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Non-technical Summary

Systemic risk is the risk that several market participants are simultaneously affected by severe losses, which then spread to the entire economy through financial interconnections among economic agents. The structure of financial networks is a key component that can either attenuate or amplify systemic risk and thus its understanding is crucial to assessing, monitoring, and regulating financial systems.

Traditional systemic risk models only account for the interbank market as the main source of shock amplification and thus assume as exogenous other potential contagion channels. There is no evidence on to what extent one could be underestimating potential systemic risk if other contagion channels, such as those from the real sector, were not taken into account.

The difficulty to incorporate the real economy in systemic risk models arises because so far there is a gap in the literature in how to model feedback effects that may occur between the real and financial sectors. The feedback between these two sectors effectively endogenizes the behavior of firms and banks as economic agents that are able to process and amplify losses.

Shocks may start in the real sector and passthrough to the financial sector, which in turn can amplify these initial shocks, thus giving form to a vicious circle by worsening even more the financial conditions to the real sector and so on. We can only model this back-and-forth mechanism by incorporating the feedback between banks and firms.

This paper develops an innovative framework to estimate systemic risk that accounts for these feedback effects between the real and financial sectors. The feedback gives rise to a negative financial accelerator among banks that is fueled by illiquidity spirals and balance-sheet contagion. While firms amplify shocks by not paying back their debt towards the financial sector, banks apply a credit crunch on the real sector.

Using loan-level data from the Brazilian credit register, we find that the feedback between the real and financial sectors is economically significant.

Sumário Não Técnico

O risco sistêmico é o risco de que vários participantes do mercado sejam simultaneamente afetados por perdas severas, que então se espalham para toda a economia através de interconexões financeiras entre agentes econômicos. A estrutura das redes financeiras é uma componente chave que pode atenuar ou amplificar o risco sistêmico e, portanto, sua compreensão é crucial para avaliar, monitorar e regulamentar os sistemas financeiros.

Modelos tradicionais de risco sistêmico apenas consideram o mercado interbancário como principal fonte de amplificação de choques e, portanto, assumem como exógenos outros potenciais canais de contágio. Não há evidências de até que ponto se estaria potencialmente subestimando o risco sistêmico se outros canais de contágio, como aqueles do setor real, não fossem levados em conta.

Há uma dificuldade de se incorporar a economia real em modelos de risco sistêmico porque até o momento inexistem modelos que quantifiquem esse *feedback* que podem ocorrer entre os setores real e financeiro. O *feedback* entre esses dois setores efetivamente endogeniza o comportamento das empresas e bancos como agentes econômicos capazes de processar e amplificar perdas.

Os choques podem começar no setor real e atingir o setor financeiro, o qual, por sua vez, pode amplificar esses choques, dando assim início a um círculo vicioso que agrava ainda mais as condições financeiras para o setor real. Só há a possibilidade de se modelar esse mecanismo em que choques vão e voltam entre os setores real e financeiro se considerarmos o mecanismo de *feedback* existente entre firmas e bancos.

Este trabalho desenvolve um *framework* inovador para estimar o risco sistêmico que incorpora os efeitos de feedback entre os setores real e financeiro. O *feedback* dá origem a um acelerador financeiro negativo que é alimentado por espirais de iliquidez e contágio de balanço. Enquanto as empresas amplificam choques por não pagarem suas dívidas oriundas do setor financeiro, os bancos restringem crédito para o setor real.

Utilizando microdados do registro de crédito brasileiro, verificamos que o *feedback* entre os setores real e financeiro é economicamente significativo.

Systemic Risk in Financial Systems: a feedback approach

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Abstract

We develop an innovative framework to estimate systemic risk that accounts for feedback effects between the real and financial sectors. We model the feedback effects through successive deterioration of borrowers' creditworthiness and illiquidity spreading, thus giving rise to a micro-level financial accelerator between firms and banks. We demonstrate that the model converges to a unique fixed point and the key role that centrality plays in shaping the level of amplification of shocks. We also provide a mathematical framework to explain systemic risk variations in time as a function of the network characteristics of economic agents. Finally, we supply empirical evidence on the economic significance of the feedback effects on comprehensive loan-level data of the Brazilian credit register. Our results corroborate the importance of incorporating new contagion channels besides the traditional interbank market in systemic risk models. Our model sheds light on the policy issue regarding risk-weighting of assets that also internalizes the costs of systemic risk.

Keywords: systemic risk, feedback, financial accelerator, financial network, transmission channel, contagion.
JEL Classification: G01, G21, G28, C63.

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1 Introduction

There is some effort within the literature to develop models that measure systemic risk, which are of utmost importance for financial supervisors and for designing policies that help mitigate its effects. Modeling the interaction between the real and the financial sector and providing evidence on how the interconnectedness within the financial markets may amplify shocks are crucial steps towards these efforts.[1]

The difficulty to incorporate the real economy in systemic risk models arises because to date there is a gap in the literature in how to model feedback effects that may occur between the real and financial sectors. The feedback between these two sectors effectively endogenizes the behavior of firms and banks as economic agents that are able to process and amplify losses. Moreover, there is a lack of empirical evidence of how shocks within the real sector may be amplified by the financial sector using micro data. Shocks may start in the real sector and passthrough to the financial sector, which in turn can amplify these initial shocks, thus giving form to a vicious circle by worsening even more the financial conditions to the real sector and so on. We can only model this back-and-forth mechanism by incorporating the feedback between banks and firms.

A recent theoretical model described in Silva et al. (2016a) considers a network of interdependencies while accounting for interbank connections, bank lending to firms, and feedback among firms and banks. The model relies on a multilayer network approach to capture the heterogeneities between banks and firms. For a given shock, Silva et al. (2016a)'s model measures financial distress of banks and firms proportionally to the reduction of the their available resources, which can occur by means of financial contagion.[2] However, the model is not clear in how economic agents propagate shocks to counterparties. In addition, the contagion dynamic is not symmetrical with respect to sources of vulnerabilities. For instance, while banks and firms are prone to financial losses both in the asset and liability sides of their balance sheets via the feedback engine, only the asset side of these economic agents is susceptible to losses in the interbank market and in the corporate trade network.

The paper contributes to the systemic risk literature in four main ways. First, we provide a microfoundation that justifies financial contagion in an economy comprising banks and firms, as well as feedback between them. Second, we enhance Silva et al. (2016a)'s feedback mechanism by incorporating several empirical evidences found in the finance literature on bank-firm relationship behavior in adverse scenarios to fine-tune

[1]In this matter, most of the research focuses on the financial sector, but relegates the importance of real sector linkages that may amplify shocks. See, for instance, Xu et al. (2016), Martínez-Jaramillo et al. (2014), Souza et al. (2016), Batiz-Zuk et al. (2015), Martínez-Jaramillo et al. (2010), and Georg (2013).

[2]This model adds to the discussion in Silva et al. (2016b), Souza et al. (2015) and Cajueiro and Tabak (2008) as it looks at both the real and the financial side of a contagion and amplification process of shocks that may hit the economy.

firms' sensitiveness to credit crunches from the financial sector. Third, using a theoretical approach, we show how idiosyncratic shocks to economic agents can develop into systemic events and also describe a strategy to decompose systemic risk variations in evolving financial networks in terms of network measures with clear economic meaning. Fourth, we apply our model to a comprehensive loan-level data extracted from the Brazilian credit register. Thus, our results corroborate the importance of incorporating new contagion channels besides the traditional interbank market in systemic risk models.

