ($i_S = 0$). So the aggregate demand equation becomes:

$$\begin{aligned} Y_S &= \mu Y_S + (1-\mu)\, bY_M + (1-\mu)(1-b)\, Y_L - \sigma\left(0 - (\mu\pi_S + (1-\mu)\, b\pi_M + (1-\mu)(1-b)\,\pi_L) - r_S^e\right) + \\ &\quad G_S - (\mu G_S + (1-\mu)\, bG_M + (1-\mu)(1-b)\, G_L) \\ Y_S &= \frac{\sigma}{(1-\mu)} r_S^e + \frac{\mu\sigma}{(1-\mu)}\pi_S + bY_M + b\sigma\pi_M + (1-b)\, Y_L + (1-b)\,\sigma\pi_L + G_S - bG_M - (1-b)\, G_L \quad \text{(B.17)} \end{aligned}$$

while the aggregate supply is given by:

$$\begin{aligned} \pi_S &= \kappa Y_S + \kappa\psi\left(\chi^I\tau_S^I - \sigma^{-1}G_S\right) + \beta\mu\pi_S + \beta(1-\mu)\, b\pi_M + \beta(1-\mu)(1-b)\,\pi_L \\ \pi_S &= \frac{\kappa}{(1-\beta\mu)} Y_S + \frac{\kappa\psi}{(1-\beta\mu)}\left(\chi^I\tau_S^I - \sigma^{-1}G_S\right) + \frac{\beta(1-\mu)\, b}{(1-\beta\mu)}\pi_M + \frac{\beta(1-\mu)(1-b)}{(1-\beta\mu)}\pi_L \quad \text{(B.18)} \end{aligned}$$

Substituting $(B.17)$ into $(B.18)$:

$$\begin{aligned} \pi_S &= \frac{\kappa}{(1-\beta\mu)}\left\{\begin{array}{l} \frac{\sigma}{(1-\mu)} r_S^e + \frac{\mu\sigma}{(1-\mu)}\pi_S + bY_M + b\sigma\pi_M + \\ (1-b)\, Y_L + (1-b)\,\sigma\pi_L + G_S - bG_M - (1-b)\, G_L \end{array}\right\} + \\ &\quad \frac{\kappa\psi}{(1-\beta\mu)}\left(\chi^I\tau_S^I - \sigma^{-1}G_S\right) + \frac{\beta(1-\mu)\, b}{(1-\beta\mu)}\pi_M + \frac{\beta(1-\mu)(1-b)}{(1-\beta\mu)}\pi_L \end{aligned}$$

$$\pi_S = \frac{1}{(1-\beta\mu)(1-\mu) - \mu\kappa\sigma}\left\{\begin{array}{l} \kappa\sigma r_S^e + (1-\mu)\left(1-\psi\sigma^{-1}\right)\kappa G_S + (1-\mu)\,\kappa\psi\chi^I\tau_S^I + \\ (1-\mu)\,\kappa\left[b\,(Y_M - G_M) + (1-b)\,(Y_L - G_L)\right] + \\ \left[\beta(1-\mu) + \kappa\sigma\right](1-\mu)\left[b\pi_M + (1-b)\,\pi_L\right] \end{array}\right\} \quad \text{(B.19)}$$

Substitute back into $(B.17)$:

$$\begin{aligned} Y_S &= \frac{\sigma}{(1-\mu)} r_S^e + G_S + b\,(Y_M - G_M) + (1-b)\,(Y_L - G_L) + \sigma\left[b\pi_M + (1-b)\,\pi_L\right] + \\ &\quad \frac{\mu\sigma}{(1-\mu)}\frac{1}{(1-\beta\mu)(1-\mu) - \mu\kappa\sigma}\left\{\begin{array}{l} \kappa\sigma r_S^e + (1-\mu)\,\kappa\psi\chi^I\tau_S^I + (1-\mu)\left(1-\psi\sigma^{-1}\right)\kappa G_S + \\ (1-\mu)\,\kappa\left[b\,(Y_M - G_M) + (1-b)\,(Y_L - G_L)\right] + \\ \left[\beta(1-\mu) + \kappa\sigma\right](1-\mu)\left[b\pi_M + (1-b)\,\pi_L\right] \end{array}\right\} \end{aligned}$$

$$Y_S = \frac{1}{(1-\beta\mu)(1-\mu) - \mu\kappa\sigma}\left\{\begin{array}{l} (1-\beta\mu)\,\sigma r_S^e + \left[(1-\beta\mu)(1-\mu) - \mu\kappa\psi\right] G_S + \mu\kappa\sigma\psi\chi^I\tau_S^I + \\ (1-\beta\mu)(1-\mu)\left[b\,(Y_M - G_M) + (1-b)\,(Y_L - G_L)\right] + \\ (1-\mu)\,\sigma\left[b\pi_M + (1-b)\,\pi_L\right] \end{array}\right\} \quad \text{(B.20)}$$

Again, since we already solved for the medium and long-run allocations (Y_M, π_M, Y_L, π_L), we can use them in equations $(B.19)$ and $(B.20)$ to obtain the short-run solutions. But I want to express these relations in terms of the fiscal and monetary instruments used in each state. Thus, defining $\Gamma_{\mu\sigma} \equiv (1-\beta\mu)(1-\mu) - \mu\kappa\sigma$ and plugging the long- and medium-run relations $(B.9)$, $(B.10)$, $(B.15)$ and $(B.16)$, into the crisis state output equation $(B.20)$, we obtain:

$$
\begin{aligned}
Y_S \;=\;& \frac{1}{\Gamma_{\mu\sigma}}\left\{(1-\beta\mu)\,\sigma r_S^e + \left[(1-\beta\mu)(1-\mu) - \mu\kappa\psi\right] G_S + \mu\kappa\sigma\psi\chi^I\tau_S^I\right\} + \\
& \frac{(1-\beta\mu)(1-\mu)}{\Gamma_{\mu\sigma}}\left\{\begin{array}{l} b\left(\left[\begin{array}{lll} \Omega_{Y_M,i_M}(\bar{r}-i_M)+ & \Omega_{Y_M,G_M}G_M+ & \Omega_{Y_M,\tau_M^I}\tau_M^I+ \\ & \Omega_{Y_M,G_L}G_L+ & \Omega_{Y_M,\tau_L^I}\tau_L^I \end{array}\right] - G_M\right) + \\ (1-b)\left(\Omega_{Y_L,G_L}G_L + \Omega_{Y_L,\tau_L^I}\tau_L^I - G_L\right) \end{array}\right\} + \\
& \frac{(1-\mu)\,\sigma}{\Gamma_{\mu\sigma}}\left\{\begin{array}{l} b\left[\begin{array}{lll} \Omega_{\pi_M,i_M}(\bar{r}-i_M)+ & \Omega_{\pi_M,G_M}G_M+ & \Omega_{\pi_M,\tau_M^I}\tau_M^I+ \\ & \Omega_{\pi_M,G_L}G_L+ & \Omega_{\pi_M,\tau_L^I}\tau_L^I \end{array}\right] + \\ (1-b)\left(\Omega_{\pi_L,G_L}G_L + \Omega_{\pi_L,\tau_L^I}\tau_L^I\right) \end{array}\right\}
\end{aligned}
$$

$$
\begin{aligned}
Y_S \;=\;& \frac{(1-\beta\mu)\,\sigma}{\Gamma_{\mu\sigma}} r_S^e + \frac{\left[(1-\beta\mu)(1-\mu)-\mu\kappa\psi\right]}{\Gamma_{\mu\sigma}} G_S + \frac{\mu\kappa\sigma\psi\chi^I}{\Gamma_{\mu\sigma}}\tau_S^I + \\
& \frac{(1-\mu)\,b\left[(1-\beta\mu)\,\Omega_{Y_M,i_M} + \sigma\Omega_{\pi_M,i_M}\right]}{\Gamma_{\mu\sigma}}(\bar{r}-i_M) + \\
& \frac{(1-\mu)\,b}{\Gamma_{\mu\sigma}}\left[(1-\beta\mu)\left(\Omega_{Y_M,G_M}-1\right) + \sigma\Omega_{\pi_M,G_M}\right] G_M + \\
& \frac{(1-\mu)\,b}{\Gamma_{\mu\sigma}}\left[(1-\beta\mu)\,\Omega_{Y_M,\tau_M^I} + \sigma\Omega_{\pi_M,\tau_M^I}\right]\tau_M^I + \\
& \frac{(1-\mu)}{\Gamma_{\mu\sigma}}\left\{\begin{array}{l} (1-\beta\mu)\left[b\Omega_{Y_M,G_L} + (1-b)\left(\Omega_{Y_L,G_L}-1\right)\right] + \\ \sigma\left[b\Omega_{\pi_M,G_L} + (1-b)\,\Omega_{\pi_L,G_L}\right] \end{array}\right\} G_L + \\
& \frac{(1-\mu)}{\Gamma_{\mu\sigma}}\left\{\begin{array}{c} (1-\beta\mu)\left[b\Omega_{Y_M,\tau_L^I} + (1-b)\,\Omega_{Y_L,\tau_L^I}\right] + \\ \sigma\left[b\Omega_{\pi_M,\tau_L^I} + (1-b)\,\Omega_{\pi_L,\tau_L^I}\right] \end{array}\right\}\tau_L^I
\end{aligned}
$$

which is summarized as:

$$
\boxed{\begin{array}{llllllll} Y_S = & \Omega_{Y_S,r_S^e} r_S^e & + & \Omega_{Y_S,G_S}G_S & + & \Omega_{Y_S,\tau_S^I}\tau_S^I & + \\ & \Omega_{Y_S,i_M}(\bar{r}-i_M) & + & \Omega_{Y_S,G_M}G_M & + & \Omega_{Y_S,\tau_M^I}\tau_M^I & + \\ & & & \Omega_{Y_S,G_L}G_L & + & \Omega_{Y_S,\tau_L^I}\tau_L^I & \end{array}} \tag{B.21}
$$

where $\Omega_{Y_S,r_S^e} \equiv \dfrac{(1-\beta\mu)\,\sigma}{\Gamma_{\mu\sigma}}$, $\Omega_{Y_S,G_S} \equiv \dfrac{\left[(1-\beta\mu)(1-\mu)-\mu\kappa\psi\right]}{\Gamma_{\mu\sigma}}$, $\Omega_{Y_S,\tau_S^I} \equiv \dfrac{\mu\kappa\sigma\psi\chi^I}{\Gamma_{\mu\sigma}}$,

$$\Omega_{Y_S,i_M} \equiv \frac{(1-\mu)\,b}{\Gamma_{\mu\sigma}}\left[(1-\beta\mu)\,\Omega_{Y_M,i_M} + \sigma\Omega_{\pi_M,i_M}\right],$$

$$\Omega_{Y_S,G_M} \equiv \frac{(1-\mu)\,b}{\Gamma_{\mu\sigma}}\left[(1-\beta\mu)\left(\Omega_{Y_M,G_M}-1\right) + \sigma\Omega_{\pi_M,G_M}\right],$$

$$\Omega_{Y_S,\tau_M^I} \equiv \frac{(1-\mu)\,b}{\Gamma_{\mu\sigma}}\left[(1-\beta\mu)\,\Omega_{Y_M,\tau_M^I} + \sigma\Omega_{\pi_M,\tau_M^I}\right],$$

$$\Omega_{Y_S,G_L} \equiv \frac{(1-\mu)}{\Gamma_{\mu\sigma}}\left\{\begin{array}{l} (1-\beta\mu)\left[b\Omega_{Y_M,G_L} + (1-b)\left(\Omega_{Y_L,G_L}-1\right)\right] + \\ \sigma\left[b\Omega_{\pi_M,G_L} + (1-b)\,\Omega_{\pi_L,G_L}\right] \end{array}\right\} \text{ and}$$

$$\Omega_{Y_S,\tau_L^I} \equiv \frac{(1-\mu)}{\Gamma_{\mu\sigma}}\left\{\begin{array}{c} (1-\beta\mu)\left[b\Omega_{Y_M,\tau_L^I} + (1-b)\,\Omega_{Y_L,\tau_L^I}\right] + \\ \sigma\left[b\Omega_{\pi_M,\tau_L^I} + (1-b)\,\Omega_{\pi_L,\tau_L^I}\right] \end{array}\right\}.$$

Finally, plugging the long- and medium-run relations $(B.9)$, $(B.10)$, $(B.15)$ and $(B.16)$, into the crisis-state inflation equation $(B.19)$, we get:

$$\pi_S = \frac{1}{\Gamma_{\mu\sigma}} \left\{ \kappa\sigma r_S^e + (1-\mu)\left(1-\psi\sigma^{-1}\right)\kappa G_S + (1-\mu)\kappa\psi\chi^I \tau_S^I \right\}$$
$$\frac{(1-\mu) b\kappa}{\Gamma_{\mu\sigma}} \left(\left[\begin{array}{lll} \Omega_{Y_M,i_M}(\overline{r}-i_M)+ & \Omega_{Y_M,G_M} G_M + & \Omega_{Y_M,\tau_M^I}\tau_M^I + \\ & \Omega_{Y_M,G_L} G_L + & \Omega_{Y_M,\tau_L^I}\tau_L^I \end{array} \right] - G_M \right) +$$
$$\frac{(1-\mu)(1-b)\kappa}{\Gamma_{\mu\sigma}} \left(\Omega_{Y_L,G_L} G_L + \Omega_{Y_L,\tau_L^I}\tau_L^I - G_L \right) +$$
$$\frac{[\beta(1-\mu)+\kappa\sigma](1-\mu) b}{\Gamma_{\mu\sigma}} \left[\begin{array}{lll} \Omega_{\pi_M,i_M}(\overline{r}-i_M)+ & \Omega_{\pi_M,G_M} G_M + & \Omega_{\pi_M,\tau_M^I}\tau_M^I + \\ & \Omega_{\pi_M,G_L} G_L + & \Omega_{\pi_M,\tau_L^I}\tau_L^I \end{array} \right] +$$
$$\frac{[\beta(1-\mu)+\kappa\sigma](1-\mu)(1-b)}{\Gamma_{\mu\sigma}} \left(\Omega_{\pi_L,G_L} G_L + \Omega_{\pi_L,\tau_L^I}\tau_L^I \right)$$