The feedback mechanism between the real and financial sectors gives rise to a micro-level financial accelerator between banks and firms that is fueled by illiquidity spirals and balance-sheet contagion.[3] The essence of our feedback engine works as follows: (i) an external shock negatively impacts the real sector; (ii) affected firms have their net worth jeopardized and thus default on their bank loans; (iii) to comply with their regulatory capital requirements, affected banks reduce bank lending through a credit crunch to the real sector; consequently (iv) firms become even more distressed due to the reduction in credit and therefore default on other bank loans, thus amplifying the initial shock.[4]

To calibrate how banks and firms react and respond to these negative events, our model incorporates several empirical evidences found in the finance literature on bank-firm relationship behavior in adverse scenarios. For instance, the extent to which banks negatively impact firms depends on (i) how proficient firms are in replacing banks that are reducing credit, (ii) how dependent firms are on bank credit to run their businesses, and (iii) how large the short-term debt of firms toward banks is. In the opposite direction, firms can reduce banks' liquidity or lead them to insolvency by (i) early-redeeming bank bonds and (ii) defaulting on bank loans.

We represent the financial and real sectors as layers of a multilayer financial network. The financial sector layer encompasses banks that interact through interbank borrowing and lending relationships. The real sector layer contains the corporate trade network, in which supplier firms grant purchases on credit to customer firms. In our setup, firms can invest in bank bonds and banks can grant loans to firms they find profitable. Banks and firms can also hold assets outside the network that they can sell off to meet their short-term needs.

Our model consists of a multiperiod economy in which at $t = 0$ economic agents exogenously set their investments, liabilities, and net worth. At $t = 1$, one or more economic agents suffer an external shock that drain some of their net worth or equities. At $t > 1$, this shock propagates and gets amplified by the financial network in the form of deterioration of banks' and firms' balance sheets. Our model assumes that the financial network does not change once the shock hits the economic agents. Therefore, our model

[3]The financial accelerator has been extensively studied from the macroeconomic viewpoint (Bernanke (1983); Bernanke and Gertler (1989); Bernanke et al. (1996)).

[4]A similar reasoning applies for an external shock that initially hits banks.

is useful to estimating systemic risk in the very short run given an external shock.

The idea of a financial accelerator that enables the feedback between banks and firms originates from the fact that the borrower amplifies an initial negative shock by further decreasing its investment and production activities. We believe that the financial accelerator is an important feature because of the following two reasons. First, banks cannot fully insulate their supply of lending in response to negative shocks, because they have finite capital buffers and regulatory capital constraints. Second, firms normally rely on bank credit to run their business, especially those that do not have access to financial markets, because equities are an expensive source of funding. Therefore, we expect the existence of this feedback mechanism between firms and banks in real financial networks.

Using loan-level data on the Brazilian financial and real sectors, we find that the feedback between the real and financial sectors is economically significant. Moreover, we find that firms respond differently to the feedback mechanism due to their contrasting profiles on the components that influence the feedback effect.[5] Our investigation highlights that models that do not consider the feedback effects could be (i) severely underestimating systemic risk and (ii) inconsistently ranking the riskiest firms and hence sectors in the economy. Considering that the financial system surveillance and policymaking depend on proper information regarding the systemic risk levels of the financial system, it becomes crucial to take into account feedback effects to effectively identify the largest sources of systemic risk of the economy.

We also demonstrate that the model presents strong theoretical properties such as the existence of a unique fixed point that depends on the network structure and the magnitude of the shock.[6] We further develop our theoretical findings taking as baseline this unique fixed point and derive useful systemic risk properties in its cross-sectional and time dimensions in evolving financial networks.[7]

In the cross-sectional component of systemic risk, we demonstrate how microeco-

[5]For instance, in our sample, a great part of bank credit that firms in the tertiary sector take are short term while firms in the oil and gas sector normally hold long-term loans. In addition, the technology sector has difficulty in substituting banks due to their small number of bank counterparties while food and beverage sector shows the opposite pattern.

[6]Our work is also in line with Acemoglu et al. (2015)'s findings who also show the existence of a unique fixed point that depends on the network structure and magnitude of the shock. However, their model focuses on the interbank market and thus does not incorporate firms nor the negative feedback mechanism that exist between banks and firms. In addition, contrasting to their work, our theoretical approach is still valid regardless of the shock magnitude.

[7]We can conceptualize systemic risk in two orthogonal but complementary dimensions: the cross-sectional dimension and the time dimension. In the first, we are concerned with how the structure of the financial system influences, and possibly amplifies, external shocks. In the time dimension, we usually look to which directions systemic risk can evolve as a result of economic agents' individual actions and other components, such as the macroeconomic cycle, that ultimately lead to variations on systemic risk of a financial system. While the cross-sectional component has been extensively studied by the literature (Acemoglu et al. (2015); di Giovanni et al. (2014); Elliott et al. (2014); Gai et al. (2011); Gandy and Veraart (2016)), the time dimension is often overlooked.

nomic shocks can develop into macroeconomic events in a system composed of banks and firms. This analysis serves to somewhat unbox the "black box" behavior of financial networks and thus show how economic interconnections between heterogeneous economic agents contribute to amplifying shocks, possibly leading to systemic events. We find that economic agents—firms or banks—that are more central in the network are the key sources of systemic risk in the financial system.[8] Our findings are useful not only to ranking and finding the key players inside financial systems but also to providing quantitative insights as to how harmful economic agents can be to the entire economy.

In the time component of systemic risk, we study how structural changes in the network affect systemic risk. All else equal, we find that the addition of a network connection, which can emerge when a bank lends to a new firm, changes systemic risk proportionally to the product between (i) the propensity of the bank of receiving shocks and (ii) the ability of the firm in amplifying shocks.[9] This is an interesting result because systemic risk would be insensitive to changes on new connections in which at least one economic agent has zero susceptibility or diffusion. In contrast, systemic risk would rise to a significant extent whenever both indices were large. Our result can be useful to identifying dangerous connections inside a financial network and thus have practical implications for monitoring and assessing financial stability.

It is worth emphasizing that the model is general. Therefore, for any risk source or different shocks to different economic agents or markets we can assess how systemic risk would evolve. In this respect, we can employ the framework to evaluate the impact of shocks on the housing market or to other economic sectors on the real economy. Note that the financial network serves as a loss amplifying medium of whatever shocks it receives. Since we allow the use of many different types of interconnections that can naturally arise between economic agents, we also contribute by developing a model that take into account multiplexes of interrelationships.

2 Methodology

In this section, we discuss our systemic risk model. We start by presenting a quick glimpse on how the model works. Then, we describe the model's microfoundations using an abstract framework of economic agents. Afterwards, we specialize this framework to

[8]Our results add to Acemoglu et al. (2015)'s findings by showing that network centrality still plays a major role even in a complex environment with economic agents with different nature and negative feedback rules.

[9]The notions of susceptibility and diffusion in the context of financial contagion have appeared in the financial network analysis literature before, such as in Drehmann and Tarashev (2013), Greenwood et al. (2015) and Silva et al. (2017). Our paper contributes to this area using a new approach, which is in using these concepts of susceptibility and diffusion to explain how systemic risk can change more or less when connections modify in the network.

an economy comprising banks and firms, where we explicitly model a stress feedback engine between them. Finally, we demonstrate useful theoretical properties of the model.

2.1 Intuition of the model

We represent the financial and real sectors as layers of a multilayer financial network. The financial sector layer encompasses banks that interact through interbank borrowing and lending relationships. The real sector layer contains the corporate trade network, in which supplier firms grant purchases on credit to customer firms. In our setup, firms invest in bank bonds and banks grant loans to firms they find profitable.

Figure 1 exhibits the multilayer network that represents our economy, where the upper and bottom layers constitute the financial and real sector layers, respectively. A link from A to B in our model embodies financial vulnerability from the former to the latter that can arise either from the asset or the liability side of economic agent A' balance sheet. Financial vulnerability generates co-movements in the balance sheets of linked economic agents and thus is an engine that fuels financial contagion in the economy.