$$\begin{aligned} \pi_S = \; & \frac{\kappa\sigma}{\Gamma_{\mu\sigma}} r_S^e + \frac{(1-\mu)\left(1-\psi\sigma^{-1}\right)\kappa}{\Gamma_{\mu\sigma}} G_S + \frac{(1-\mu)\kappa\psi\chi^I}{\Gamma_{\mu\sigma}} \tau_S^I + \\ & \frac{(1-\mu) b}{\Gamma_{\mu\sigma}} \left[\kappa\Omega_{Y_M,i_M} + [\beta(1-\mu)+\kappa\sigma]\Omega_{\pi_M,i_M} \right](\overline{r}-i_M) + \\ & \frac{(1-\mu) b}{\Gamma_{\mu\sigma}} \left[\kappa\left(\Omega_{Y_M,G_M}-1\right) + [\beta(1-\mu)+\kappa\sigma]\Omega_{\pi_M,G_M} \right] G_M + \\ & \frac{(1-\mu) b}{\Gamma_{\mu\sigma}} \left[\kappa\Omega_{Y_M,\tau_M^I}\tau_M^I + [\beta(1-\mu)+\kappa\sigma]\Omega_{\pi_M,\tau_M^I} \right] \tau_M^I + \\ & \frac{(1-\mu)}{\Gamma_{\mu\sigma}} \left\{ \begin{array}{l} \kappa\left[b\Omega_{Y_M,G_L} + (1-b)\left(\Omega_{Y_L,G_L}-1\right)\right] + \\ {[\beta(1-\mu)+\kappa\sigma]}\left[b\Omega_{\pi_M,G_L} + (1-b)\Omega_{\pi_L,G_L}\right] \end{array} \right\} G_L \\ & \frac{(1-\mu)}{\Gamma_{\mu\sigma}} \left\{ \begin{array}{l} \kappa\left[b\Omega_{Y_M,\tau_L^I} + (1-b)\Omega_{Y_L,\tau_L^I}\right] + \\ {[\beta(1-\mu)+\kappa\sigma]}\left[b\Omega_{\pi_M,\tau_L^I} + (1-b)\Omega_{\pi_L,\tau_L^I}\right] \end{array} \right\} \tau_L^I \end{aligned}$$

which can be summarized as:

$$\boxed{\begin{array}{lllllll} \pi_S = & \Omega_{\pi_S,r_S^e} r_S^e & + & \Omega_{\pi_S,G_S} G_S & + & \Omega_{\pi_S,\tau_S^I}\tau_S^I & + \\ & \Omega_{\pi_S,i_M}(\overline{r}-i_M) & + & \Omega_{\pi_S,G_M} G_M & + & \Omega_{\pi_S,\tau_M^I}\tau_M^I & + \\ & & & \Omega_{\pi_S,G_L} G_L & + & \Omega_{\pi_S,\tau_L^I}\tau_L^I & \end{array}} \tag{B.22}$$

where $\Omega_{\pi_S,r_S^e} \equiv \frac{\kappa\sigma}{\Gamma_{\mu\sigma}} r_S^e$, $\Omega_{\pi_S,G_S} \equiv \frac{(1-\mu)\left(1-\psi\sigma^{-1}\right)\kappa}{\Gamma_{\mu\sigma}}$, $\Omega_{\pi_S,\tau_S^I} \equiv \frac{(1-\mu)\kappa\psi\chi^I}{\Gamma_{\mu\sigma}}$,

$\Omega_{\pi_S,i_M} \equiv \frac{(1-\mu) b}{\Gamma_{\mu\sigma}} \left[\kappa\Omega_{Y_M,i_M} + [\beta(1-\mu)+\kappa\sigma]\Omega_{\pi_M,i_M}\right]$,

$\Omega_{\pi_S,G_M} \equiv \frac{(1-\mu) b}{\Gamma_{\mu\sigma}} \left[\kappa\left(\Omega_{Y_M,G_M}-1\right) + [\beta(1-\mu)+\kappa\sigma]\Omega_{\pi_M,G_M}\right]$,

$\Omega_{\pi_S,\tau_M^I} \equiv \frac{(1-\mu) b}{\Gamma_{\mu\sigma}} \left[\kappa\Omega_{Y_M,\tau_M^I}\tau_M^I + [\beta(1-\mu)+\kappa\sigma]\Omega_{\pi_M,\tau_M^I}\right]$,

$\Omega_{\pi_S,G_L} \equiv \frac{(1-\mu)}{\Gamma_{\mu\sigma}} \left\{ \begin{array}{l} \kappa\left[b\Omega_{Y_M,G_L} + (1-b)\left(\Omega_{Y_L,G_L}-1\right)\right] + \\ {[\beta(1-\mu)+\kappa\sigma]}\left[b\Omega_{\pi_M,G_L} + (1-b)\Omega_{\pi_L,G_L}\right] \end{array} \right\}$ and

$\Omega_{\pi_S,\tau_L^I} \equiv \frac{(1-\mu)}{\Gamma_{\mu\sigma}} \left\{ \begin{array}{l} \kappa\left[b\Omega_{Y_M,\tau_L^I} + (1-b)\Omega_{Y_L,\tau_L^I}\right] + \\ {[\beta(1-\mu)+\kappa\sigma]}\left[b\Omega_{\pi_M,\tau_L^I} + (1-b)\Omega_{\pi_L,\tau_L^I}\right] \end{array} \right\}$. ■

Proof. (Proposition 1 - Part III - ZLB is binding) To find out the condition under which a shock $r_t^e = r_S^e$ makes the zero lower bound binding in the short run, we need to verify the combination of parameters and policy choices that would imply a negative nominal interest rate during the crisis if the monetary authority uses rule (3.3). Eggertsson (2011b) derives this condition under the assumption that once the shock is no longer active ($r_t^e = \bar{r}$), the economy goes back to the long-run $\left(\hat{Y}_t = \pi_t = 0\right)$.

However, I want to obtain a general condition that accounts for the effects that future expected policies have on short-run output and inflation. Hence I use the medium and long-run solution allocations derived in Part II of this proof, where the $\Omega's$ are defined.

It is then necessary to solve for the short-run allocations using the system formed by the (IS) and aggregate supply equations, together with the monetary policy rule:

$$\begin{aligned}
(IS) \quad \hat{Y}_t &= E_t\hat{Y}_{t+1} - \sigma\left(i_t - E_t\pi_{t+1} - r_t^e\right) + \left(\hat{G}_t - E_t\hat{G}_{t+1}\right) \\
(AS) \quad \pi_t &= \kappa\hat{Y}_t + \kappa\psi\left(\chi^I\hat{\tau}_t^I - \sigma^{-1}\hat{G}_t\right) + \beta E_t\pi_{t+1} \\
(MP) \quad i_t &= \max\left\{0, r_t^e + \phi_\pi\pi_t + \phi_y\hat{Y}_t\right\}
\end{aligned}$$

given the expectations regarding future values of inflation, output and government spending:

$$\begin{aligned}
E_t\hat{Y}_{t+1} &= \mu Y_S + (1-\mu)\,bY_M + (1-\mu)(1-b)\,Y_L \\
E_t\pi_{t+1} &= \mu\pi_S + (1-\mu)\,b\pi_M + (1-\mu)(1-b)\,\pi_L \\
E_t\hat{G}_{t+1} &= \mu G_S + (1-\mu)\,bG_M + (1-\mu)(1-b)\,G_L
\end{aligned}$$

Substituting these expectations and the monetary policy rule into the (IS) equation, we obtain the aggregate demand relation:

$$\begin{aligned}
Y_S = {} & \mu Y_S + (1-\mu)\,bY_M + (1-\mu)(1-b)\,Y_L - \sigma\left(r_S^e + \phi_\pi\pi_S + \phi_y Y_S\right) + \\
& \sigma\left(\mu\pi_S + (1-\mu)\,b\pi_M + (1-\mu)(1-b)\,\pi_L\right) + \sigma r_S^e + G_S \\
& - \left(\mu G_S + (1-\mu)\,bG_M + (1-\mu)(1-b)\,G_L\right)
\end{aligned}$$

$$\begin{aligned}
Y_S = {} & -\frac{(\phi_\pi - \mu)}{\left(1-\mu+\sigma\phi_y\right)}\sigma\pi_S + \frac{(1-\mu)}{\left(1-\mu+\sigma\phi_y\right)}G_S + \frac{(1-\mu)\,b}{\left(1-\mu+\sigma\phi_y\right)}\left(Y_M - G_M\right) + \\
& \frac{(1-\mu)\,b\sigma}{\left(1-\mu+\sigma\phi_y\right)}\pi_M + \frac{(1-\mu)(1-b)}{\left(1-\mu+\sigma\phi_y\right)}\left(Y_L - G_L\right) + \frac{(1-\mu)(1-b)\,\sigma}{\left(1-\mu+\sigma\phi_y\right)}\pi_L
\end{aligned}$$

Doing the same for the aggregate supply equation:

$$\pi_S = \kappa Y_S + \kappa\psi\left(\chi^I\tau_S^I - \sigma^{-1}G_S\right) + \beta\left(\mu\pi_S + (1-\mu)\,b\pi_M + (1-\mu)(1-b)\,\pi_L\right)$$

$$\pi_S = \frac{\kappa}{(1-\beta\mu)}Y_S + \frac{\kappa\psi}{(1-\beta\mu)}\left(\chi^I\tau_S^I - \sigma^{-1}G_S\right) + \frac{\beta(1-\mu)\,b}{(1-\beta\mu)}\pi_M + \frac{\beta(1-\mu)(1-b)}{(1-\beta\mu)}\pi_L$$

Substituting the aggregate demand into the aggregate supply we obtain an expression for the short-run inflation:

$$\begin{aligned}
\pi_S = {} & \frac{\kappa}{(1-\beta\mu)}\left[\begin{array}{l}
-\frac{(\phi_\pi - \mu)}{\left(1-\mu+\sigma\phi_y\right)}\sigma\pi_S + \frac{(1-\mu)}{\left(1-\mu+\sigma\phi_y\right)}G_S + \\
\frac{(1-\mu)\,b}{\left(1-\mu+\sigma\phi_y\right)}\left(Y_M - G_M\right) + \frac{(1-\mu)\,b\sigma}{\left(1-\mu+\sigma\phi_y\right)}\pi_M + \\
+\frac{(1-\mu)(1-b)}{\left(1-\mu+\sigma\phi_y\right)}\left(Y_L - G_L\right) + \frac{(1-\mu)(1-b)\,\sigma}{\left(1-\mu+\sigma\phi_y\right)}\pi_L
\end{array}\right] + \\
& \frac{\kappa\psi}{(1-\beta\mu)}\left(\chi^I\tau_S^I - \sigma^{-1}G_S\right) + \frac{\beta(1-\mu)\,b}{(1-\beta\mu)}\pi_M + \frac{\beta(1-\mu)(1-b)}{(1-\beta\mu)}\pi_L
\end{aligned}$$

$$\pi_S = \frac{(1-\mu)\left(1-\psi\sigma^{-1}\right)-\phi_y\psi}{\Gamma_{\mu\phi}}\kappa G_S + \frac{\left(1-\mu+\sigma\phi_y\right)\kappa\psi}{\Gamma_{\mu\phi}}\chi^I\tau_S^I + \frac{(1-\mu)\kappa}{\Gamma_{\mu\phi}}\left[b\left(Y_M-G_M\right)+(1-b)\left(Y_L-G_L\right)\right] + \frac{(1-\mu)\left[\beta\left(1-\mu+\sigma\phi_y\right)+\kappa\sigma\right]}{\Gamma_{\mu\phi}}\left[b\pi_M+(1-b)\pi_L\right] \tag{B.23}$$

where $\Gamma_{\mu\phi} \equiv (1-\beta\mu)\left(1-\mu+\sigma\phi_y\right)+(\phi_\pi-\mu)\kappa\sigma$.

Plugging back into the aggregate demand equation we obtain an expression for the short-run output:

$$Y_S = -\frac{(\phi_\pi-\mu)}{\left(1-\mu+\sigma\phi_y\right)}\sigma\left[\begin{array}{l}\frac{(1-\mu)\left(1-\psi\sigma^{-1}\right)-\phi_y\psi}{\Gamma_{\mu\phi}}\kappa G_S + \frac{\left(1-\mu+\sigma\phi_y\right)\kappa\psi}{\Gamma_{\mu\phi}}\chi^I\tau_S^I + \\ \frac{(1-\mu)\kappa}{\Gamma_{\mu\phi}}\left[b\left(Y_M-G_M\right)+(1-b)\left(Y_L-G_L\right)\right] + \\ \frac{(1-\mu)\left[\beta\left(1-\mu+\sigma\phi_y\right)+\kappa\sigma\right]}{\Gamma_{\mu\phi}}\left[b\pi_M+(1-b)\pi_L\right]\end{array}\right] + \frac{(1-\mu)}{\left(1-\mu+\sigma\phi_y\right)}G_S + \frac{(1-\mu)}{\left(1-\mu+\sigma\phi_y\right)}\left[b\left(Y_M-G_M\right)+(1-b)\left(Y_L-G_L\right)\right] + \frac{(1-\mu)\sigma}{\left(1-\mu+\sigma\phi_y\right)}\left[b\pi_M+(1-b)\pi_L\right]$$

$$Y_S = \frac{\left[(1-\mu)(1-\beta\mu)+(\phi_\pi-\mu)\kappa\psi\right]}{\Gamma_{\mu\phi}}G_S - \frac{(\phi_\pi-\mu)\kappa\sigma\psi}{\Gamma_{\mu\phi}}\chi^I\tau_S^I + \frac{(1-\beta\mu)(1-\mu)}{\Gamma_{\mu\phi}}\left[b\left(Y_M-G_M\right)+(1-b)\left(Y_L-G_L\right)\right] + \frac{(1-\beta\phi_\pi)(1-\mu)\sigma}{\Gamma_{\mu\phi}}\left[b\pi_M+(1-b)\pi_L\right] \tag{B.24}$$

Plugging both short-run output and inflation into monetary policy rule (3.3) , we have that the implied nominal interest rate is given by:

$$i_S = r_S^e + \frac{(1-\mu)\left(1-\psi\sigma^{-1}\right)\phi_\pi\kappa+\left[(1-\mu)(1-\beta\mu)-\mu\kappa\psi\right]\phi_y}{\Gamma_{\mu\phi}}G_S + \frac{(1-\mu)\phi_\pi+\mu\phi_y\sigma}{\Gamma_{\mu\phi}}\kappa\psi\chi^I\tau_S^I + \frac{(1-\mu)\left[\phi_\pi\kappa+(1-\beta\mu)\phi_y\right]}{\Gamma_{\mu\phi}}\left[b\left(Y_M-G_M\right)+(1-b)\left(Y_L-G_L\right)\right] + \frac{(1-\mu)\left[\phi_y\sigma+\left(\beta(1-\mu)+\kappa\sigma\right)\phi_\pi\right]}{\Gamma_{\mu\phi}}\left[b\pi_M+(1-b)\pi_L\right] \tag{B.25}$$