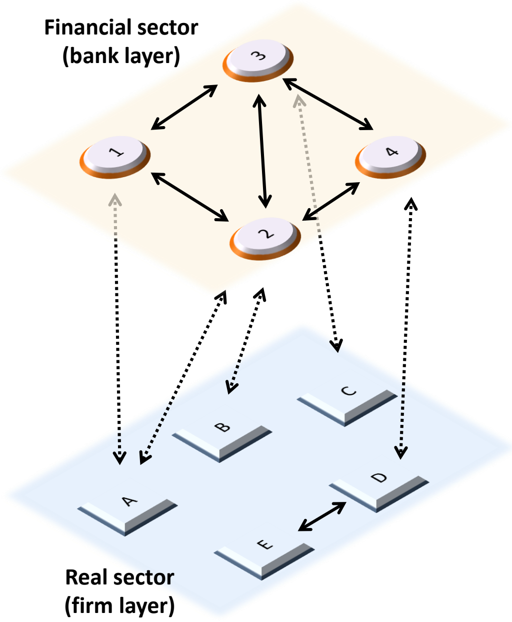


Figure 1: *A multilayer financial network with two layers: bank (circle) and firm (square) layers. Edges represent financial vulnerability between two economic agents. We represent links connecting economic agents of the same nature with continuous arrowed lines and links connecting the real and financial sectors, which are responsible for the feedback mechanism, with dashed arrowed lines.*

In respect to the financial sector layer, banks interconnect to satisfy their liquidity needs and also to profit from financial operations in which they enjoy comparative advantage.[10] Therefore, in the asset side, interbank links indicate financial vulnerability due to

[10]For instance, banks can group together and use certain financial instruments to obtain mutual benefits with cost savings and increased profits. For instance, banks that are competitive in lending to the non-

investments in other bank counterparties. For instance, bank 3 lends to bank 1 in Fig. 1 and thus is subject to loan default. In the liability side, we model funding vulnerability that emerges from the need of meeting short-term obligations. Whenever bank counterparties are under financial distress, they may not roll over their short-term debt to debtor banks. Consequently, these debtor banks may run into liquidity problems if they have insufficient cash to honor their short-term obligations. For instance, bank 1 has a funding vulnerability to bank 3 when the exposure is due in the short term and bank 3 does not roll over that debt.

In regard to the real sector layer, firms also have cross-exposures to meet their liquidity needs or even to match different cashflow seasonality.[11] Thus, in the asset side, interfirm links denote credit on services or goods that supplier firms grant to customer firms. In Fig. 1, the supplier firm *E* grants credit to customer firm *D* and hence is susceptible to credit default of that firm. In the liability side, we also consider the funding vulnerability between two firms that becomes important when firms face unexpected liquidity needs. In Fig. 1, firm *D* has a funding vulnerability toward firm *E* when that debt is due in the short term.

Linking the real and financial sector layers, banks and firms also interconnect to boost their profits. On the one side, firms invest in bank bonds that they find profitable, making them susceptible to bond default of that counterparty in the asset side of their balance sheet. For instance, in Fig. 1, firm *A* bought bonds issued by bank 1 and therefore such investment becomes dependent on the financial soundness of bank 1. In the liability side, firms are vulnerable to credit crunches of the financial sector. A credit crunch may lead firms to readjust their project portfolio and thus can impact their profits. In Fig. 1, firm *A* is susceptible to a credit crunch of bank 1. On the other side, banks lend credit to firms so that the latter can finance their projects and hence maintain their operational activities toward their customers. Thus, there is a vulnerability in the asset side of banks' balance sheets in view of the loan. In Fig. 1, bank 1 lends to firm *A* and therefore the former bears the risk of a loan default. In the liability side, banks are susceptible to early redemptions of their bonds that the real sector acquires. In Fig. 1, firm *B* bought bonds issued by bank 2 and thus the latter is susceptible to early redemption of those bonds.

We give particular emphasis on the stress feedback between the real and financial

financial sector and do not have the same ability as fundraisers can borrow funds from banks with excess of liquidity, thus obtaining the necessary fund to supply credit to the non-financial sector. In light of this association, both banks would be acting in their business lines that they possibly enjoy comparative advantage and would therefore have larger credit portfolios.

[11]For instance, a farmer that plants soybeans and needs machinery may expect payment inflows in fixed periods of the year, while a firm that produces an industrialized product whose inputs depend on soybeans may expect a continuous cashflow. In this setup, knowing that the farmer has a continuous demand for soybeans by industrialized firms, another firm that produces cropping machinery may grant on credit tractors to the farmer with the expectation of payment whenever the soybeans is sold. In general, inputs with different specificities and seasonality generate interfirm trade networks.

sectors. This feedback gives rise to a micro-level financial accelerator that amplifies negative shocks. Our framework evaluates how the net worth or equities of banks and firms deteriorate as a consequence of an external shock that triggers financial contagion. We consider that the network topology is exogenous and does not change during contagion, such that the model is useful to estimating the short-run consequences in the economy of a negative shock. We look at how the feedback between banks and firms through an example. Suppose firm A in Fig. 1 suffers a negative shock that reduces its net worth.[12] Thus, the unexpected reduction in firm A's net worth causes financial distress and leads to consequences both for the firm itself and also to creditor counterparties as follows:

- Firm A early-redeems bond investments issued by banks to raise cash and cover up the negative impacts of the external event.
- Creditor banks reprice down their loans granted to firm A in view of the reduction in firm A's net worth to reflect the increase in the loan default probability.

These two events jeopardize creditor banks both in the asset side, due to the repricing mechanism, and also in the liability side, due to the reduction in available funding. In the illustrative network in Fig. 1, firm A negatively affects the balance sheets of banks 1 and 2. While the loss on the bank loan directly reflects on a reduction in banks' net worth, the amount that is early-redeemed by firm A does not necessarily lead banks into losses. Banks will suffer losses only when firm A early-redeems a large amount of bonds, because this event will force banks into firesales of illiquid assets to make up cash for their short-term obligations. To prevent violating regulatory capital constraints, banks reduce their risk-weighted assets by replacing lending to the real sector by more secured assets, such as government bonds, and thus the real sector suffers a credit crunch.[13]

Firms absorb the credit restriction of the financial sector in heterogeneous ways. In Fig. 1, the newly distressed banks 1 and 2 reduce credit to their customer firms A and B. Based on empirical observations in the literature, we model the credit crunch that the real sector suffers as a function of three components:

- the amount of short-term debt that firms have with their creditor banks. The larger the amount of the short-term debt, the more firms will be exposed when banks do

[12] Our model captures any shocks that reflect on the net worth of banks and firms. For instance, firms can have their net worth changed by natural disasters that may devastate machinery or production inputs or by sudden increases in interest rates that may push down asset prices and weaken balance sheets.

[13] Empirical evidences show that reduction in banks' net worth is an important component that explains credit crunches on the real sector. See, for instance, Bernanke et al. (1996) and Iyer et al. (2014). We can also see this credit crunch as a consequence of the reduction of firms' net worth that depresses their creditworthiness as borrowers, as their ability to post collateral to reduce the operation risk diminishes. See, for instance, Bernanke (1983), who discusses how the creditworthiness of borrowers acts as a channel through which financial crises can affect the real economy, thereby preventing the efficient allocation of credit.

not roll over these debts. To make up cash and meet their short-term obligations, firms will be forced to firesale assets with prices below fundamentals, incurring thus in monetary losses.

- the proficiency of firms in replacing bank counterparties by others that are reducing credit. Firms that do not rely much on bank credit to finance their projects will be less affected than those firms that are strongly reliant on bank credit.[14]

- the strength of the relationship lending history between firms and banks. We capture here the information asymmetry that arises as a financial friction between banks and firms. A central function of banks is to screen and monitor borrowers, thereby overcoming information and incentive problems. By developing expertise in gathering relevant information, as well as by maintaining ongoing relationships with customers, banks and similar intermediaries develop informational capital. Good long-term relationships between one firm and one bank can generate benefits to both counterparties.[15] Therefore, in our model, the more lengthy the relationship history between firms and banks is, the less substitutable that bank would be to the firm in case of a credit crunch.

The reduction of credit to the real sector has a downside effect in that it reduces profitability of firms with less resources to invest, thereby inflicting more financial distress to these firms. As a result, this negative effect then bounces back to the financial sector in the form of further deterioration of loans and early redemption of firm bonds, thus completing the negative cycle. Our main contribution in this work is not in only designing this feedback mechanism that links the financial and real sectors, but also in showing that it is economically significant for systemic risk analysis.