Substituting the expressions for output and inflation in the medium and long runs derived in Part II of this proof (equations $(B.9)$, $(B.10)$, $(B.15)$ and $(B.16)$), we obtain that the nominal interest rate implied by the monetary policy rule is negative, and the zero lower bound is binding, if the shock respects the following condition:

$$\boxed{r_S^e < -\Theta_{G_S}G_S - \Theta_{\tau_S^I}\tau_S^I - \Theta_{\bar{r}}\left(\bar{r}-i_M\right) - \Theta_{G_M}G_M - \Theta_{\tau_M^I}\tau_M^I - \Theta_{G_L}G_L - \Theta_{\tau_L^I}\tau_L^I} \quad \text{(Condition } (C4))$$

where the coefficients are given by

$$\Theta_{G_S} \equiv \frac{(1-\mu)\left(1-\psi\sigma^{-1}\right)\phi_\pi\kappa+\left[(1-\mu)(1-\beta\mu)-\mu\kappa\psi\right]\phi_y}{\Gamma_{\mu\phi}},$$

$$\Theta_{\tau_S^I} \equiv \frac{(1-\mu)\,\phi_\pi + \mu\phi_y\sigma}{\Gamma_{\mu\phi}}\kappa\psi\chi^I,\ \Theta_{\bar{r}} \equiv \frac{(1-\mu)\,b}{\Gamma_{\mu\phi}}\left\{\begin{array}{c}\left[\phi_\pi\kappa + (1-\beta\mu)\,\phi_y\right]\Omega_{Y_M,i_M} + \\ \left[\phi_y\sigma + (\beta(1-\mu)+\kappa\sigma)\,\phi_\pi\right]\Omega_{\pi_M,i_M}\end{array}\right\},$$

$$\Theta_{G_M} \equiv \frac{(1-\mu)\,b}{\Gamma_{\mu\phi}}\left\{\begin{array}{c}\left[\phi_\pi\kappa + (1-\beta\mu)\,\phi_y\right](\Omega_{Y_M,G_M}-1) + \\ \left[\phi_y\sigma + (\beta(1-\mu)+\kappa\sigma)\,\phi_\pi\right]\Omega_{\pi_M,G_M}\end{array}\right\},$$

$$\Theta_{\tau_M^I} \equiv \frac{(1-\mu)\,b}{\Gamma_{\mu\phi}}\left\{\begin{array}{c}\left[\phi_\pi\kappa + (1-\beta\mu)\,\phi_y\right]\Omega_{Y_M,\tau_M^I} + \\ \left[\phi_y\sigma + (\beta(1-\mu)+\kappa\sigma)\,\phi_\pi\right]\Omega_{\pi_M,\tau_M^I}\end{array}\right\},$$

$$\Theta_{G_L} \equiv \frac{(1-\mu)}{\Gamma_{\mu\phi}}\left\{\begin{array}{c}\left[\phi_\pi\kappa + (1-\beta\mu)\,\phi_y\right]\left[b\Omega_{Y_M,G_L} + (1-b)(\Omega_{Y_L,G_L}-1)\right] + \\ \left[\phi_y\sigma + (\beta(1-\mu)+\kappa\sigma)\,\phi_\pi\right]\left[b\Omega_{\pi_M,G_L} + (1-b)\,\Omega_{\pi_L,G_L}\right]\end{array}\right\} \text{ and}$$

$$\Theta_{\tau_L^I} \equiv \frac{(1-\mu)}{\Gamma_{\mu\phi}}\left\{\begin{array}{c}\left[\phi_\pi\kappa + (1-\beta\mu)\,\phi_y\right]\left[b\Omega_{Y_M,\tau_L^I} + (1-b)\,\Omega_{Y_L,\tau_L^I}\right] + \\ \left[\phi_y\sigma + (\beta(1-\mu)+\kappa\sigma)\,\phi_\pi\right]\left[b\Omega_{\pi_M,\tau_L^I} + (1-b)\,\Omega_{\pi_L,\tau_L^I}\right]\end{array}\right\}. \blacksquare$$

Proposition 2 *Assume that the nominal interest rate zero lower bound is binding in the short run* ($t \in [T_0, T_{exit})$) *and that the following conditions hold:*

$(C1)\quad (1-\mu)(1-\beta\mu) - \mu\kappa\sigma > 0$

$(C2)\quad (1-b)(1-\beta b) - b\kappa\sigma > 0$

$(C3)\quad \phi_\pi + \frac{(1-\beta)}{\kappa}\phi_y > 1$

$(C4')\quad r_S^e < -\Theta_{G_S}G_S - \Theta_{\tau_S^I}\tau_S^I - \Theta_{G_M}^T G_M - \Theta_{\tau_M^I}^T \tau_M^I$

If there is a transitional state generated by the fiscal authority keeping the stimulus provided in the short run for a few periods after the crisis is over ($\forall t \in [T_{exit}, T_M)$)*, while monetary policy returns to rule* (3.3) *as soon as* r_t^e *returns to* $\bar{r}$*, the solutions in each state can be obtained backward as follows:*

(i) In the long run, there is a locally unique bounded solution ($\forall t > T_M$)*, with* $r_t^e = \bar{r}$*, such that* $i_L = \bar{r}$ *and* $Y_L = \pi_L = 0$*.*

(ii) There is a locally unique bounded medium-run solution ($\forall t \in [T_{exit}, T_M)$)*, with* $r_t^e = \bar{r}$*, such that*

$$Y_M^T = \Omega_{Y_M,G_M}^T G_M + \Omega_{Y_M,\tau_M^I}^T \tau_M^I \quad (4.10)$$

$$\pi_M^T = \Omega_{\pi_M,G_M}^T G_M + \Omega_{\pi_M,\tau_M^I}^T \tau_M^I \quad (4.11)$$

$$i_M^T = \bar{r} + \phi_\pi \pi_M^T + \phi_y Y_M^T \quad (4.12)$$

(ii) In the short run ($\forall t \in [T_0, T_{exit})$)*, with* $r_t^e = r_S^e$*, there is a locally unique bounded solution, such that*

$$Y_S^T = \Omega_{Y_S,r_S^e} r_S^e + \Omega_{Y_S,G_S} G_S + \Omega_{Y_S,\tau_S^I}\tau_S^I + \Omega_{Y_S,G_M}^T G_M + \Omega_{Y_S,\tau_M^I}^T \tau_M^I \quad (4.13)$$

$$\pi_S^T = \Omega_{\pi_S,r_S^e} r_S^e + \Omega_{\pi_S,G_S} G_S + \Omega_{\pi_S,\tau_S^I}\tau_S^I + \Omega_{\pi_S,G_M}^T G_M + \Omega_{\pi_S,\tau_M^I}^T \tau_M^I \quad (4.14)$$

$$i_S = 0 \quad (4.15)$$

where the analytical expressions for the coefficients $\Omega_{i,j}$*,* $i \in \{Y_S, \pi_S\}$ *and* $j \in \{r_S^e, G_S, \tau_S^I\}$*, and* Θ_k*,* $k \in \{G_S, \tau_S^I\}$*, are the same as those defined in Proposition 1. The expressions for the coefficients* $\Omega_{m,n}^T$*,* $m \in \{Y_S, \pi_S, Y_M, \pi_M\}$ *and* $n \in \{G_M, \tau_M^I\}$*, and* Θ_h^T*,* $k \in \{G_M, \tau_M^I\}$*, are defined in the Appendix and depend on the structural parameters.*

The proof of this proposition follows the same procedure adopted for Proposition 1. It is divided into three parts. The first part discusses solutions' determinacy in each state. The second part derives the allocations for output and inflation in each state. Finally, the third part derives the condition under which the zero lower bound is binding in the short run.

Proof. (Proposition 2 - Part I - Determinacy) The derivation of the conditions for determinacy and existence of a solution in this model is analogous to that presented in Proposition 1. We will have the crisis state ($s_t = 1$), where the shock hits the economy ($r_1^e = r_S^e$) and the zero lower bound is binding ($i_1 = 0$), and the transitional state ($s_t = 2$), where the shock is no longer active ($r_2^e = \bar{r}$), but the fiscal authority keeps the stimulus implemented during the crisis, while monetary policy goes back to following the rule ($i_2 = \bar{r} + \phi_\pi \pi_2 + \phi_x Y_2$). In the long run ($s_t = 3$), fiscal instruments return to their steady states $\left(\hat{G}_t = \hat{\tau}_t^I = 0, \forall t > T_M\right)$ implying $Y_L = \pi_L = 0$, which simplifies the derivations.

The solutions is then defined by the set of equations formed, respectively, by the (IS) equation (3.1), the aggregate supply relation (3.2) and the monetary policy rule (3.3):

$$\hat{Y}_t = E_t\hat{Y}_{t+1} - \sigma\left(i_t - E_t\pi_{t+1} - r_t^e\right) + \left(\hat{G}_t - E_t\hat{G}_{t+1}\right) \tag{B.26}$$

$$\pi_t = \kappa\hat{Y}_t + \kappa\psi\left(\chi^I\hat{\tau}_t^I - \sigma^{-1}\hat{G}_t\right) + \beta E_t\pi_{t+1} \tag{B.27}$$

$$i_t = \begin{cases} 0, & s_t = 1\ (Short\ \ run) \\ \bar{r} + \phi_\pi\pi_2 + \phi_y Y_2, & s_t = 2\ (Medium\ \ run) \\ \bar{r}, & s_t = 3\ (Long\ \ run) \end{cases} \tag{B.28}$$

The probability transition matrix is the same as in $(B.4)$:

$$\Pi \equiv \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \begin{bmatrix} \mu & (1-\mu)\,b & (1-\mu)\,(1-b) \\ 0 & b & (1-b) \\ 0 & 0 & 1 \end{bmatrix} \tag{B.29}$$

The state-contingent expectations with respect to output and inflation are given by:

$$\begin{aligned} E_t\hat{Y}_{t+1} &= E\left[Y_{t+1}|s_t = j, \Omega_t^{-s}\right] = p_{j1}E\left[Y_{1t+1}|\Omega_t^{-s}\right] + p_{j2}E\left[Y_{2t+1}|\Omega_t^{-s}\right] \\ E_t\pi_{t+1} &= E\left[\pi_{t+1}|s_t = j, \Omega_t^{-s}\right] = p_{j1}E\left[\pi_{1t+1}|\Omega_t^{-s}\right] + p_{j2}E\left[\pi_{2t+1}|\Omega_t^{-s}\right] \end{aligned}$$

Expectations regarding government spending are given by:

$$E_t\hat{G}_{t+1} = E\left[G_{t+1}|s_t = j, \Omega_t^{-s}\right] = p_{j1}G_1 + p_{j2}G_2$$

The (IS) and (AS) equations can be written as:

$$\begin{aligned} (IS_j) \quad Y_{jt} = \; & p_{j1}E_tY_{1t+1} + p_{j2}E_tY_{2t+1} + \sigma\left(p_{j1}E_t\pi_{1t+1} + p_{j2}E_t\pi_{2t+1}\right) + \\ & \left(p_{j1}G_1 + p_{j2}G_2\right) - G_j - \sigma i_{jt} + \sigma r_{jt}^e \\ (AS_j) \quad \pi_{jt} = \; & \beta\left(p_{j1}E_t\pi_{1t+1} + p_{j2}E_t\pi_{2t+1}\right) + \kappa Y_{jt} + \kappa\psi\left(\chi^I\tau_j^I - \sigma^{-1}G_j\right) \end{aligned}$$

where we already used $E_tY_{3t+1} = E_t\pi_{3t+1} = 0$.Define state-contingent forecast errors for each state j:

$$\begin{aligned} \eta_{jt+1}^\pi &= \pi_{jt+1} - E_t\pi_{jt+1} \Rightarrow E_t\pi_{jt+1} = \pi_{jt+1} - \eta_{jt+1}^\pi \\ \eta_{jt+1}^Y &= Y_{jt+1} - E_tY_{jt+1} \Rightarrow E_tY_{jt+1} = Y_{jt+1} - \eta_{jt+1}^Y \end{aligned}$$

and use them to eliminate the conditional expectations and rewrite (IS_j) and (AS_j) as:

$$\begin{array}{lll} (IS_j) & Y_{jt} = & p_{j1}\left(Y_{1t+1}-\eta^Y_{1t+1}\right)+p_{j2}\left(Y_{2t+1}-\eta^Y_{2t+1}\right)+\sigma\left(p_{j1}\left(\pi_{1t+1}-\eta^\pi_{1t+1}\right)+p_{j2}\left(\pi_{2t+1}-\eta^\pi_{2t+1}\right)\right)+ \\ & & \left(p_{j1}G_1+p_{j2}G_2\right)-G_j-\sigma i_{jt}+\sigma r^e_{jt} \\ (AS_j) & \pi_{jt} = & \beta p_{j1}\left(\pi_{1t+1}-\eta^\pi_{1t+1}\right)+\beta p_{j2}\left(\pi_{2t+1}-\eta^\pi_{2t+1}\right)+\kappa Y_{jt}+\kappa\psi\left(\chi^I\tau^I_j-\sigma^{-1}G_j\right) \end{array}$$

Hence, we can write the system of equations $(B.26)-(B.28)$ for each state, incorporating the monetary policy stance into the (IS) equation to get the aggregate demand relation. In the first state (short run) we have:

$$\begin{array}{ll} (AD_1) & p_{11}Y_{1t+1}+p_{12}Y_{2t+1}+\sigma\left(p_{11}\pi_{1t+1}+p_{12}\pi_{2t+1}\right)=Y_{1t}+\left(p_{11}\eta^Y_{1t+1}+p_{12}\eta^Y_{2t+1}\right)+ \\ & \sigma\left(p_{11}\eta^\pi_{1t+1}+p_{12}\eta^\pi_{2t+1}\right)-\sigma r^e_S-\left[\left(p_{11}-1\right)G_1+p_{12}G_2\right] \\ (AS_1) & \beta\left(p_{11}\pi_{1t+1}+p_{12}\pi_{2t+1}\right)=\pi_{1t}-\kappa Y_{1t}+\beta\left(p_{11}\eta^\pi_{1t+1}+p_{12}\eta^\pi_{2t+1}\right)-\kappa\psi\left(\chi^I\tau^I_1-\sigma^{-1}G_1\right) \end{array}$$

In the second state (medium run) we have:

$$\begin{array}{ll} (AD_2) & p_{21}Y_{1t+1}+p_{22}Y_{2t+1}+\sigma\left(p_{21}\pi_{1t+1}+p_{22}\pi_{2t+1}\right)=\sigma\phi_\pi\pi_{2t}+\left(1+\sigma\phi_y\right)Y_{2t}+ \\ & \left(p_{21}\eta^Y_{1t+1}+p_{22}\eta^Y_{2t+1}\right)+\sigma\left(p_{21}\eta^\pi_{1t+1}+p_{22}\eta^\pi_{2t+1}\right)-\left[p_{21}G_1+\left(p_{22}-1\right)G_2\right] \\ (AS_2) & \beta\left(p_{21}\pi_{1t+1}+p_{22}\pi_{2t+1}\right)=\pi_{2t}-\kappa Y_{2t}+\beta\left(p_{21}\eta^\pi_{1t+1}+p_{22}\eta^\pi_{2t+1}\right)-\kappa\psi\left(\chi^I\tau^I_2-\sigma^{-1}G_2\right) \end{array}$$

The complete system can be expressed in matrix form

$$AX_{t+1}=BX_t+A\eta_t+Ce_t$$

$$X_t\equiv\begin{bmatrix}\pi_{1t}\\ Y_{1t}\\ \pi_{2t}\\ Y_{2t}\end{bmatrix}\qquad \eta_t\equiv\begin{bmatrix}\eta^\pi_{1t+1}\\ \eta^Y_{1t+1}\\ \eta^\pi_{2t+1}\\ \eta^Y_{2t+1}\end{bmatrix}\qquad e_t\equiv\begin{bmatrix}r^e_S\\ G_1\\ \tau^I_1\\ G_2\\ \tau^I_2\end{bmatrix}$$

$$A=\begin{bmatrix}\mu & (1-\mu)b\\ 0 & b\end{bmatrix}\otimes\begin{bmatrix}\beta & 0\\ \sigma & 1\end{bmatrix}\qquad B=\begin{bmatrix}1 & -\kappa & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -\kappa\\ 0 & 0 & \sigma\phi_\pi & \left(1+\sigma\phi_y\right)\end{bmatrix}$$

$$C\equiv\begin{bmatrix}0 & \kappa\psi\sigma^{-1} & -\kappa\psi\chi^I & 0 & 0\\ -\sigma & (1-\mu) & 0 & -(1-\mu)b & 0\\ 0 & 0 & 0 & \kappa\psi\sigma^{-1} & -\kappa\psi\chi^I\\ 0 & 0 & 0 & (1-b) & 0\end{bmatrix}$$

Since A is invertible, we can write the system as:

$$X_{t+1}=A^{-1}BX_t+\eta_t+A^{-1}Ce_t$$

where

$$A^{-1}B=\begin{bmatrix}\frac{1}{\beta\mu} & -\frac{\kappa}{\beta\mu} & -\frac{(1-\mu)}{\beta\mu} & \frac{(1-\mu)\kappa}{\beta\mu}\\ -\frac{\sigma}{\beta\mu} & \frac{\beta+\kappa\sigma}{\beta\mu} & \frac{\sigma(1-\mu)(1-\beta\phi_\pi)}{\beta\mu} & -\frac{(1-\mu)\left[\beta\left(1+\sigma\phi_y\right)+\kappa\sigma\right]}{\beta\mu}\\ 0 & 0 & \frac{1}{\beta b} & -\frac{\kappa}{\beta b}\\ 0 & 0 & \frac{\sigma(\beta\phi_\pi-1)}{\beta b} & \frac{\beta\left(1+\sigma\phi_y\right)+\kappa\sigma}{\beta b}\end{bmatrix}$$

Again, we can check if the eigenvalues of matrix $A^{-1}B$ are outside the unit circle by analyzing the eigenvalues of the block matrices Z_i in the diagonal.

$$Z_1 \equiv \begin{bmatrix} \frac{1}{\beta\mu} & -\frac{\kappa}{\beta\mu} \\ -\frac{\sigma}{\beta\mu} & \frac{\beta+\kappa\sigma}{\beta\mu} \end{bmatrix} \qquad Z_2 \equiv \begin{bmatrix} \frac{1}{\beta b} & -\frac{\kappa}{\beta b} \\ \frac{(\beta\phi_\pi - 1)}{\beta b} & \frac{\beta\left(1+\sigma\phi_y\right)+\kappa\sigma}{\beta b} \end{bmatrix}$$

Matrix Z_1 is the same from the proof of Part I of Proposition 1. Thus we need Condition $(C1)$ to hold to the short-run solution to be determinate. We can then use the result in Proposition C.1 from Woodford (2003) to verify the determinacy conditions for matrix Z_2, reminding that all parameters $\left(\beta,\kappa,\sigma,\mu,b,\phi_\pi,\phi_y\right)$ are positive. For matrix Z_2 we have:

$$tr\left(Z_2\right) = \frac{1+\beta\left(1+\sigma\phi_y\right)+\kappa\sigma}{\beta b} \qquad det\left(Z_2\right) = \frac{1+\sigma\left(\phi_y+\kappa\phi_\pi\right)}{\beta b^2}$$

Since all parameters are positive, the conditions in Case II of Woodford's proposition are not satisfied. So I verify the conditions in Case I:

$$(a)\, det\left(Z_2\right) = \frac{1+\sigma\left(\phi_y+\kappa\phi_\pi\right)}{\beta b^2} > 1$$

$$(b)\, det\left(Z_2\right) - tr\left(Z_2\right) > -1$$

$$\begin{aligned} det\left(Z_2\right) - tr\left(Z_2\right) &= \frac{1+\sigma\left(\phi_y+\kappa\phi_\pi\right)}{\beta b^2} - \frac{1+\beta\left(1+\sigma\phi_y\right)+\kappa\sigma}{\beta b} > -1 \\ &\Rightarrow \left[(1-b)(1-\beta b) - b\kappa\sigma\right] + \left[(1-\beta b)\sigma\phi_y + \kappa\sigma\phi_\pi\right] > 0 \end{aligned}$$

The first term in this expression is the one from Condition $(C2)$ which imposes it should be bigger than zero. The second part is similar to Condition $(C3)$, but not the same. But given that $\phi_\pi > 1$ and $\phi_y > 0$, $\kappa > 0$, $\beta \in [0,1]$ and $b \in [0,1]$, we have that the second term is also positive. In fact, it is easy to see that Condition $(C3)$ guarantees $\left[(1-\beta b)\sigma\phi_y + \kappa\sigma\phi_\pi\right] > 0$. Thus, if Conditions $(C2)$ and $(C3)$ hold, then $\left[det\left(Z_2\right) - tr\left(Z_2\right) > -1\right]$.

$$(c)\, det\left(Z_3\right) + tr\left(Z_3\right) > -1$$

$$det\left(Z_3\right) + tr\left(Z_3\right) = \frac{b + (1+\beta b)\left(1+\sigma\phi_y\right) + \left(\phi_\pi + b\right)\kappa\sigma}{\beta b^2} > -1$$

Hence, the conditions for Case I are satisfied for matrix Z_2 and we need Conditions $(C1)$ – $(C3)$ to hold to guarantee that the solution is determinate. ■

Proof. (Proposition 2 - Part II - Output and Inflation Allocations in Each State) Again we need to obtain solutions for output and inflation in each state, working backward from the long-run to the short-run allocations. As was done in Part II of the proof of Proposition 1, we need to solve the system formed by the (IS) and aggregate supply equations, together with the specifications for monetary and fiscal policies in each state.

Since I am assuming that the fiscal stimulus is temporary, the long-run solution is given by $Y_L = \pi_L = 0$. So we only need to solve for the short-run and medium-run allocations.

Medium-Run Solution Allocations (Transitional State)

Given that the monetary authority follows the rule (3.3) in the transitional state, and the zero lower bound is no longer binding, the medium-run nominal interest rate is given by

$i^T_M = \overline{r} + \phi_\pi \pi^T_M + \phi_y Y^T_M$, where $\left(Y^T_M, \pi^T_M\right)$ represent output and inflation in the transitional state under this rule. Expectations in the medium run are given by

$$\begin{aligned} E_t\hat{Y}_{t+1} &= bY^T_M + (1-b)\times 0 = bY^T_M \\ E_t\pi_{t+1} &= b\pi^T_M + (1-b)\times 0 = b\pi^T_M \\ E_t\hat{G}_{t+1} &= bG_M + (1-b)\times 0 = bG_M \end{aligned}$$

Using these expectations and the monetary policy rule into the (IS) equation, the aggregate demand equation becomes:

$$Y^T_M = bY^T_M + (G_M - bG_M) - \sigma\left(\overline{r} + \phi_\pi \pi^T_M + \phi_y Y^T_M - b\pi^T_M - \overline{r}\right)$$

$$\left(1 - b + \sigma\phi_y\right) Y^T_M = -\left(\phi_\pi - b\right)\sigma\pi^T_M + (1-b)\, G_M$$

$$Y^T_M = -\frac{(\phi_\pi - b)}{\left(1-b+\sigma\phi_y\right)}\sigma\pi^T_M + \frac{(1-b)}{\left(1-b+\sigma\phi_y\right)}G_M \tag{B.30}$$

while the aggregate supply is given by

$$\pi^T_M = \frac{\kappa}{(1-\beta b)}Y^T_M + \frac{\kappa\psi}{(1-\beta b)}\left(\chi^I\tau^I_M - \sigma^{-1}G_M\right) \tag{B.31}$$

Substituting the aggregate demand expression into the aggregate supply:

$$\pi^T_M = \frac{\kappa}{(1-\beta b)}\left[-\frac{(\phi_\pi - b)}{\left(1-b+\sigma\phi_y\right)}\sigma\pi^T_M + \frac{(1-b)}{\left(1-b+\sigma\phi_y\right)}G_M\right] + \frac{\kappa\psi}{(1-\beta b)}\left(\chi^I\tau^I_M - \sigma^{-1}G_M\right)$$

$$\pi^T_M = \frac{1}{(1-\beta b)\left(1-b+\sigma\phi_y\right) + (\phi_\pi - b)\kappa\sigma}\left\{\left[(1-b)\left(1-\psi\sigma^{-1}\right) - \psi\phi_y\right]\kappa G_M + \left(1-b+\sigma\phi_y\right)\kappa\psi\chi^I\tau^I_M\right\}$$

Substituting back into the aggregate demand equation:

$$\begin{aligned} Y^T_M \;=\; & -\frac{(\phi_\pi - b)\,\sigma}{\left(1-b+\sigma\phi_y\right)}\frac{1}{(1-\beta b)\left(1-b+\sigma\phi_y\right) + (\phi_\pi - b)\kappa\sigma}\left\{\begin{array}{l} \left[(1-b)\left(1-\psi\sigma^{-1}\right) - \psi\phi_y\right]\kappa G_M + \\ \left(1-b+\sigma\phi_y\right)\kappa\psi\chi^I\tau^I_M \end{array}\right\} + \\ & \frac{(1-b)}{\left(1-b+\sigma\phi_y\right)}G_M \end{aligned}$$

$$Y^T_M = \frac{1}{(1-\beta b)\left(1-b+\sigma\phi_y\right) + (\phi_\pi - b)\kappa\sigma}\left\{\begin{array}{l} \left[(1-b)(1-\beta b) + (\phi_\pi - b)\kappa\psi\right]G_M + \\ -(\phi_\pi - b)\kappa\sigma\psi\chi^I\tau^I_M \end{array}\right\}$$

Defining $\Gamma_{b\phi} \equiv (1-\beta b)\left(1-b+\sigma\phi_y\right) + (\phi_\pi - b)\kappa\sigma$, the medium-run solution allocations can be summarized as follows:

$$\boxed{Y^T_M = \Omega^T_{Y_M,G_M}G_M + \Omega^T_{Y_M,\tau^I_M}\tau^I_M} \tag{B.32}$$

where $\Omega^T_{Y_M,G_M} \equiv \dfrac{(1-b)(1-\beta b) + (\phi_\pi - b)\kappa\psi}{\Gamma_{b\phi}}$ and $\Omega^T_{Y_M,\tau^I_M} \equiv -\dfrac{(\phi_\pi - b)\kappa\sigma\psi\chi^I}{\Gamma_{b\phi}}$

$$\boxed{\pi^T_M = \Omega^T_{\pi_M,G_M}G_M + \Omega^T_{\pi_M,\tau^I_M}\tau^I_M} \tag{B.33}$$

where $\Omega^T_{\pi_M,G_M} \equiv \dfrac{\left[(1-b)\left(1-\psi\sigma^{-1}\right) - \psi\phi_y\right]\kappa}{\Gamma_{b\phi}}$ and $\Omega^T_{\pi_M,\tau^I_M} \equiv \dfrac{\left(1-b+\sigma\phi_y\right)\kappa\psi\chi^I}{\Gamma_{b\phi}}$

Short-Run Solution Allocations (Crisis State)

Now we can solve for the short-run allocations. First, we need to compute expectations during the crisis state:

$$\begin{aligned} E_t\hat{Y}_{t+1} &= \mu Y_S^T + (1-\mu)\, bY_M^T + (1-\mu)(1-b)\times 0 = \mu Y_S^T + (1-\mu)\, bY_M^T \\ E_t\pi_{t+1} &= \mu\pi_S^T + (1-\mu)\, b\pi_M^T + (1-\mu)(1-b)\times 0 = \mu\pi_S^T + (1-\mu)\, b\pi_M^T \\ E_t\hat{G}_{t+1} &= \mu G_S + (1-\mu)\, bG_M + (1-\mu)(1-b)\times 0 = \mu G_S + (1-\mu)\, bG_M \end{aligned}$$

The solution for output and inflation in the crisis state is analogous to equations $(B.19)$ and $(B.20)$ from Part II of Proposition 1, substituting $Y_L = \pi_L = 0$:

$$Y_S^T = \frac{1}{\Gamma_{\mu\sigma}}\left\{\begin{array}{l} (1-\beta\mu)\,\sigma r_S^e + [(1-\beta\mu)(1-\mu) - \mu\kappa\psi]\, G_S + \mu\kappa\sigma\psi\chi^I\tau_S^I + \\ (1-\beta\mu)(1-\mu)\, b\,(Y_M - G_M) + (1-\mu)\,\sigma b\pi_M \end{array}\right\}$$

$$\pi_S^T = \frac{1}{\Gamma_{\mu\sigma}}\left\{\begin{array}{l} \kappa\sigma r_S^e + (1-\mu)\left(1-\psi\sigma^{-1}\right)\kappa G_S + (1-\mu)\,\kappa\psi\chi^I\tau_S^I \\ (1-\mu)\,\kappa b\,(Y_M - G_M) + [\beta(1-\mu) + \kappa\sigma](1-\mu)\, b\pi_M \end{array}\right\}$$

Substituting the medium-run solution allocations $(B.32)$ and $(B.33)$ into these short-run relations, we obtain crisis-state output and inflation as a function of the fiscal instruments, which can be summarized as:

$$\boxed{\begin{array}{lllll} Y_S^T = & \Omega_{Y_S,r_S^e} r_S^e & + \;\Omega_{Y_S,G_S} G_S & + \;\Omega_{Y_S,\tau_S^I}\tau_S^I & + \\ & & + \;\Omega^T_{Y_S,G_M} G_M & + \;\Omega^T_{Y_S,\tau_M^I}\tau_M^I & \end{array}} \quad \text{(B.34)}$$

where $\Omega_{Y_S,r_S^e} \equiv \dfrac{(1-\beta\mu)\,\sigma}{\Gamma_{\mu\sigma}}$, $\Omega_{Y_S,G_S} \equiv \dfrac{[(1-\beta\mu)(1-\mu) - \mu\kappa\psi]}{\Gamma_{\mu\sigma}}$, $\Omega_{Y_S,\tau_S^I} \equiv \dfrac{\mu\kappa\sigma\psi\chi^I}{\Gamma_{\mu\sigma}}$,

$\Omega^T_{Y_S,G_M} \equiv \dfrac{(1-\mu)\, b}{\Gamma_{\mu\sigma}}\left[(1-\beta\mu)\left(\Omega^T_{Y_M,G_M} - 1\right) + \sigma\Omega^T_{\pi_M,G_M}\right]$ and