2.2 Microfoundations of the model

In this section, we derive the economic microfoundations of our model and treat banks and firms indistinctively as economic agents.

Although our model consists of a single-period economy, we represent the shock propagation process as a dynamic system that may take several iterations before converging. Since our model does not have a time component, because it captures the immediate effect of an external shock, the model is only suitable to estimating the very short-term systemic risk consequences of that event.

[14] In this component, smaller firms will be more affected because they usually do not have access to financial markets and hence cannot issue bonds nor commercial papers (Holmstrom and Tirole (1997)).

[15] For instance, banks can reduce monitoring costs due to reduction in information asymmetry and consequently firms can enjoy lower interest rates. For more researches dealing with the effect of long relationships between banks and firms and the amount of lending, see Bernanke (1983), Gorton and Pennacchi (1990), Dahiya et al. (2003), Temizsoy et al. (2015), and Finger and Lux (2017).

For mathematical convenience, we assume that $t = 0$ represents the economy prior to the shock, where t indexes the current iteration of the dynamic system. An exogenous shock hits one or more economic agents at $t = 1$. The dynamic system takes as input this shock and therefore iterates from $t = 1$ onwards. If t_c is the iteration of convergence, we are interested in analyzing the economy at $t = t_c$, which represents the very short-term consequences of the shock that took place at iteration $t = 1$.

The balance sheet of economic agent i at iteration t consists of three elements:

1. *Assets*: we subdivide the economic agent i's assets at time t, $\mathbf{A}_i(t)$, as assets inside the financial network $\mathbf{A}_i^{(\text{in})}(t)$, which include investments in banks and firms, and external assets $\mathbf{A}_i^{(\text{out})}(t)$, which encompass liquid assets (cash) as well as illiquid assets (such as vehicles and properties). Therefore, $\mathbf{A}_i(t) = \mathbf{A}_i^{(\text{in})}(t) + \mathbf{A}_i^{(\text{out})}(t)$ and $\mathbf{A}_i(t) > 0$.

2. *Liabilities*: we subdivide liabilities of economic agent i at time t, $\mathbf{L}_i(t)$, as inside-network short-term liabilities $\mathbf{L}_i^{(\text{in-st})}(t)$, inside-network long-term liabilities $\mathbf{L}_i^{(\text{in-lt})}(t)$, and outside-network liabilities $\mathbf{L}_i^{(\text{out})}(t)$. Thus, $\mathbf{L}_i(t) = \mathbf{L}_i^{(\text{in-st})}(t) + \mathbf{L}_i^{(\text{in-lt})}(t) + \mathbf{L}_i^{(\text{out})}(t)$ and $\mathbf{L}_i(t) > 0$.

3. *Equities or net worth*: they embody the economic agent's residual value once all liabilities are properly settled, such that $\mathbf{E}_i(t) = \mathbf{A}_i(t) - \mathbf{L}_i(t)$. When $\mathbf{E}_i(t) = 0$, we say that economic agent has defaulted at time t. If $\mathbf{E}_i > 0$, then economic agent i is active. Though equities can achieve negative values by the fundamental accounting equation, our model assumes that economic agents that reach $\mathbf{E}_i(t) = 0$ are immediately put into liquidation and hence are removed from the contagion process, in such a way that $\mathbf{E}_i(t)$ remains at zero in the short run.

Financial assets held by economic agents inside the financial network always have reciprocal entries: for every asset held by an economic agent i, there is a corresponding liability registered by a counterparty j within the same network. Therefore, these inside-network assets interconnect balance sheets and are sources of contagion both in the asset and liability sides of the involved counterparties. In the asset side, our model considers that economic agents can face losses in the inside-network investments due to financial deterioration of borrowers. In the liability side, economic agents are subject to liquidity issues that can drive them into losses when they have to sell off outside-network illiquid assets to meet short-term obligations.[16]

Overall equation: Our model respects the fundamental accounting equation and is concerned with how the net worth or equities of economic agents deteriorate when external negative events occur. Spillover effects can occur through the network links. We model

[16]Martínez-Jaramillo et al. (2010) also provide a similar dynamics.

two risk sources that can reduce economic agent i's net worth: the counterparty and the funding risks. To identify these sources, first consider the differential version of the fundamental accounting equation:

$$\begin{aligned}\Delta \mathbf{E}_i(t) &= \Delta \mathbf{A}_i(t) - \Delta \mathbf{L}_i(t) \\ &= \left[\Delta \mathbf{A}_i^{(\mathrm{in})}(t) + \Delta \mathbf{A}_i^{(\mathrm{out})}(t)\right] - \Delta \mathbf{L}_i(t),\end{aligned} \tag{1}$$

in which we make explicit variations of assets that are inside and outside the financial network. The net worth of economic agent i can reduce in view of (i) direct impacts on its inside-network assets and (ii) indirect impacts on its outside-network assets. Hence, we rewrite the previous equation as:

$$\begin{aligned}\Delta \mathbf{E}_i(t) &= \Delta \mathbf{A}_i^{(\mathrm{in})}(t) + \left[\Delta \mathbf{A}_i^{(\mathrm{out})}(t) - \Delta \mathbf{L}_i(t)\right] \\ &= \Delta \mathbf{E}_i^{(\mathrm{ct})}(t) + \Delta \mathbf{E}_i^{(\mathrm{f})}(t)\end{aligned} \tag{2}$$

in which $\Delta \mathbf{E}_i^{(\mathrm{ct})}(t) = \Delta \mathbf{A}_i^{(\mathrm{in})}(t)$ and $\Delta \mathbf{E}_i^{(\mathrm{f})} = \Delta \mathbf{A}_i^{(\mathrm{out})}(t) - \Delta \mathbf{L}_i(t)$ indicate potential losses due to counterparty risk and funding risk, respectively.

Losses due to counterparty risk: Losses due to counterparty risk arise because economic agents internalize in their balance sheets changes in the debtors' creditworthiness due to fluctuations in the default probability. Figure 2(a) illustrates this mechanism. Since creditor i monitors its investments, we assume that it is able to see or at least estimate the net worth conditions of debtors, say j in the figure. If there is a downfall in j's net worth, then i will reprice down its investments toward j to reflect the new deteriorated credit quality (increased default probability). The reduction in value of this investment reflects in a subsequent reduction in i's net worth as well. In another round, creditors of i will also see the change in i's creditworthiness and will also reprice down their investment, and so on.

This repricing mechanism only assumes that economic agents locally monitor their debtors, i.e., their direct neighbors in the network, to mitigate risk. This is consistent with the fact that (i) apart from the local knowledge of the neighbors (creditors or debtors), the network topology is unknown from the viewpoint of each economic agent and (ii) information asymmetry increases as a function of the (shortest) distance of two economic agents in the network. In particular, economic agents have relatively strong incentives to monitor direct neighbors, while this behavior significantly decreases for neighbors of neighbors and so forth.