$\Omega^T_{Y_S,\tau_M^I} \equiv \dfrac{(1-\mu)\, b}{\Gamma_{\mu\sigma}}\left[(1-\beta\mu)\,\Omega^T_{Y_M,\tau_M^I} + \sigma\Omega^T_{\pi_M,\tau_M^I}\right]$

$$\boxed{\begin{array}{lllll} \pi_S^T = & \Omega_{\pi_S,r_S^e} r_S^e & + \;\Omega_{\pi_S,G_S} G_S & + \;\Omega_{\pi_S,\tau_S^I}\tau_S^I & + \\ & & + \;\Omega^T_{\pi_S,G_M} G_M & + \;\Omega^T_{\pi_S,\tau_M^I}\tau_M^I & \end{array}} \quad \text{(B.35)}$$

where $\Omega_{\pi_S,r_S^e} \equiv \dfrac{\kappa\sigma}{\Gamma_{\mu\sigma}} r_S^e$, $\Omega_{\pi_S,G_S} \equiv \dfrac{(1-\mu)\left(1-\psi\sigma^{-1}\right)\kappa}{\Gamma_{\mu\sigma}}$, $\Omega_{\pi_S,\tau_S^I} \equiv \dfrac{(1-\mu)\,\kappa\psi\chi^I}{\Gamma_{\mu\sigma}}$,

$\Omega^T_{\pi_S,G_M} \equiv \dfrac{(1-\mu)\, b}{\Gamma_{\mu\sigma}}\left[\kappa\left(\Omega^T_{Y_M,G_M} - 1\right) + [\beta(1-\mu) + \kappa\sigma]\,\Omega^T_{\pi_M,G_M}\right]$ and

$\Omega^T_{\pi_S,\tau_M^I} \equiv \dfrac{(1-\mu)\, b}{\Gamma_{\mu\sigma}}\left[\kappa\Omega^T_{Y_M,\tau_M^I}\tau_M^I + [\beta(1-\mu) + \kappa\sigma]\,\Omega^T_{\pi_M,\tau_M^I}\right]$. ■

Proof. (Proposition 2 - Part III - ZLB is binding) With the new allocations defined for output and inflation in the short and medium runs, we need to derive again the relation between the shock and the fiscal instruments that guarantee that the zero lower bound is binding in the short run (Condition $(C4')$). The idea follows that used to derive Condition $(C4)$ in Proposition 1, remembering that the long-run allocation is given by $Y_L = \pi_L = 0$. We can already use the result from equation $(B.25)$ and substitute for output and inflation in the transitional state ($(B.32)$ and $(B.33)$) to obtain:

$$
\begin{aligned}
i_S &= r_S^e + \frac{(1-\mu)\left(1-\psi\sigma^{-1}\right)\phi_\pi\kappa + [(1-\mu)(1-\beta\mu) - \mu\kappa\psi]\,\phi_y}{\Gamma_{\mu\phi}} G_S + \frac{(1-\mu)\,\phi_\pi + \mu\phi_y\sigma}{\Gamma_{\mu\phi}} \kappa\psi\chi^I\tau_S^I + \\
&\quad \frac{(1-\mu)\left[\phi_\pi\kappa + (1-\beta\mu)\,\phi_y\right]}{\Gamma_{\mu\phi}} b\left(\left(\Omega^T_{Y_M,G_M} - 1\right) G_M + \Omega^T_{Y_M,\tau^I_M}\tau^I_M\right) + \\
&\quad \frac{(1-\mu)\left[\phi_y\sigma + (\beta(1-\mu) + \kappa\sigma)\,\phi_\pi\right]}{\Gamma_{\mu\phi}} b\left(\Omega^T_{\pi_M,G_M} G_M + \Omega^T_{\pi_M,\tau^I_M}\tau^I_M\right)
\end{aligned}
$$

$$
\begin{aligned}
i_S &= r_S^e + \frac{(1-\mu)\left(1-\psi\sigma^{-1}\right)\phi_\pi\kappa + [(1-\mu)(1-\beta\mu) - \mu\kappa\psi]\,\phi_y}{\Gamma_{\mu\phi}} G_S + \frac{(1-\mu)\,\phi_\pi + \mu\phi_y\sigma}{\Gamma_{\mu\phi}} \kappa\psi\chi^I\tau_S^I + \\
&\quad \frac{(1-\mu)\,b}{\Gamma_{\mu\phi}} \begin{bmatrix} \left[\phi_\pi\kappa + (1-\beta\mu)\,\phi_y\right]\left(\Omega^T_{Y_M,G_M} - 1\right) + \\ \left[\phi_y\sigma + (\beta(1-\mu) + \kappa\sigma)\,\phi_\pi\right]\Omega^T_{\pi_M,G_M} \end{bmatrix} G_M + \\
&\quad \frac{(1-\mu)\,b}{\Gamma_{\mu\phi}} \begin{bmatrix} \left[\phi_\pi\kappa + (1-\beta\mu)\,\phi_y\right]\Omega^T_{Y_M,\tau^I_M} + \\ \left[\phi_y\sigma + (\beta(1-\mu) + \kappa\sigma)\,\phi_\pi\right]\Omega^T_{\pi_M,\tau^I_M} \end{bmatrix} \tau^I_M
\end{aligned}
$$

Thus, the nominal interest rate implied by the monetary policy rule is negative, and the zero lower bound is binding, if the shock respects the following condition:

$$
\boxed{r_S^e < -\Theta_{G_S} G_S - \Theta_{\tau^I_S}\tau^I_S - \Theta^T_{G_M} G_M - \Theta^T_{\tau^I_M}\tau^I_M} \qquad \text{(Condition } (C4')\text{)}
$$

where the coefficients are Θ_{G_S} and $\Theta_{\tau^I_S}$ are the same as those defined in Part III of the proof of Proposition 1 and the other coefficients are given by

$$
\Theta^T_{G_M} \equiv \frac{(1-\mu)\,b}{\Gamma_{\mu\phi}} \left\{ \begin{matrix} \left[\phi_\pi\kappa + (1-\beta\mu)\,\phi_y\right]\left(\Omega^T_{Y_M,G_M} - 1\right) + \\ \left[\phi_y\sigma + (\beta(1-\mu) + \kappa\sigma)\,\phi_\pi\right]\Omega^T_{\pi_M,G_M} \end{matrix} \right\} \text{ and}
$$

$$
\Theta^T_{\tau^I_M} \equiv \frac{(1-\mu)\,b}{\Gamma_{\mu\phi}} \left\{ \begin{matrix} \left[\phi_\pi\kappa + (1-\beta\mu)\,\phi_y\right]\Omega^T_{Y_M,\tau^I_M} + \\ \left[\phi_y\sigma + (\beta(1-\mu) + \kappa\sigma)\,\phi_\pi\right]\Omega^T_{\pi_M,\tau^I_M} \end{matrix} \right\}. \blacksquare
$$

Proposition 3 *The welfare loss function in this model is given by*

$$
L_t \equiv \sum_{t=0}^{\infty}\beta^t \left\{\pi_t^2 + \lambda_y\left(\hat{Y}_t - \Gamma\hat{G}_t\right)^2 + \lambda_g\hat{G}_t^2\right\} \tag{4.16}
$$

where $\Gamma \equiv \dfrac{\sigma^{-1}\gamma}{\sigma^{-1}+\omega}$, $\lambda_y \equiv \dfrac{\kappa}{\theta}$ *and* $\lambda_g \equiv \lambda_y\Gamma(1-\gamma-\Gamma)$. *Given the probability structure of the model, this function can be expressed in present discounted terms as*

$$
L^{PDV} = \left\{ \begin{matrix} \dfrac{1}{1-\beta\mu}\left(\pi_S^2 + \lambda_y(Y_S - \Gamma G_S)^2 + \lambda_g G_S^2\right) + \\ \dfrac{\beta(1-\mu)\,b}{(1-\beta\mu)(1-\beta b)}\left(\pi_M^2 + \lambda_y(Y_M - \Gamma G_M)^2 + \lambda_g G_M^2\right) + \\ \dfrac{\beta(1-\mu)(1-b)}{(1-\beta)(1-\beta\mu)(1-\beta b)}\left(\pi_L^2 + \lambda_y(Y_L - \Gamma G_L)^2 + \lambda_g G_L^2\right) \end{matrix} \right\} \tag{4.17}
$$

The proof of this proposition is divided into two parts. In the first part, I derive the expression for the welfare loss function, while the second part shows how I obtain the expression for the present discounted value of the loss function.

Proof. (Proposition 3 - Part I - Welfare Loss Function) - The welfare loss function is obtained through a second order Taylor expansion of the utility function, according to Woodford (2003) and Eggertsson (2011a). The utility function in this model is given by:

$$E\left\{\sum_{t=0}^{\infty}\beta^{t}\left[u\left(C_{t},\xi_{t}\right)+g\left(G_{t},\xi_{t}\right)-\int_{0}^{1}\upsilon\left(h_{t}\left(i\right),\xi_{t}\right)di\right]\right\}$$

I use the aggregate resource constraint ($Y_t = C_t + G_t$) to substitute out for consumption and the production function ($y_t(i) = h_t(i)$) to substitute out for labor. The utility function becomes

$$E\left\{\sum_{t=0}^{\infty}\beta^{t}\left[u\left(Y_{t}-G_{t},\xi_{t}\right)+g\left(G_{t},\xi_{t}\right)-\int_{0}^{1}\upsilon\left(y_{t}\left(i\right),\xi_{t}\right)di\right]\right\}$$

The Taylor expansion is performed in each term of the utility function separately. The first term yields:

$$\begin{aligned}u\left(Y_{t}-G_{t},\xi_{t}\right) &\approx \bar{u}+u_{c}\tilde{Y}_{t}-u_{c}\tilde{G}_{t}+u_{\xi}\xi_{t}+\tfrac{1}{2}u_{cc}\tilde{Y}_{t}^{2}-\tfrac{1}{2}u_{cc}\tilde{G}_{t}^{2}+\\ &\quad u_{c\xi}\xi_{t}\tilde{Y}_{t}-u_{cc}\tilde{Y}_{t}\tilde{G}_{t}-u_{c\xi}\xi_{t}\tilde{G}_{t}+\tfrac{1}{2}\xi_{t}'u_{\xi\xi}\xi_{t}+\mathcal{O}\left(\|\xi\|^{3}\right)\end{aligned}$$

where $\bar{u} \equiv u\left(\bar{Y};0\right)$, $\tilde{Y}_t \equiv Y_t - \bar{Y}$and $\tilde{G}_t \equiv G_t - \bar{G}$ and assuming that the fluctuations in $\tilde{Y}_t$ are of order $\mathcal{O}\left(\|\xi\|\right)$. This can be written as:

$$\begin{aligned}u\left(Y_{t}-G_{t},\xi_{t}\right) &\approx \bar{u}+u_{c}\overline{Y}\left(\hat{Y}_{t}+\tfrac{1}{2}\hat{Y}_{t}^{2}\right)-u_{c}\bar{G}\left(\hat{G}_{t}+\tfrac{1}{2}\hat{G}_{t}^{2}\right)+u_{\xi}\xi_{t}+\tfrac{1}{2}u_{cc}\bar{Y}^{2}\left(\hat{Y}_{t}+\tfrac{1}{2}\hat{Y}_{t}^{2}\right)^{2}\\ &\quad -\tfrac{1}{2}u_{cc}\bar{G}^{2}\left(\hat{G}_{t}+\tfrac{1}{2}\hat{G}_{t}^{2}\right)^{2}+u_{c\xi}\xi_{t}\bar{Y}\left(\hat{Y}_{t}+\tfrac{1}{2}\hat{Y}_{t}^{2}\right)-u_{cc}\bar{Y}\bar{G}\left(\hat{Y}_{t}+\tfrac{1}{2}\hat{Y}_{t}^{2}\right)\left(\hat{G}_{t}+\tfrac{1}{2}\hat{G}_{t}^{2}\right)\\ &\quad -u_{c\xi}\xi_{t}\bar{G}\left(\hat{G}_{t}+\tfrac{1}{2}\hat{G}_{t}^{2}\right)+\tfrac{1}{2}\xi_{t}'u_{\xi\xi}\xi_{t}+\mathcal{O}\left(\|\xi\|^{3}\right)\\ &= \bar{u}+u_{c}\bar{Y}\left(\hat{Y}_{t}+\tfrac{1}{2}\hat{Y}_{t}^{2}\right)-u_{c}\bar{G}\left(\hat{G}_{t}+\tfrac{1}{2}\hat{G}_{t}^{2}\right)+u_{\xi}\xi_{t}+\tfrac{1}{2}u_{cc}\bar{Y}^{2}\hat{Y}_{t}^{2}-\tfrac{1}{2}u_{cc}\bar{G}^{2}\hat{G}_{t}^{2}\\ &\quad +u_{c\xi}\bar{Y}\xi_{t}\hat{Y}_{t}-u_{cc}\bar{Y}\bar{G}\hat{Y}_{t}\hat{G}_{t}-u_{c\xi}\bar{G}\xi_{t}\hat{G}_{t}+\tfrac{1}{2}\tilde{\xi}_{t}'u_{\xi\xi}\tilde{\xi}_{t}+\mathcal{O}\left(\|\xi\|^{3}\right)\end{aligned}$$

where I substitute $\tilde{Y}_t$ in terms of $\hat{Y}_t \equiv \log\left(Y_t/\bar{Y}\right)$, using the Taylor series expansion

$$\frac{Y_{t}}{\bar{Y}}=1+\hat{Y}_{t}+\tfrac{1}{2}\hat{Y}_{t}^{2}+\mathcal{O}\left(\|\xi\|^{3}\right)$$

Analogously, I substituted $\tilde{G}_t$ in terms of $\hat{G}_t \equiv \log\left(G_t/\bar{G}\right)$. This can be conveniently written as:

$$\begin{aligned}u\left(Y_{t}-G_{t},\xi_{t}\right) &= u_{c}\bar{Y}\left[\hat{Y}_{t}+\tfrac{1}{2}\hat{Y}_{t}^{2}+\tfrac{1}{2}\frac{u_{cc}\bar{Y}}{u_{c}}\hat{Y}_{t}^{2}+\frac{u_{c\xi}\xi_{t}}{u_{c}}\hat{Y}_{t}\right]-u_{c}\bar{G}\left[\hat{G}_{t}+\tfrac{1}{2}\hat{G}_{t}^{2}+\tfrac{1}{2}\frac{u_{cc}\bar{Y}}{u_{c}}\frac{\bar{G}}{\bar{Y}}\hat{G}_{t}^{2}+\frac{u_{c\xi}\xi_{t}}{u_{c}}\hat{G}_{t}\right]\\ &\quad -u_{c}\bar{Y}\frac{u_{cc}\bar{Y}}{u_{c}}\frac{\bar{G}}{\bar{Y}}\hat{Y}_{t}\hat{G}_{t}+t.i.p.+\mathcal{O}\left(\|\xi\|^{3}\right)\\ &= u_{c}\bar{Y}\left[\hat{Y}_{t}+\tfrac{1}{2}\left(1-\sigma^{-1}\right)\hat{Y}_{t}^{2}+\sigma^{-1}d_{t}\hat{Y}_{t}\right]-u_{c}\bar{G}\left[\hat{G}_{t}+\tfrac{1}{2}\left(1-\sigma^{-1}\gamma\right)\hat{G}_{t}^{2}+\sigma^{-1}d_{t}\hat{G}_{t}\right]\\ &\quad +u_{c}\bar{Y}\sigma^{-1}\gamma\hat{Y}_{t}\hat{G}_{t}+t.i.p.+\mathcal{O}\left(\|\xi\|^{3}\right)\end{aligned}$$

where $\sigma^{-1} \equiv -u_{cc}\overline{Y}/u_c$, $d_t \equiv \sigma u_{c\xi}\xi_t/u_c$, $\gamma \equiv \bar{G}/\bar{Y}$ and $t.i.p.$ are the terms independent of policy.

The expansion of the second term gives:

$$g(G_t,\xi_t) \approx \bar{g} + g_G\tilde{G}_t + g_\xi\xi_t + \tfrac{1}{2}g_{GG}\tilde{G}_t^2 + g_{G\xi}\xi_t\tilde{G}_t + \tfrac{1}{2}\xi_t' u_{\xi\xi}\xi_t + \mathcal{O}\left(\|\xi\|^3\right)$$

$$= \begin{array}{l} \bar{g} + g_G\bar{G}\left(\hat{G}_t + \tfrac{1}{2}\hat{G}_t^2\right) + g_\xi\xi_t + \tfrac{1}{2}g_{GG}\bar{G}^2\left(\hat{G}_t + \tfrac{1}{2}\hat{G}_t^2\right)^2 + \\ g_{G\xi}\xi_t\bar{G}\left(\hat{G}_t + \tfrac{1}{2}\hat{G}_t^2\right) + \tfrac{1}{2}\xi_t' u_{\xi\xi}\xi_t + \mathcal{O}\left(\|\xi\|^3\right) \end{array}$$

$$= g_G\bar{G}\left(\hat{G}_t + \tfrac{1}{2}\hat{G}_t^2\right) + \tfrac{1}{2}g_{GG}\bar{G}^2\hat{G}_t^2 + g_{G\xi}\bar{G}\xi_t\hat{G}_t + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)$$

$$= g_G\bar{G}\left[\hat{G}_t + \tfrac{1}{2}\hat{G}_t^2 + \tfrac{1}{2}\frac{g_{GG}\bar{G}}{g_G}\hat{G}_t^2 + \frac{g_{G\xi}\xi_t}{g_G}\hat{G}_t\right] + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)$$

$$= g_G\bar{G}\left[\hat{G}_t + \tfrac{1}{2}\left(1-\sigma_G^{-1}\right)\hat{G}_t^2 + \sigma_G^{-1}d_t^G\hat{G}_t\right] + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)$$

where $\sigma_G^{-1} \equiv -g_{GG}\bar{G}/g_G$, $d_t^G \equiv \sigma_G g_{G\xi}\xi_t/g_G$.

Finally, the expansion of the third term yields:

$$\upsilon(y_t(i),\xi_t) \approx \bar{\upsilon} + \upsilon_y\tilde{y}_t(i) + \upsilon_\xi\xi_t + \tfrac{1}{2}\upsilon_{yy}\tilde{y}_t(i)^2 + \upsilon_{y\xi}\xi_t\tilde{y}_t(i) + \tfrac{1}{2}\xi_t'\upsilon_{\xi\xi}\xi_t + \mathcal{O}\left(\|\xi\|^3\right)$$

$$= \begin{array}{l} \bar{\upsilon} + \upsilon_y\bar{Y}\left(\hat{y}_t(i) + \tfrac{1}{2}\hat{y}_t(i)^2\right) + \tfrac{1}{2}\upsilon_{yy}\left[\bar{Y}\left(\hat{y}_t(i) + \tfrac{1}{2}\hat{y}_t(i)^2\right)\right]^2 + \\ \upsilon_{y\xi}\xi_t\bar{Y}\left(\hat{y}_t(i) + \tfrac{1}{2}\hat{y}_t(i)^2\right) + \upsilon_\xi\xi_t + \tfrac{1}{2}\xi_t'\upsilon_{\xi\xi}\xi_t + \mathcal{O}\left(\|\xi\|^3\right) \end{array}$$

$$= \bar{Y}\upsilon_y\hat{y}_t(i) + \tfrac{1}{2}\bar{Y}\left[\upsilon_y + \upsilon_{yy}\bar{Y}\right]\hat{y}_t(i)^2 + \upsilon_{y\xi}\bar{Y}\xi_t\hat{y}_t(i) + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)$$

$$= \bar{Y}\upsilon_y\left\{\hat{y}_t(i) + \tfrac{1}{2}\left[1 + \frac{\upsilon_{yy}}{\upsilon_y}\bar{Y}\right]\hat{y}_t(i)^2 + \frac{\upsilon_{y\xi}}{\upsilon_y}\xi_t\hat{y}_t(i)\right\} + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)$$

where $\hat{y}_t(i) \equiv \log\left(y_t(i)/\bar{Y}\right)$. We can use the first order conditions of the household's problem to get:

$$\frac{\upsilon_y\left(\bar{Y},\xi\right)}{u_c\left(\bar{Y}-\bar{G},\xi\right)} = \frac{\theta-1}{\theta}\left(1-\bar{\tau}^I\right) = 1-\Phi \Rightarrow \upsilon_y = (1-\Phi)\,u_c$$

where $\frac{\theta-1}{\theta}$ is the desired mark up as a result of the suppliers market power. Woodford (2003) explains that the parameter $\Phi > 0$, which is assumed to be of order $\mathcal{O}(\|\xi\|)$, summarizes the overall distortion in the steady-state output level as a result of both taxes and market power. Substituting into the expansion above, we get:

$$\upsilon(y_t(i),\xi_t) = \bar{Y}(1-\Phi)\,u_c\left\{\hat{y}_t(i) + \tfrac{1}{2}\left[1 + \frac{\bar{Y}\upsilon_{yy}}{\upsilon_y}\right]\hat{y}_t(i)^2 + \frac{\bar{Y}\upsilon_{yy}}{\bar{Y}\upsilon_{yy}}\frac{\upsilon_{y\xi}}{\upsilon_y}\xi_t\hat{y}_t(i)\right\} + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)$$

$$= \bar{Y}u_c\left\{(1-\Phi)\,\hat{y}_t(i) + \tfrac{1}{2}\left[1 + \frac{\bar{Y}\upsilon_{yy}}{\upsilon_y}\right]\hat{y}_t(i)^2 + \frac{\bar{Y}\upsilon_{yy}}{\upsilon_y}\frac{\upsilon_{y\xi}\xi_t}{\bar{Y}\upsilon_{yy}}\hat{y}_t(i)\right\} + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)$$

$$= \bar{Y}u_c\left\{(1-\Phi)\,\hat{y}_t(i) + \tfrac{1}{2}\left[1+\omega\right]\hat{y}_t(i)^2 - \omega q_t\hat{y}_t(i)\right\} + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)$$

where $\omega = \bar{Y}\upsilon_{yy}/\upsilon_y$ and $q_t \equiv -\upsilon_{y\xi}\xi_t/\left(\bar{Y}\upsilon_{yy}\right)$. Integrating this expression over the differentiated good i yields:

$$\int_0^1 v\left(y_t\left(i\right),\xi_t\right)di = \bar{Y}u_c \left\{ \begin{array}{l} \left(1-\Phi\right)E_i\hat{y}_t\left(i\right) - \omega q_t E_i\hat{y}_t\left(i\right) + \\ \frac{1}{2}\left[1+\omega\right]\left[\left(E_i\hat{y}_t\left(i\right)\right)^2 + var_i\hat{y}_t\left(i\right)\right] \end{array} \right\} + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)$$

$$= \bar{Y}u_c \left\{ \begin{array}{l} \left(1-\Phi\right)\left[\hat{Y}_t - \frac{1}{2}\left[1-\theta^{-1}\right]var_i\hat{y}_t\left(i\right)\right] - \omega q_t\left[\hat{Y}_t - \frac{1}{2}\left[1-\theta^{-1}\right]var_i\hat{y}_t\left(i\right)\right] + \\ \frac{1}{2}\left[1+\omega\right]\left[\left(\hat{Y}_t - \frac{1}{2}\left[1-\theta^{-1}\right]var_i\hat{y}_t\left(i\right)\right)^2 + var_i\hat{y}_t\left(i\right)\right] \end{array} \right\} + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)$$

$$= \bar{Y}u_c \left\{ \begin{array}{l} \left(1-\Phi\right)\hat{Y}_t + \frac{1}{2}\left[1+\omega\right]\hat{Y}_t^2 - \omega q_t\hat{Y}_t \\ -\frac{1}{2}\left[1-\theta^{-1}\right]var_i\hat{y}_t\left(i\right) + \frac{1}{2}\left[1+\omega\right]var_i\hat{y}_t\left(i\right) \end{array} \right\} + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)$$

$$= \bar{Y}u_c\left\{\left(1-\Phi\right)\hat{Y}_t + \frac{1}{2}\left[1+\omega\right]\hat{Y}_t^2 - \omega q_t\hat{Y}_t + \frac{1}{2}\left[\theta^{-1}+\omega\right]var_i\hat{y}_t\left(i\right)\right\} + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)$$

where $E_i\hat{y}_t\left(i\right)$ denotes the mean value of $\hat{y}_t\left(i\right)$ across all differentiated goods at date t and $var_i\hat{y}_t\left(i\right)$ is the corresponding variance. The last line makes use of the Taylor series approximation of the aggregate $Y_t \equiv \left[\int_0^1 y_t\left(i\right)^{\frac{\theta-1}{\theta}}di\right]^{\frac{\theta}{\theta-1}}$

$$\hat{Y}_t = E_i\hat{y}_t\left(i\right) + \frac{1}{2}\left(1-\theta\right)^{-1}var_i\hat{y}_t\left(i\right) + \mathcal{O}\left(\|\xi\|^3\right)$$

to eliminate $E_i\hat{y}_t\left(i\right)$.

We can use that, in steady state, the marginal utility of consumption must be equal to the marginal utility of government spending ($u_c = g_G$) and assume that the intertemporal elasticities of substitutions of public and private spending are equal ($\sigma = \sigma_G$). Putting the first and second terms together:

$$\begin{array}{c} u_c\bar{Y}\left[\hat{Y}_t + \frac{1}{2}\left(1-\sigma^{-1}\right)\hat{Y}_t^2 + \sigma^{-1}d_t\hat{Y}_t\right] - u_c\bar{Y}\gamma\left[\hat{G}_t + \frac{1}{2}\left(1-\sigma^{-1}\gamma\right)\hat{G}_t^2 + \sigma^{-1}d_t\hat{G}_t\right] \\ +u_c\bar{Y}\sigma^{-1}\gamma\hat{Y}_t\hat{G}_t + u_c\bar{Y}\gamma\left[\hat{G}_t + \frac{1}{2}\left(1-\sigma^{-1}\right)\hat{G}_t^2 + \sigma^{-1}d_t^G\hat{G}_t\right] + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right) \end{array}$$

$$= \begin{array}{l} u_c\bar{Y}\left[\hat{Y}_t + \frac{1}{2}\left(1-\sigma^{-1}\right)\hat{Y}_t^2 + \sigma^{-1}d_t\hat{Y}_t\right] + u_c\bar{Y}\sigma^{-1}\gamma\hat{Y}_t\hat{G}_t \\ -u_c\bar{Y}\gamma\left[\frac{1}{2}\sigma^{-1}\left(1-\gamma\right)\hat{G}_t^2 + \sigma^{-1}\left(d_t - d_t^G\right)\hat{G}_t\right] + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right) \end{array}$$

$$= u_c\bar{Y}\left\{\hat{Y}_t + \frac{1}{2}\left(1-\sigma^{-1}\right)\hat{Y}_t^2 + \sigma^{-1}d_t\hat{Y}_t + \sigma^{-1}\gamma\hat{Y}_t\hat{G}_t - \frac{\gamma\sigma^{-1}\left(1-\gamma\right)}{2}\hat{G}_t^2\right\} + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)$$

$$= u_c\bar{Y}\left\{\frac{1}{2}\left(1-\sigma^{-1}\right)\hat{Y}_t^2 + \left(1+\sigma^{-1}d_t + \sigma^{-1}\gamma\hat{G}_t\right)\hat{Y}_t - \frac{\gamma\sigma^{-1}\left(1-\gamma\right)}{2}\hat{G}_t^2\right\} + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)$$

where I assume that the preference shock ξ_t enters the utility function of private and public consumption in the same way, which implies $d_t = d_t^G$.

Combining this last expression with the third term we obtain:

$$U_t = \begin{array}{l} u_c\bar{Y}\left\{\frac{1}{2}\left(1-\sigma^{-1}\right)\hat{Y}_t^2 + \left(1+\sigma^{-1}d_t+\sigma^{-1}\gamma\hat{G}_t\right)\hat{Y}_t - \frac{\gamma\sigma^{-1}\left(1-\gamma\right)}{2}\hat{G}_t^2\right\} \\ -\bar{Y}u_c\left\{\left(1-\Phi\right)\hat{Y}_t + \frac{1}{2}\left[1+\omega\right]\hat{Y}_t^2 - \omega q_t\hat{Y}_t + \frac{1}{2}\left[\theta^{-1}+\omega\right]var_i\hat{y}_t\left(i\right)\right\} + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)\end{array}$$

$$= -\frac{u_c\bar{Y}}{2}\left\{\begin{array}{l}\left(\sigma^{-1}+\omega\right)\hat{Y}_t^2 - 2\left[\sigma^{-1}d_t+\omega q_t+\sigma^{-1}\gamma\hat{G}_t+\Phi\right]\hat{Y}_t + \\ \gamma\sigma^{-1}\left(1-\gamma\right)\hat{G}_t^2 + \left[\theta^{-1}+\omega\right]var_i\hat{y}_t\left(i\right)\end{array}\right\} + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)$$

Assuming that the government removes monopolistic distortions and distortions from income taxation in the steady state through a subsidy $\mu^w \equiv \frac{\theta-1}{\theta}\left(1-\overline{\tau}^I\right)$, then $\Phi = 0$, which implies $\upsilon_y = u_c$. Consequently we have that $\sigma^{-1}d_t + \omega q_t = 0$. This removes the linear terms in the expression above:

$$U_t = -\frac{u_c\bar{Y}}{2}\left\{\begin{array}{l}\left(\sigma^{-1}+\omega\right)\left[\hat{Y}_t^2 - 2\frac{\sigma^{-1}\gamma}{\sigma^{-1}+\omega}\hat{G}_t\hat{Y}_t + \frac{\sigma^{-1}\gamma}{\sigma^{-1}+\omega}\left(1-\gamma\right)\hat{G}_t^2\right] \\ +\left[\theta^{-1}+\omega\right]var_i\hat{y}_t\left(i\right)\end{array}\right\} + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)$$

$$= -\frac{u_c\bar{Y}}{2}\left\{\begin{array}{l}\left(\sigma^{-1}+\omega\right)\left(\hat{Y}_t - \frac{\sigma^{-1}\gamma}{\sigma^{-1}+\omega}\hat{G}_t\right)^2 + \left[\left(1-\gamma\right) - \frac{\sigma^{-1}\gamma}{\sigma^{-1}+\omega}\right]\sigma^{-1}\gamma\hat{G}_t^2 \\ +\left[\theta^{-1}+\omega\right]var_i\hat{y}_t\left(i\right)\end{array}\right\} + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)$$