We use the following assumption to model how economic agents price inside-

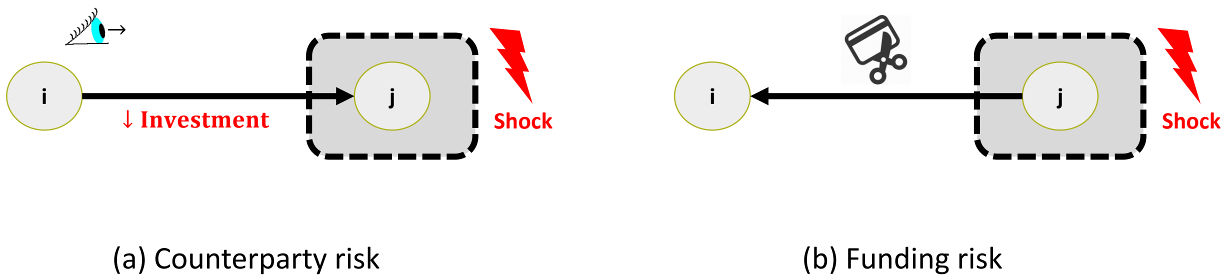


Figure 2: Local implications in the network when an external shock hits economic agent j. All neighbors of j, which we abstractly reduce to a single neighbor i, can be potentially affected due to (a) counterparty risk and (b) funding risk. (a) Counterparty risk: Since economic agents monitor their investments, economic agent i can observe j's net worth. Since j received a negative shock, i reprices down the exposure towards j to reflect the downfall in j's creditworthiness (increased default probability). (b) Funding risk: From the other viewpoint, j will potentially need to make up more cash and may not roll over short-term credit or grant new loans at the adverse moment. Hence, it cuts short-term credit to the debtor counterparty i.

network investments:

Assumption 1. *Economic agents reprice down investments solely based on the net worth variation (creditworthiness) of debtors in a linear fashion.*[17]

Using Assumption 1. we assume that creditor i reprices down such investment using a linear decaying function on debtor j's net worth, i.e.:

$$\mathbf{A}_{ij}^{(\text{in})}(t+1) = \begin{cases} \mathbf{A}_{ij}^{(\text{in})}(t)\frac{\mathbf{E}_j(t)}{\mathbf{E}_j(t-1)}, \text{if } j \in \mathscr{A}(t) \\ 0, \text{if } j \notin \mathscr{A}(t) \end{cases} \tag{3}$$

in which $\mathbf{A}_{ij}^{(\text{in})}(t)$ represents investment value of creditor i in borrower j at iteration t, $\mathbf{E}_j(t)$ is the net worth of j at iteration t, and $\mathscr{A}(t) = \{i \in \mathscr{S} : \mathbf{E}_i(t) > 0\}$ is the set of economic agents that are still active at iteration t, where $\mathscr{S}$ is the set of all economic agents in the economy. We assume that economic agent i reprices the investment $\mathbf{A}_{ij}^{(\text{in})}$ at iteration $t+1$ according to the latest momentum of debtor j's net worth, i.e., $\mathbf{E}_j(t-1)$ and $\mathbf{E}_j(t)$. If debtor j suffers a downfall in its equities, then $\frac{\mathbf{E}_j(t)}{\mathbf{E}_j(t-1)} < 1$, such that $\mathbf{A}_{ij}^{(\text{in})}(t+1) < \mathbf{A}_{ij}^{(\text{in})}(t)$.

The investment $\mathbf{A}_{ij}^{(\text{in})}(t+1)$ has a positive value only if the borrower is still active at iteration t. If the borrower defaults, then the investment becomes worthless in the market in the short-term. This hypothesis is consistent with the fact that, although the financial asset may be recoverable to some extent in the medium or long run, it is unlikely that

[17] Bardoscia et al. (2015) also use a linear decaying function to simulate a similar investment repricing mechanism. Appendix A shows that the model also accepts non-linear functional forms of transmitting risk. The way risk spills over from one economic agent to another has a direct effect on our systemic risk estimates.

creditor i will receive back any cash in the short term because defaulted economic agents must undergo liquidation processes that involve time-consuming judicial intervention.

Considering that i may have multiple debtors in the network, the total net worth loss due to counterparty risk is the sum of the losses of each these inside-network investments, i.e.:

$$
\begin{aligned}
\Delta \mathbf{E}_i^{(\mathrm{ct})}(t+1) &= \mathbf{A}_i^{(\mathrm{in})}(t+1) - \mathbf{A}_i^{(\mathrm{in})}(t) \\
&= \sum_{j \in \mathscr{A}(t)} \mathbf{A}_{ij}^{(\mathrm{in})}(t+1) - \sum_{j \in \mathscr{A}(t-1)} \mathbf{A}_{ij}^{(\mathrm{in})}(t) \\
&= \sum_{j \in \mathscr{A}(t-1)} \left[\mathbf{A}_{ij}^{(\mathrm{in})}(t+1) - \mathbf{A}_{ij}^{(\mathrm{in})}(t) \right] - \sum_{j \in \mathscr{A}(t-1) \setminus \mathscr{A}(t)} \mathbf{A}_{ij}^{(\mathrm{in})}(t+1), \qquad (4)
\end{aligned}
$$

in which $\mathscr{A}(t-1) \setminus \mathscr{A}(t)$ indicates the residual set of economic agents that were active at $t-1$ but defaulted at time t. The second term in (4) results in 0 because all debtors j are in default at iteration t. In this way,

$$
\Delta \mathbf{E}_i^{(\mathrm{ct})}(t+1) = \sum_{j \in \mathscr{A}(t-1)} \frac{\mathbf{A}_{ij}^{(\mathrm{in})}(t)}{\mathbf{E}_j(t-1)} \left[\mathbf{E}_j(t) - \mathbf{E}_j(t-1) \right]. \qquad (5)
$$

If we recursively replace $\mathbf{A}_{ij}^{(\mathrm{in})}(t)$ in (5) as a function of $\mathbf{A}_{ij}^{(\mathrm{in})}(t-1)$ using (3), we can value $\mathbf{A}_{ij}^{(\mathrm{in})}(t)$ in terms of the original face value of the investment $\mathbf{A}_{ij}^{(\mathrm{in})}(0)$ as follows:

$$
\begin{aligned}
\Delta \mathbf{E}_i^{(\mathrm{ct})}(t+1) &= \sum_{j \in \mathscr{A}(t-1)} \mathbf{A}_{ij}^{(\mathrm{in})}(1) \frac{\mathbf{E}_j(1)}{\mathbf{E}_j(0)} \frac{\mathbf{E}_j(2)}{\mathbf{E}_j(1)} \cdots \frac{\mathbf{E}_j(t-1)}{\mathbf{E}_j(t-2)} \frac{1}{\mathbf{E}_j(t-1)} \left[\mathbf{E}_j(t) - \mathbf{E}_j(t-1) \right] \\
&= \sum_{j \in \mathscr{A}(t-1)} \frac{\mathbf{A}_{ij}^{(\mathrm{in})}(0)}{\mathbf{E}_j(0)} \left[\mathbf{E}_j(t) - \mathbf{E}_j(t-1) \right] \qquad (6)
\end{aligned}
$$

in which $\mathbf{A}_{ij}^{(\mathrm{in})}(0) = \mathbf{A}_{ij}^{(\mathrm{in})}(1)$ because equities only change at $t = 1$ onward and therefore assets can only change for $t \geq 2$ in view of (3). Equation (6) quantifies variations of economic agent i's net worth exclusively as a function of the exogenous variables $\mathbf{A}_{ij}^{(\mathrm{in})}(0)$ and $\mathbf{E}_j(0)$.

Losses due to funding risk: We assume that economic agents perform precautionary liquidity hoarding as they approach insolvency. The hoarding is performed by not rolling over outstanding short-term credit to debtor neighbors in the network (credit crunch).[18]

[18]During the global financial crisis, liquidity hoarding was widely observed. The behavioral hypothesis we use here is consistent with the work of Gai et al. (2011) and Acharya and Skeie (2011). They show that

We measure the distance to insolvency by looking at how far from zero the stressed net worth of an economic agent is. To link the extent of the credit crunch and the distance to insolvency, we use the following assumption in our model.

Assumption 2. *The extent of liquidity hoarding linear relates to how close one is to insolvency.*

Example 1. *To illustrate, consider Figure 2(b). Suppose that economic agent i has an outstanding short-term debt of 100 dollars to j in Fig. 2(b). If j suffers a shock that reduces by 40% its net worth, then it cuts credit (credit crunch) by the same amount, in a linear way, i.e., 40 dollars. In this way, i will have to make up cash to pay back* 60 *dollars in the short term.*

Generally, if $\mathbf{L}_{ij}^{(\text{in-st})}$ is the outstanding short-term liability of i toward j, then the potential cash outflow in the short term of i to j can range from 0 to $\mathbf{L}_{ij}^{(\text{in-st})}$. Due to Assumption 2, if economic agent j is undistressed, then it fully rolls over the credit to i and no liquidity exposures arise in the short term. In contrast, if j defaults, then it does not roll over any credit and consequently i must pay back all the debt in the short term.

In view of that, we can express the liquidity exposure of i to j as $\alpha_{ij}\mathbf{L}_{ij}^{(\text{in-st})} \in [0, \mathbf{L}_{ij}^{(\text{in-st})}]$, in which $\alpha_{ij} \in [0,1]$ captures the idiosyncrasies between i and j that can either exacerbate or attenuate the consequences of a credit crunch of j on i's financial health. Such term depends on:

- The liquidity conditions of debtor i and creditor j. The more liquid the creditor j is, the less likely it is to restrain credit to its counterparties. In addition, the more liquid the debtor i is, the less affected it will be from a credit crunch of creditor j. For example, if i is liquid enough to withstand the credit restriction, then $\alpha_{ij} = 0$, such that there are no liquidity exposures and therefore no funding risk of i to j.

- The ability of economic agent i to replace the funding counterparty j by another one. The more substitutable j is to i, the less affected i will be from a credit crunch of j.

To estimate losses due to funding risk, we first simplify (2) as follows:

$$\begin{aligned}
\Delta\mathbf{E}_i^{(\text{f})}(t+1) &= \Delta\mathbf{A}_i^{(\text{out})}(t+1) - \Delta\mathbf{L}_i(t+1) \\
&= \Delta\mathbf{A}_i^{(\text{out})}(t+1) \\
&= \Delta\mathbf{A}_i^{(\text{out-liq})}(t+1) + \mathbf{A}_i^{(\text{out-illiq})}(t+1) \\
&= \Delta\mathbf{A}_i^{(\text{out-illiq})}(t+1).
\end{aligned} \tag{7}$$

banks use liquidity hoarding as a way to control their uncertainty over their ability to roll over their own debt or even to survive.

In the first transition, we use the fact that economic agent i always registers liabilities with their face values, regardless of the financial state of their creditors. In this way, $\mathbf{L}_i(t+1) = \mathbf{L}_i(t)$ because the network links do not change in the model, such that $\Delta\mathbf{L}_i(t+1) = 0, \forall t \geq 0$. In the second transition, we decompose the outside-network assets in terms of their liquidity, in which $\mathbf{A}_i^{(\text{out-liq})}(t+1)$ and $\mathbf{A}_i^{(\text{out-illiq})}(t+1)$ denote the liquid and illiquid outside-network assets, respectively. The last transition accounts for the fact that only illiquid outside-network assets are subject to losses.

Equation (7) states that losses due to funding risk occur in view of negative variations of illiquid outside-network assets. This risk component comes into play when economic agent i has to firesale illiquid assets to liquidate its short-term liabilities, thus incurring losses.

The extent of these losses relates to the healthiness of the direct neighbors of i. If they are under increasing distress, they will hoard liquidity and hence will apply a credit crunch to i in view of Assumption 2. The influence of each neighbor j on i's potential net worth loss is estimated by $\alpha_{ij}\mathbf{L}_{ij}^{(\text{in-st})}$. Then, the losses in illiquid outside-network assets of i vary as a function of the net worth or equities of its creditors j, i.e.:

$$\begin{aligned}\Delta\mathbf{E}_i^{(\text{f})}(t+1) &= \Delta\mathbf{A}_i^{(\text{out-illiq})}(t+1) \\ &= \sum_{j \in \mathscr{A}(t-1)} \alpha_{ij}\mathbf{L}_{ij}^{(\text{in-st})}(t)\left[\frac{\mathbf{E}_j(t) - \mathbf{E}_j(t-1)}{\mathbf{E}_j(t-1)}\right]. \end{aligned} \tag{8}$$

But $\mathbf{L}_{ij}^{(\text{in-st})}(t)$ varies linearly accordingly to Assumption 2, such that it follows the same functional form as in (3). Therefore, Equation (8) simplifies to:

$$\Delta\mathbf{E}_i^{(\text{f})}(t+1) = \sum_{j \in \mathscr{A}(t-1)} \frac{\alpha_{ij}\mathbf{L}_{ij}^{(\text{in-st})}(0)}{\mathbf{E}_j(0)}\left[\mathbf{E}_j(t) - \mathbf{E}_j(t-1)\right], \tag{9}$$

in which $\mathbf{L}_{ij}^{(\text{in-st})}(0)$ is the initial short-term liabilities of i to j that is exogenous to the model.

Abstract shock propagation dynamic: Joining both risk sources that can force losses on the equities or net worth of economic agent i as in (2), we get:

$$
\begin{aligned}
\Delta\mathbf{E}_i(t+1) &= \Delta\mathbf{E}_i^{(\mathrm{ct})}(t+1) + \Delta\mathbf{E}_i^{(\mathrm{f})}(t+1) \\
&= \sum_{j\in\mathscr{A}(t-1)} \frac{\mathbf{A}_{ij}^{(\mathrm{in})}(0)}{\mathbf{E}_j(0)} \left[\mathbf{E}_j(t) - \mathbf{E}_j(t-1)\right] + \frac{\alpha_{ij}\mathbf{L}_{ij}^{(\mathrm{in\text{-}st})}(0)}{\mathbf{E}_j(0)} \left[\mathbf{E}_j(t) - \mathbf{E}_j(t-1)\right]
\end{aligned}
\tag{10}
$$

in which we replace $\Delta\mathbf{E}_i^{(\mathrm{ct})}$ and $\Delta\mathbf{E}_i^{(\mathrm{f})}$ by the expressions in (6) and (9). Noting that $\Delta\mathbf{E}_i(t+1) = \mathbf{E}_i(t+1) - \mathbf{E}_i(t)$ and isolating $\mathbf{E}_i(t+1)$, we get:

$$
\mathbf{E}_i(t+1) = \max\left[0, \mathbf{E}_i(t) + \sum_{j\in\mathscr{A}(t-1)} \frac{\mathbf{A}_{ij}^{(\mathrm{in})}(0)}{\mathbf{E}_j(0)} [\mathbf{E}_j(t) - \mathbf{E}_j(t-1)] + \frac{\alpha_{ij}\mathbf{L}_{ij}^{(\mathrm{in\text{-}st})}(0)}{\mathbf{E}_j(0)} [\mathbf{E}_j(t) - \mathbf{E}_j(t-1)]\right]
\tag{11}
$$

in which we have used the fact that losses cannot surpass the zero bound, in such a way that $\mathbf{E}_i(t+1) \geq 0$.[19]

Let us define the economic agent i's financial stress level $\mathbf{s}_i(t)$, which will be an important endogenous variable to estimate systemic risk, as:

$$
\mathbf{s}_i(t) = \frac{\mathbf{E}_i(0) - \mathbf{E}_i(t)}{\mathbf{E}_i(0)},
\tag{12}
$$

in which $\mathbf{E}_i(0)$ represents the initial and exogenous equities or net worth of economic agent i. The numerator $\mathbf{E}_i(0) - \mathbf{E}_i(t)$ quantifies the losses of economic agent i up to iteration t. If economic agent i has not suffered any losses, then $\mathbf{s}_i(t) = 0$. If economic agent is in default, then $\mathbf{s}_i(t) = 1$. Once in default, the economic agent i cannot suffer further losses and therefore its net worth cannot be negative. This implies that $\mathbf{s}_i(t) \in [0,1]$. In-between values represent partial financial distress.

For systemic risk purposes, we are interested in rewriting (11) in terms of economic agent i's financial stress $\mathbf{s}_i(t)$ as in (12). To accomplish that, we multiply by -1 (11), add $\mathbf{E}_i(0)$, then divide by $\mathbf{E}_i(0) > 0$ to obtain:[20]

[19] Later on, we will see that $\max[.]$ operator prevents defaulted economic agents from propagating further stress in the network. In our model, this would correspond to creditors recognizing losses that were larger than their actual exposure due to the repricing mechanism via the borrower's creditworthiness. We prevent such phenomenon by fixing the net worth lower bound at 0.