The price dispersion term $var_i\hat{y}_t\left(i\right)$ can be written as a function of inflation using the demand condition for good i, $y_t\left(i\right) = Y_t\left[p_t\left(i\right)/P_t\right]^{-\theta}$ which yields:

$$\log y_t\left(i\right) = \log Y_t - \theta\left(\log p_t\left(i\right) - \log P_t\right)$$

Thus,

$$var_i \log\hat{y}_t\left(i\right) = \theta^2 var_i \log p_t\left(i\right)$$

Substituting into the expression for U_t:

$$U_t = -\frac{u_c\bar{Y}}{2}\left\{\begin{array}{l}\left(\sigma^{-1}+\omega\right)\left(\hat{Y}_t - \frac{\sigma^{-1}\gamma}{\sigma^{-1}+\omega}\hat{G}_t\right)^2 + \left[\left(1-\gamma\right) - \frac{\sigma^{-1}\gamma}{\sigma^{-1}+\omega}\right]\sigma^{-1}\gamma\hat{G}_t^2 \\ +\theta\left[1+\omega\theta\right]var_i\log p_t\left(i\right)\end{array}\right\} + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)$$

Define $\Delta_t \equiv var_i \log p_t\left(i\right)$, which can be written in recursive form as

$$\Delta_t = \alpha\Delta_{t-1} + \frac{\alpha}{1-\alpha}\pi_t^2 + \mathcal{O}\left(\|\xi\|^3\right)$$

Iterating backwards to time $t = 0$:

$$\Delta_t = \alpha^{t+1}\Delta_{-1} + \frac{\alpha}{1-\alpha}\sum_{s=0}^{t}\alpha^{t-s}\pi_s^2 + \mathcal{O}\left(\|\xi\|^3\right)$$

Taking the discounted value of these terms $\forall t \geq 0$ we obtain:

$$\sum_{t=0}^{\infty}\beta^t\Delta_t = \frac{\alpha}{\left(1-\alpha\right)\left(1-\alpha\beta\right)}\sum_{t=0}^{\infty}\beta^t\pi_t^2 + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)$$

Substitute it back in the loss function and rearranging:

$$\sum_{t=0}^{\infty}\beta^t U_t = -\frac{\overline{Y}u_c}{2}\sum_{t=0}^{\infty}\beta^t \left\{ \begin{array}{l} \frac{\alpha\theta\left(1+\omega\theta\right)}{\left(1-\alpha\right)\left(1-\alpha\beta\right)}\pi_t^2 + \left(\sigma^{-1}+\omega\right)\left(\hat{Y}_t - \frac{\sigma^{-1}\gamma}{\sigma^{-1}+\omega}\hat{G}_t\right)^2 + \\ \left[\left(1-\gamma\right) - \frac{\sigma^{-1}\gamma}{\sigma^{-1}+\omega}\right]\sigma^{-1}\gamma\hat{G}_t^2 \end{array} \right\} + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)$$

$$= -\frac{\overline{Y}u_c}{2}\frac{\alpha\theta\left(1+\omega\theta\right)}{\left(1-\alpha\right)\left(1-\alpha\beta\right)}\sum_{t=0}^{\infty}\beta^t \left\{ \begin{array}{l} \pi_t^2 + \\ \frac{\left(1-\alpha\right)\left(1-\alpha\beta\right)}{\alpha}\frac{\left(\sigma^{-1}+\omega\right)}{\left(1+\omega\theta\right)}\frac{1}{\theta}\left(\hat{Y}_t - \frac{\sigma^{-1}\gamma}{\sigma^{-1}+\omega}\hat{G}_t\right)^2 + \\ \frac{\left(1-\alpha\right)\left(1-\alpha\beta\right)}{\alpha}\frac{\left(\sigma^{-1}+\omega\right)}{\left(1+\omega\theta\right)}\frac{1}{\theta}\left[\left(1-\gamma\right) - \frac{\sigma^{-1}\gamma}{\sigma^{-1}+\omega}\right]\frac{\sigma^{-1}\gamma}{\left(\sigma^{-1}+\omega\right)}\hat{G}_t^2 \end{array} \right\}$$

$$+t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)$$

Define $\Omega \equiv \frac{\overline{Y}u_c}{2}\frac{\alpha\theta\left(1+\omega\theta\right)}{\left(1-\alpha\right)\left(1-\alpha\beta\right)}$, $\kappa \equiv \frac{\left(1-\alpha\right)\left(1-\alpha\beta\right)}{\alpha}\frac{\left(\sigma^{-1}+\omega\right)}{\left(1+\omega\theta\right)}$ and $\Gamma \equiv \frac{\sigma^{-1}\gamma}{\sigma^{-1}+\omega}$

$$\sum_{t=0}^{\infty}\beta^t U_t = -\Omega\sum_{t=0}^{\infty}\beta^t\left\{\pi_t^2 + \lambda_y\left(\hat{Y}_t - \Gamma\hat{G}_t\right)^2 + \lambda_G\hat{G}_t^2\right\} + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)$$

where $\lambda_y \equiv \frac{\kappa}{\theta}$ and $\lambda_g \equiv \lambda_y\Gamma\left[1-\gamma-\Gamma\right]$. This loss function is similar to that presented by Woodford (2011). ■

Proof. (Proposition 3 - Part II - PDV Welfare Loss Function) The probability structure of the model can be used to write the infinite summation in the loss function in terms of present discounted values. Before the crisis, the economy is at the steady state, where $\left(\pi_t = \hat{Y}_t = \hat{G}_t = 0\right)$, so welfare losses will be zero until the crisis hits. At the initial crisis state $(t = T_0)$, this function assumes the value $S = \left[\pi_S^2 + \lambda_y\left(Y_S - \Gamma G_S\right)^2 + \lambda_g G_S^2\right]$. In the following period, with probability μ, the shock will still be effective, and the economy will have welfare S again. With probability $(1-\mu)(1-b)$ it will go to the long run[41] and welfare will be $L = \left[\pi_L^2 + \lambda_y\left(Y_L - \Gamma G_L\right)^2 + \lambda_g G_L^2\right]$. With probability $(1-\mu)\,b$ there will be a transitional state due to a combination of monetary and fiscal policies, even when the zero lower bound is no longer binding. In this state, welfare will assume the value[42] $M = \left[\pi_M^2 + \lambda_y\left(Y_M - \Gamma G_M\right)^2 + \lambda_g G_M^2\right]$. After the economy enters the transitional state, it is expected to stay there in the following period with probability b, with welfare losses M. With probability $(1-b)$ the economy goes to the long run and welfare will be L.

Expanding the summation of the loss function for a few periods using the definitions for S, M and L, we can analyze the pattern:

[41] In order to obtain a general expression for the function, it is assumed that the economy does not necessarily go back to the steady state in the long run. This happens when policies implemented during the crisis are expected to be permanent. If they are temporary, the long-run allocations go back to $(\pi_L = Y_L = G_L = 0)$, and it is possible to simplify the expression even more.

[42] Note that the medium-run values for output, inflation and government spending here will be different depending on the combination of monetary and fiscal policies chosen. But to derive the present value of welfare losses their actual values do not matter at this point.

$$L^{PDV} = \underbrace{S}_{t=0} + \underbrace{\beta \begin{bmatrix} \mu S + (1-\mu) bM + \\ (1-\mu)(1-b) L \end{bmatrix}}_{t=1} + \underbrace{\beta^2 \begin{bmatrix} \mu^2 S + (1-\mu) b [\mu + b] M + \\ (1-\mu)(1-b)[1+\mu+b] L \end{bmatrix}}_{t=2} +$$

$$\underbrace{\beta^3 \begin{bmatrix} \mu^3 S + (1-\mu) b \left[\mu^2 + b^2 + \mu b\right] M + \\ (1-\mu)(1-b) \left[1+\mu+\mu^2+b+b^2+\mu b\right] L \end{bmatrix}}_{t=3} +$$

$$\underbrace{\beta^4 \begin{bmatrix} \mu^4 S + (1-\mu) b \left[\mu^3 + b^3 + \mu b (\mu + b)\right] M + \\ (1-\mu)(1-b) \left[1+\mu+\mu^2+\mu^3+b+b^2+b^3+\mu b(1+\mu+b)\right] L \end{bmatrix}}_{t=4} +$$

$$\underbrace{\beta^5 \begin{bmatrix} \mu^5 S + (1-\mu) b \left[\mu^4 + b^4 + \mu b \left(\mu b + \mu^2 + b^2\right)\right] M + \\ (1-\mu)(1-b) \begin{bmatrix} 1+\mu+\mu^2+\mu^3+\mu^4+b+b^2+b^3+b^4+ \\ \mu b \left(1+\mu+\mu^2+b+b^2+\mu b\right) \end{bmatrix} L \end{bmatrix}}_{t=5} +$$

$$\underbrace{\beta^6 \begin{bmatrix} \mu^6 S + (1-\mu) b \left[\mu^5 + b^5 + \mu b \left(\mu^3 + b^3 + \mu b (\mu + b)\right)\right] M \\ (1-\mu)(1-b) \begin{bmatrix} 1+\mu+\mu^2+\mu^3+\mu^4+\mu^5+b+b^2+b^3+b^4+b^5+ \\ \mu b \left(1+\mu+\mu^2+\mu^3+b+b^2+b^3+\mu b(1+\mu+b)\right) \end{bmatrix} L \end{bmatrix}}_{t=6} + \cdots$$

Distributing the $\beta' s$and grouping similar terms:

$$L^{PDV} = \left(1+\beta\mu+\beta^2\mu^2+\beta^3\mu^3+\beta^4\mu^4+\beta^5\mu^5+\beta^6\mu^6+\beta^7\mu^7+\cdots\right) S +$$

$$\left\{ \beta(1-\mu) b \begin{bmatrix} 1+\beta[\mu+b]+\beta^2\left[\mu^2+b^2+\mu b\right]+\beta^3\left[\mu^3+b^3+\mu b(\mu+b)\right]+ \\ \beta^4\left[\mu^4+b^4+\mu b\left(\mu b+\mu^2+b^2\right)\right]+\beta^5\left[\mu^5+b^5+\mu b\left(\mu^3+b^3+\mu b(\mu+b)\right)\right]+ \\ \beta^6\left[\mu^6+b^6+\mu b\left(\mu^4+b^4+\mu b\left(\mu^2+b^2+\mu b\right)\right)\right]+\ldots \end{bmatrix} \right\} M +$$

$$\left\{ \beta(1-\mu)(1-b) \begin{bmatrix} 1+\beta[1+\mu+b]+\beta^2\left[1+\mu+\mu^2+b+b^2+\mu b\right]+ \\ \beta^3\left[1+\mu+\mu^2+\mu^3+b+b^2+b^3+\mu b(1+\mu+b)\right]+ \\ \beta^4 \begin{bmatrix} 1+\mu+\mu^2+\mu^3+\mu^4+b+b^2+b^3+b^4+ \\ \mu b\left(1+\mu+\mu^2+b+b^2+\mu b\right) \end{bmatrix} + \\ \beta^5 \begin{bmatrix} 1+\mu+\mu^2+\mu^3+\mu^4+\mu^5+b+b^2+b^3+b^4+b^5+ \\ \mu b\left(1+\mu+\mu^2+\mu^3+b+b^2+b^3+\mu b(1+\mu+b)\right) \end{bmatrix} + \\ \beta^6 \begin{bmatrix} \begin{pmatrix} 1+\mu+\mu^2+\mu^3+\mu^4+\mu^5+\mu^6+ \\ b+b^2+b^3+b^4+b^5+b^6 \end{pmatrix} + \\ \mu b \begin{pmatrix} 1+\mu+\mu^2+\mu^3+\mu^4+b+b^2+b^3+b^4+ \\ \mu b\left(1+\mu+\mu^2+b+b^2+\mu b\right) \end{pmatrix} + \ldots \end{bmatrix} + \ldots \end{bmatrix} \right\} L$$

Rearranging:

$$
\begin{aligned}
L^{PDV} &= \left(\begin{array}{l} 1+\beta\mu+\beta^2\mu^2+ \\ \beta^3\mu^3+\beta^4\mu^4+\beta^5\mu^5+ \\ \beta^6\mu^6+\beta^7\mu^7+... \end{array}\right) S + \\
&\left\{ \beta\left(1-\mu\right)b \left[\begin{array}{l} 1+\beta\mu+\beta^2\mu^2+\beta^3\mu^3+\beta^4\mu^4+\beta^5\mu^5+\beta^6\mu^6+ \\ \beta b+\beta^2b^2+\beta^3b^3+\beta^4b^4+\beta^5b^5+\beta^6b^6+ \\ \beta^2\mu b\left(\begin{array}{l} 1+\beta\mu+\beta^2\mu^2+\beta^3\mu^3+\beta^4\mu^4+ \\ \beta b+\beta^2b^2+\beta^3b^3+\beta^4b^4+ \\ \beta^2\mu b\left(\begin{array}{l} 1+\beta\mu+\beta^2\mu^2+ \\ \beta b+\beta^2b^2+ \\ \beta^2\mu b+\ldots \end{array}\right) \end{array}\right) \end{array}\right] \right\} M + \\
&\left\{ \beta\left(1-\mu\right)\left(1-b\right) \left[\begin{array}{l} 1+\beta+\beta^2+\beta^3+\beta^4+\beta^5+\beta^6+ \\ \beta\mu\left(\begin{array}{l} 1+\beta+\beta^2+\beta^3+\beta^4+\beta^5+ \\ \beta\mu\left(\begin{array}{l} 1+\beta+\beta^2+\beta^3+\beta^4+ \\ \beta\mu\left(\begin{array}{l} 1+\beta+\beta^2+\beta^3+ \\ \beta\mu\left(\begin{array}{l} 1+\beta+\beta^2+ \\ \beta\mu\left(1+\beta+\beta\mu\right)+\ldots \end{array}\right)+\ldots \end{array}\right) \end{array}\right) \end{array}\right)+ \\
\beta b\left(\begin{array}{l} 1+\beta+\beta^2+\beta^3+\beta^4+\beta^5+ \\ \beta b\left(\begin{array}{l} 1+\beta+\beta^2+\beta^3+\beta^4+ \\ \beta b\left(\begin{array}{l} 1+\beta+\beta^2+\beta^3+ \\ \beta b\left(\begin{array}{l} 1+\beta+\beta^2+ \\ \beta b\left(1+\beta+\beta b\right)+\ldots \end{array}\right) \end{array}\right) \end{array}\right) \end{array}\right)+ \\
\beta^2\mu b\left(\begin{array}{l} 1+\beta+\beta^2+\beta^3+\beta^4+ \\ \beta\mu\left(\begin{array}{l} 1+\beta+\beta^2+\beta^3+ \\ \beta\mu\left(\begin{array}{l} 1+\beta+\beta^2+ \\ \beta\mu\left(1+\beta+\beta\mu+\ldots\right) \end{array}\right) \end{array}\right) \\ \beta b\left(\begin{array}{l} 1+\beta+\beta^2+\beta^3+ \\ \beta b\left(\begin{array}{l} 1+\beta+\beta^2+ \\ \beta b\left(1+\beta+\beta b+\ldots\right) \end{array}\right) \end{array}\right) \\ \beta^2\mu b\left(\begin{array}{l} 1+\beta+\beta^2+\ldots \\ \beta\mu\left(1+\beta+\beta\mu+\right) \\ \beta b\left(1+\beta+\beta b\right)+ \\ \beta^2\mu b+\ldots \end{array}\right) \end{array}\right)+... \end{array}\right] \right\} L
\end{aligned}
$$