[20] We assume that all economic agents are active at time 0, i.e., they have positive equities or net worth.

$$
\begin{aligned}
\mathbf{s}_i(t+1) &= \min\left[1, \mathbf{s}_i(t) + \sum_{j\in\mathscr{A}(t-1)} \frac{\mathbf{A}_{ij}^{(\text{in})}(0)}{\mathbf{E}_i(0)}\left[\mathbf{s}_j(t)-\mathbf{s}_j(t-1)\right] + \frac{\alpha_{ij}\mathbf{L}_{ij}^{(\text{in-st})}(0)}{\mathbf{E}_i(0)}\left[\mathbf{s}_j(t)-\mathbf{s}_j(t-1)\right]\right] \\
&= \min\left[1, \mathbf{s}_i(t) + \sum_{j\in\mathscr{A}(t-1)} \frac{\mathbf{A}_{ij}^{(\text{in})}(0)}{\mathbf{E}_i(0)}\Delta\mathbf{s}_j(t) + \frac{\alpha_{ij}\mathbf{L}_{ij}^{(\text{in-st})}(0)}{\mathbf{E}_i(0)}\Delta\mathbf{s}_j(t)\right] \\
&= \min\left[1, \mathbf{s}_i(t) + \sum_{j\in\mathscr{S}} \frac{\mathbf{A}_{ij}^{(\text{in})}(0)}{\mathbf{E}_i(0)}\Delta\mathbf{s}_j(t) + \frac{\alpha_{ij}\mathbf{L}_{ij}^{(\text{in-st})}(0)}{\mathbf{E}_i(0)}\Delta\mathbf{s}_j(t)\right] \\
&= \min\left[1, \mathbf{s}_i(t) + \sum_{j\in\mathscr{S}} \mathbf{V}_{ij}(\text{AS})\Delta\mathbf{s}_j(t) + \mathbf{V}_{ij}(\text{LS})\Delta\mathbf{s}_j(t)\right]. \qquad (13)
\end{aligned}
$$

In the first equation, we use the fact that financial stress is non-decreasing and that its upper bound is 1. In the second transition, we substitute the exogenous set $\mathscr{S}$ for the endogenous set $\mathscr{A}(t-1)$ because $\Delta\mathbf{s}_j(t) = 0, \forall j : \mathbf{E}_j(t) = 0$ in view of the max[.] operator in (11). Thus, we do not need to keep track of the set of active banks in our dynamics.

The terms $\mathbf{V}(\text{AS})$ and $\mathbf{V}(\text{LS})$ in the last line of (13) are the vulnerability matrices that numerically translate how financial contagion spills over and impacts economic agents from their asset and liability sides, respectively. These matrices are exogenous once we have knowledge of the financial network and the economic agents' equities or net worth. We evaluate the (i,j)th of $\mathbf{V}(\text{AS})$ and $\mathbf{V}(\text{LS})$ as follows:

$$
\mathbf{V}_{ij}(\text{AS}) = \frac{\mathbf{A}_{ij}^{(\text{in})}(0)}{\mathbf{E}_i(0)}, \qquad (14)
$$

$$
\mathbf{V}_{ij}(\text{LS}) = \frac{\alpha_{ij}\mathbf{L}_{ij}^{(\text{in-st})}(0)}{\mathbf{E}_i(0)}, \qquad (15)
$$

i.e., $\mathbf{V}_{ij}(\text{AS})$ and $\mathbf{V}_{ij}(\text{LS})$ provides how economic agent i's net worth is sensitive to deterioration of the counterparty i's balance sheet.

Equation (13) supplies the shock propagation dynamic to estimate systemic risk in an abstract economy composed of possibly heterogenous economic agents. Note that all variables are exogenous at iteration $t = 0$ and that $\mathbf{s}_i(t)$ is the only endogenous variable for $t > 0$, permitting us to computationally evolve the dynamics in (13).

Systemic risk estimation: We use the financial stress variable (see (12)) to proxy the financial soundness of economic agents, because it numerically supplies the distance to insolvency of economic agents. We represent the systemic risk that an external event can potentially cause as the amount of net worth reduction it provokes in each of the economic agents in the system.

Assume that the shock propagation dynamic settles down and thus Equation (13) converges for a sufficiently large $t = t_c \gg 1$ when the economy experiences an external shock.[21] Our system accepts shocks that directly hit the net worth or equities of economic agents. Denote the external shock vector as $\mathbf{s}(1) \in \mathscr{S} \times 1$, in which $\mathbf{s}_i(1) > 0$ if the external shock affects economic agent i's net worth and $\mathbf{s}_i(1) = 0$ otherwise.

Then, we compute systemic risk of the external shock $\mathbf{s}(1)$ using the following expression:

$$SR(\mathbf{s}(1)) = \sum_{i \in \mathscr{S}} (\mathbf{s}_i(t_c) - \mathbf{s}_i(1))\, v_i, \tag{16}$$

in which v_i denotes the economic importance of economic agent i to the overall economy. We do not simply sum up the net worth loss of each of the economic agents, because their role to the economy as a whole may differ. This form of aggregating net worth loss to compose systemic risk has also been performed by Bardoscia et al. (2015) and Battiston et al. (2012). Observe that we remove the stress that the initial shock $\mathbf{s}(1)$ causes to the economy. In this way, our systemic risk measure explicitly captures the additional loss in the system, apart from that caused by the initial shock.

The economic importance v simply aggregates the stress levels of all economic agents when the dynamic system reaches equilibrium. It does not interfere in the stress propagation dynamic of the financial system. Therefore, it enters only in a post-processing step. It is interesting that such parameter be exogenous, because it gives flexibility to users, such as regulators, who could weigh economic agents differently based on some criteria they are willing to analyze. For instance, if they are willing to understand the impact on all credit to individuals and firms granted by banks, they could weigh banks accordingly to the volume of outstanding credit to these persons and firms. Conversely, if she wants to understand the impact of defaults inside the banking system, one could use the total liabilities inside the network, and so on. In the paper, as we are dealing with the importance of each economic agent in the economy as a whole, we use the size of the economic agents, i.e., their total assets.

In the next section, we specialize this model for an economy comprising the real and financial sectors. In addition, we innovate in the systemic risk estimation by explicitly bringing in stress feedback between these two sectors.

2.3 Systemic risk model: real and financial sectors with feedback

We break up the set of economic agents $\mathscr{S}$ to two disjoint sets $\mathscr{B}$ and $\mathscr{F}$, which indicate the set of banks and firms, respectively, such that $\mathscr{S} = \mathscr{B} \bigcup \mathscr{F}$. We also divide

[21] We show in the next sections that the model always converges to a unique fixed point regardless of the external shock and the network topology. Then, we can assume that $t_c < \infty$ with no loss of generality.

the economic agents' stress levels vector $\mathbf{s}(t) \in \mathscr{S} \times 1$ (see (12)) as $\mathbf{b}(t) \in \mathscr{B} \times 1$ and $\mathbf{f}(t) \in \mathscr{F} \times 1$, which represent the stress levels of banks and firms, respectively.

Specialized shock propagation dynamic: The idea here is to redefine (13) in terms of stress levels of banks and firms. For that, we first divide inside-network assets $\mathbf{A}_{ij}^{(\text{in})}$ as follows:

- If $i \in \mathscr{B}$ and $j \in \mathscr{B}$, then $\mathbf{A}_{ij}^{(\text{in})} = \mathbf{A}_{ij}^{(\text{bank-bank})}$ is an interbank loan.
- If $i \in \mathscr{B}$ and $j \in \mathscr{F}$, then $\mathbf{A}_{ij}^{(\text{in})} = \mathbf{A}_{ij}^{(\text{bank-firm})}$ represents credit that banks grant to firms.
- If $i \in \mathscr{F}$ and $j \in \mathscr{B}$, then $\mathbf{A}_{ij}^{(\text{in})} = \mathbf{A}_{ij}^{(\text{firm-bank})}$ indicates investments of firms in bonds issued by banks.
- If $i \in \mathscr{F}$ and $j \in \mathscr{F}$, then $\mathbf{A}_{ij}^{(\text{in})} = \mathbf{A}_{ij}^{(\text{firm-firm})}$ supplies purchases on credit that the supplier firm i grants to the customer firm j.