Grouping similar terms one more time:

$$
\begin{aligned}
L^{PDV} &= \left(1+\beta\mu+\beta^2\mu^2+\beta^3\mu^3+\beta^4\mu^4+\beta^5\mu^5+\beta^6\mu^6+\beta^7\mu^7+\ldots\right) S + \\
&\left\{\begin{array}{l} \beta\left(1-\mu\right)b\left(1+\beta^2\mu b+\left(\beta^2\mu b\right)^2+\ldots\right) \\ \left[\begin{array}{l} \left(1+\beta\mu+\beta^2\mu^2+\beta^3\mu^3+\beta^4\mu^4+\beta^5\mu^5+\beta^6\mu^6+\ldots\right)+ \\ \beta b\left(1+\beta b+\beta^2b^2+\beta^3b^3+\beta^4b^4+\beta^5b^5+\ldots\right) \end{array}\right] \end{array}\right\} M + \\
&\left\{\begin{array}{l} \beta\left(1-\mu\right)\left(1-b\right)\left(1+\beta+\beta^2+\beta^3+\beta^4+\beta^5+\beta^6+\ldots\right)\left(1+\beta^2\mu b+\left(\beta^2\mu b\right)^2+\ldots\right)\times \\ \left[\begin{array}{l} 1+\beta\mu\left(1+\beta\mu+\left(\beta\mu\right)^2+\left(\beta\mu\right)^3+\left(\beta\mu\right)^4+\ldots\right)+ \\ \beta b\left(1+\beta b+\left(\beta b\right)^2+\left(\beta b\right)^3+\left(\beta b\right)^4+\ldots\right)+ \end{array}\right] \end{array}\right\} L
\end{aligned}
$$

We can write the summations as:

$$
\begin{aligned}
L^{PDV} &= \left[\sum_{t=0}^{\infty}(\beta\mu)^{t}\right] S + \\
& \left\{\beta(1-\mu)\, b\left(\sum_{t=0}^{\infty}\left(\beta^{2}\mu b\right)^{t}\right)\left[\left(\sum_{t=0}^{\infty}(\beta\mu)^{t}\right)+\left(\beta b\sum_{t=0}^{\infty}(\beta b)^{t}\right)\right]\right\} M \\
& \left\{\beta(1-\mu)(1-b)\left(\sum_{t=0}^{\infty}\beta^{t}\right)\left(\sum_{t=0}^{\infty}\left(\beta^{2}\mu b\right)^{t}\right)\left[1+\beta\mu\left(\sum_{t=0}^{\infty}(\beta\mu)^{t}\right)+\beta b\left(\sum_{t=0}^{\infty}(\beta b)^{t}\right)\right]\right\} L
\end{aligned}
$$

Using the properties of infinite summations to get a more straightforward expression:

$$
L^{PDV}=\frac{1}{1-\beta\mu} S+\left\{\frac{\beta(1-\mu)\, b}{1-\beta^{2}\mu b}\left[\frac{1}{1-\beta\mu}+\frac{\beta b}{1-\beta b}\right]\right\} M+\left\{\frac{\beta}{1-\beta}\frac{(1-\mu)(1-b)}{1-\beta^{2}\mu b}\left[1+\frac{\beta\mu}{1-\beta\mu}+\frac{\beta b}{1-\beta b}\right]\right\} L
$$

Which gives the final expression for the welfare loss function in present discounted value terms:

$$
\boxed{
\begin{aligned}
L^{PDV} = \; & \frac{1}{1-\beta\mu}\left(\pi_{S}^{2}+\lambda_{y}\left(Y_{S}-\Gamma G_{S}\right)^{2}+\lambda_{g} G_{S}^{2}\right)+ \\
& \frac{\beta(1-\mu)\, b}{(1-\beta\mu)(1-\beta b)}\left(\pi_{M}^{2}+\lambda_{y}\left(Y_{M}-\Gamma G_{M}\right)^{2}+\lambda_{g} G_{M}^{2}\right)+ \\
& \frac{\beta(1-\mu)(1-b)}{(1-\beta)(1-\beta\mu)(1-\beta b)}\left(\pi_{L}^{2}+\lambda_{y}\left(Y_{L}-\Gamma G_{L}\right)^{2}+\lambda_{g} G_{L}^{2}\right)
\end{aligned}
} \tag{B.36}
$$

■

Proposition 4 *Assuming Conditions* $(C1)-(C4)$ *hold, if the monetary authority keeps the nominal interest rate fixed at an optimal level* i_M^* *in the transitional state, it picks this level by solving the following minimization problem*

$$
\begin{aligned}
& \min_{\{i_M\}} L^{PDV} \\
& s.t. \quad Y_S, \pi_S, Y_M, \pi_M, Y_L, \pi_L \\
& \qquad\;\; i_M \geq 0
\end{aligned}
$$

where L^{PDV} *is defined in Proposition 3 and the levels of output and inflation in each state are given by Proposition 1. The solution to this problem yields an optimal nominal interest rate given by*

$$
i_M^* = \begin{cases} i_M^{opt} & , \text{ if } i_M^{opt} > 0 \\ 0 & , \text{ otherwise} \end{cases}
$$

$$
i_M^{opt} = \bar{r} + \frac{1}{\Omega^*_{i_M,\bar{r}}}\left\{\Omega^*_{i_M,r_S^e} r_S^e + \Omega^*_{i_M,G_S} G_S + \Omega^*_{i_M,\tau_S^I}\tau_S^I + \Omega^*_{i_M,G_M} G_M + \Omega^*_{i_M,\tau_M^I}\tau_M^I\right\} \tag{4.18}
$$

where the analytical expressions for the coefficients $\Omega^*_{i_M,j}$, $j \in \{\bar{r}, r_S^e, G_S, \tau_S^I, G_M, \tau_M^I\}$, *are defined in the appendix and depend on the structural parameters and the coefficients* $\left(\Omega' s\right)$ *from Proposition 1.*

Proof. Since the long-run allocations do not depend on the medium-run nominal interest rate, the problem that the monetary authority needs to solve can be expressed as:

$$\min_{\{i_M\}} \frac{1}{1-\beta\mu}\left\{\left(\pi_S^2+\lambda_y\left(Y_S-\Gamma G_S\right)^2+\lambda_g G_S^2\right)+\frac{\beta\left(1-\mu\right)b}{\left(1-\beta b\right)}\left(\pi_M^2+\lambda_y\left(Y_M-\Gamma G_M\right)^2+\lambda_g G_M^2\right)\right\}$$

$s.t.$

$$Y_S=\Omega_{Y_S,r_S^e}r_S^e+\Omega_{Y_S,G_S}G_S+\Omega_{Y_S,\tau_S^I}\tau_S^I+\Omega_{Y_S,i_M}\left(\bar{r}-i_M\right)+\Omega_{Y_S,G_M}G_M+\Omega_{Y_S,\tau_M^I}\tau_M^I$$

$$\pi_S=\Omega_{\pi_S,r_S^e}r_S^e+\Omega_{\pi_S,G_S}G_S+\Omega_{\pi_S,\tau_S^I}\tau_S^I+\Omega_{\pi_S,i_M}\left(\bar{r}-i_M\right)+\Omega_{\pi_S,G_M}G_M+\Omega_{\pi_S,\tau_M^I}\tau_M^I$$

$$Y_M=\Omega_{Y_M,i_M}\left(\bar{r}-i_M\right)+\Omega_{Y_M,G_M}G_M+\Omega_{Y_M,\tau_M^I}\tau_M^I$$

$$\pi_M=\Omega_{\pi_M,i_M}\left(\bar{r}-i_M\right)+\Omega_{\pi_M,G_M}G_M+\Omega_{\pi_M,\tau_M^I}\tau_M^I$$

$$i_M\geq 0$$

Substituting the restrictions into the objective function:

$$\min_{\{i_M\}} \frac{1}{1-\beta\mu}\left[\begin{array}{l}\left(\begin{array}{l}\Omega_{\pi_S,r_S^e}r_S^e+\Omega_{\pi_S,G_S}G_S+\Omega_{\pi_S,\tau_S^I}\tau_S^I+\\ \Omega_{\pi_S,i_M}\left(\bar{r}-i_M\right)+\Omega_{\pi_S,G_M}G_M+\Omega_{\pi_S,\tau_M^I}\tau_M^I\end{array}\right)^2+\\ \lambda_y\left(\begin{array}{l}\Omega_{Y_S,r_S^e}r_S^e+\left(\Omega_{Y_S,G_S}-\Gamma\right)G_S+\Omega_{Y_S,\tau_S^I}\tau_S^I+\\ \Omega_{Y_S,i_M}\left(\bar{r}-i_M\right)+\Omega_{Y_S,G_M}G_M+\Omega_{Y_S,\tau_M^I}\tau_M^I\end{array}\right)^2+\\ \lambda_g G_S^2\end{array}\right]$$

$$+\frac{\beta\left(1-\mu\right)b}{\left(1-\beta\mu\right)\left(1-\beta b\right)}\left[\begin{array}{l}\left(\Omega_{\pi_M,i_M}\left(\bar{r}-i_M\right)+\Omega_{\pi_M,G_M}G_M+\Omega_{\pi_M,\tau_M^I}\tau_M^I\right)^2+\\ \lambda_y\left(\Omega_{Y_M,i_M}\left(\bar{r}-i_M\right)+\left(\Omega_{Y_M,G_M}-\Gamma\right)G_M+\Omega_{Y_M,\tau_M^I}\tau_M^I\right)^2+\\ \lambda_g G_M^2\end{array}\right]$$

The first order condition with respect to i_M is given by:

$$\begin{array}{l}-\frac{1}{1-\beta\mu}\left(\begin{array}{l}\Omega_{\pi_S,r_S^e}r_S^e+\Omega_{\pi_S,G_S}G_S+\Omega_{\pi_S,\tau_S^I}\tau_S^I+\\ \Omega_{\pi_S,i_M}\left(\bar{r}-i_M\right)+\Omega_{\pi_S,G_M}G_M+\Omega_{\pi_S,\tau_M^I}\tau_M^I\end{array}\right)\Omega_{\pi_S,i_M}\\ -\frac{1}{1-\beta\mu}\lambda_y\left(\begin{array}{l}\Omega_{Y_S,r_S^e}r_S^e+\left(\Omega_{Y_S,G_S}-\Gamma\right)G_S+\Omega_{Y_S,\tau_S^I}\tau_S^I+\\ \Omega_{Y_S,i_M}\left(\bar{r}-i_M\right)+\Omega_{Y_S,G_M}G_M+\Omega_{Y_S,\tau_M^I}\tau_M^I\end{array}\right)\Omega_{Y_S,i_M}\\ -\frac{\beta\left(1-\mu\right)b}{\left(1-\beta\mu\right)\left(1-\beta b\right)}\left(\Omega_{\pi_M,i_M}\left(\bar{r}-i_M\right)+\Omega_{\pi_M,G_M}G_M+\Omega_{\pi_M,\tau_M^I}\tau_M^I\right)\Omega_{\pi_M,i_M}\\ -\frac{\beta\left(1-\mu\right)b}{\left(1-\beta\mu\right)\left(1-\beta b\right)}\lambda_y\left(\Omega_{Y_M,i_M}\left(\bar{r}-i_M\right)+\left(\Omega_{Y_M,G_M}-\Gamma\right)G_M+\Omega_{Y_M,\tau_M^I}\tau_M^I\right)\Omega_{Y_M,i_M}=0\end{array}$$

which can be written as:

$$i_M^{opt}=\bar{r}+\frac{1}{\Omega^*_{i_M,\bar{r}}}\left\{\Omega^*_{i_M,r_S^e}r_S^e+\Omega^*_{i_M,G_S}G_S+\Omega^*_{i_M,\tau_S^I}\tau_S^I+\Omega^*_{i_M,G_M}G_M+\Omega^*_{i_M,\tau_M^I}\tau_M^I\right\}$$

where $\Omega^*_{i_M,\bar{r}}\equiv\left[\left(\Omega_{\pi_S,i_M}\right)^2+\lambda_y\left(\Omega_{Y_S,i_M}\right)^2+\frac{\beta(1-\mu)b}{(1-\beta b)}\left(\left(\Omega_{\pi_M,i_M}\right)^2+\lambda_y\left(\Omega_{Y_M,i_M}\right)^2\right)\right]$,

$\Omega^*_{i_M,r_S^e}\equiv\left[\Omega_{\pi_S,i_M}\Omega_{\pi_S,r_S^e}+\lambda_y\Omega_{Y_S,i_M}\Omega_{Y_S,r_S^e}\right]$, $\Omega^*_{i_M,G_S}\equiv\left[\Omega_{\pi_S,i_M}\Omega_{\pi_S,G_S}+\lambda_y\Omega_{Y_S,i_M}\left(\Omega_{Y_S,G_S}-\Gamma\right)\right]$,

$\Omega^*_{i_M,\tau_S^I}\equiv\left[\Omega_{\pi_S,i_M}\Omega_{\pi_S,\tau_S^I}+\lambda_y\Omega_{Y_S,i_M}\Omega_{Y_S,\tau_S^I}\right]$,

$\Omega^*_{i_M,G_M}\equiv\left[\Omega_{\pi_S,i_M}\Omega_{\pi_S,G_M}+\lambda_y\Omega_{Y_S,i_M}\Omega_{Y_S,G_M}+\frac{\beta(1-\mu)b}{(1-\beta b)}\left(\Omega_{\pi_M,i_M}\Omega_{\pi_M,G_M}+\lambda_y\Omega_{Y_M,i_M}\left(\Omega_{Y_M,G_M}-\Gamma\right)\right)\right]$

and $\Omega^*_{i_M,\tau_M^I}\equiv\left[\Omega_{\pi_S,i_M}\Omega_{\pi_S,\tau_M^I}+\lambda_y\Omega_{Y_S,i_M}\Omega_{Y_S,\tau_M^I}+\frac{\beta(1-\mu)b}{(1-\beta b)}\left(\Omega_{\pi_M,i_M}\Omega_{\pi_M,\tau_M^I}+\lambda_y\Omega_{Y_M,i_M}\Omega_{Y_M,\tau_M^I}\right)\right]$.

Thus, to also satisfy the non-negativity condition ($i_M\geq 0$), we have that the optimal nominal interest rate is given by

$$i_M^*=\begin{cases} i_M^{opt} & \text{, if } i_M^{opt}>0\\ 0 & \text{, otherwise}\end{cases}$$

■