Given that banks and firms experience an external shock that jeopardizes their net worth at $t = 1$, then, for $t > 1$, we can specialize the shock propagation dynamics in (13) as follows:

$$\mathbf{b}_i(t+1) = \min\left[1, \mathbf{b}_i(t) + \sum_{j \in \mathscr{B}} \mathbf{V}_{ij}^{(\text{bank-bank})} \Delta\mathbf{b}_j(t) + \sum_{u \in \mathscr{F}} \mathbf{V}_{iu}^{(\text{bank-firm})} \Delta\mathbf{f}_u(t)\right], \tag{17}$$

$$\mathbf{f}_k(t+1) = \min\left[1, \mathbf{f}_k(t) + \sum_{u \in \mathscr{F}} \mathbf{V}_{ku}^{(\text{firm-firm})} \Delta\mathbf{f}_u(t) + \sum_{j \in \mathscr{B}} \mathbf{V}_{kj}^{(\text{firm-bank})} \Delta\mathbf{b}_j(t)\right], \tag{18}$$

in which we divide the vulnerability matrices $\mathbf{V}(\text{AS})$ and $\mathbf{V}(\text{LS})$ in (13) as follows:

$$\mathbf{V}_{ij}^{(\text{bank-bank})} = \mathbf{V}_{ij}^{(\text{bank-bank})}(\text{AS}) + \mathbf{V}_{ij}^{(\text{bank-bank})}(\text{LS}), \tag{19}$$

$$\mathbf{V}_{ku}^{(\text{firm-firm})} = \mathbf{V}_{ku}^{(\text{firm-firm})}(\text{AS}) + \mathbf{V}_{ku}^{(\text{firm-firm})}(\text{LS}), \tag{20}$$

$$\mathbf{V}_{iu}^{(\text{bank-firm})} = \mathbf{V}_{iu}^{(\text{bank-firm})}(\text{AS}) + \mathbf{V}_{iu}^{(\text{bank-firm})}(\text{LS}), \tag{21}$$

$$\mathbf{V}_{kj}^{(\text{firm-bank})} = \mathbf{V}_{kj}^{(\text{firm-bank})}(\text{AS}) + \mathbf{V}_{kj}^{(\text{firm-bank})}(\text{LS}). \tag{22}$$

$\forall i, j \in \mathscr{B}$ and $k, u \in \mathscr{F}$. The idea is to define the matrices (19)–(22) using the functional forms in (14) and (15). Apart from the term $\mathbf{E}(0)$, which is the net worth of banks or firms, we simply need to suitably define the term $\mathbf{A}^{(\text{in})}$ in (14) and the terms α and $\mathbf{L}^{(\text{in-st})}$ in (15) in view of our application.

Terms $\mathbf{A}^{(\text{in})}$ and $\mathbf{L}^{(\text{in-st})}$: Table 1 summarizes how we define $\mathbf{A}^{(\text{in})}$ and $\mathbf{L}^{(\text{in-st})}$.

Table 1: Sources of vulnerabilities in the asset and liability sides of banks' and firms' balance sheets.

Creditor	Debtor	Notation	Creditor-side vulnerability: Asset side (investment)	Creditor-side vulnerability: Liability side (funding)
Firm k	Bank j	$\mathbf{V}_{kj}^{(\text{firm-bank})}$	Bank bonds $\mathbf{A}_{kj}^{(\text{firm-bank})}$	Bank credit crunch of outstanding short-term debt $\mathbf{L}_{kj}^{(\text{in-st})}$
Bank i	Firm u	$\mathbf{V}_{iu}^{(\text{bank-firm})}$	Bank loan $\mathbf{A}_{iu}^{(\text{bank-firm})}$	Early redemption of bank bonds $\mathbf{L}_{iu}^{(\text{in-st})}$
Firm k	Firm u	$\mathbf{V}_{ku}^{(\text{firm-firm})}$	Purchase on credit $\mathbf{A}_{ku}^{(\text{firm-firm})}$	Early redemption of firm securities $\mathbf{L}_{ku}^{(\text{in-st})}$
Bank i	Bank j	$\mathbf{V}_{ij}^{(\text{bank-bank})}$	Interbank loan $\mathbf{A}_{ij}^{(\text{bank-bank})}$	Interbank credit crunch of outstanding short-term $\mathbf{L}_{ij}^{(\text{in-st})}$

Term α_{kj}: This coefficient measures to what extent credit crunches of j would affect the net worth of k. While it is difficult to exactly precise this impact, we use several evidences in the banking and firm literature to estimate such event. Suppose here that k denotes a firm and j represents a bank.[22] Then, we compute α_{kj} as follows:

$$\alpha_{kj} = \phi_k \phi_j \left[1 - \rho_{kj}\right] \tag{23}$$

$\forall k \in \mathscr{F}$ and $j \in \mathscr{B}$. The coefficients ϕ_k and ϕ_j indicate the level of illiquidity of firm k and bank j, and $\rho_{kj} \in [0,1]$ is a proxy for firm k's ability to replace bank j by another bank. Therefore, the expression $\left[1 - \rho_{kj}\right]$ gives us a sense of the inability of firm k to replace bank j by another counterparty in its funding portfolio.

We first discuss how to compute the inability of firms to switch between banks. Equation (23) considers that the more difficult it is to replace the funding counterparty j, the more firm k will suffer if bank j restrains credit. When bank j is perfectly substitutable, then $\rho_{kj}^{(\text{firm})} = 1$ such that no losses due to funding risk arise. When bank j is singular and cannot be substituted, then $\rho_{kj}^{(\text{firm})} = 0$ such that losses due to funding risk become proportional to the short-term loan that firm k owes to bank j (see (15)).

We calibrate the firms' ability to substitute banks as a product of two components:

$$\rho_{kj} = \left[1 - \lambda_k\right] \left[1 - \mathbf{RL}_{kj}\right], \tag{24}$$

in which $\lambda_k \in [0,1]$ represents firm k's dependency on bank financing and $\mathbf{RL}_{kj}$ stands for its relationship lending history with bank j. The intuition behind (24) is that firm k has more facility to substitute bank j when it is less dependent on bank credit and neither it has bank j as one of its main sources of external financing. Observe that $\lambda_k = 0$ implies $\mathbf{RL}_{kj} = 0, \forall j \in \mathscr{B}$. Given that firm k depends on bank credit to finance its projects, then it becomes more difficult to replace bank counterparties that firm k maintains most of its financial transactions.

[22] We can apply a similar reasoning for other combinations of economic agents.

We compute firm k's dependency on bank credit using the expression:[23]

$$\lambda_k = \frac{\text{bank debt}_k}{\text{debt}_k + \text{equity}_k} \tag{25}$$

in which bank $\text{debt}_k = \sum_{j \in \mathscr{B}} \mathbf{A}_{jk}^{(\text{bank-firm})}$ is the total bank debt of firm k, $\text{equity}_k = \mathbf{E}_k(0)$ is the total equities that firm k issues to shareholders, and debt_k represents the total external financing of firm k, which includes bank debt, issued private debt securities, and accounts payables. From the fundamental accounting equation, the expression $\text{debt}_k + \text{equity}_k$ equates to firm k's total assets.

When $\lambda_k = 1$, firm k's sources of financing are exclusively composed of bank loans. As we consider firm equities as the main resources firms use to absorb losses from financial contagion, then $\lambda_k < 1$ for an active or non-defaulted firm in view of $e_k > 0$. When $\lambda_k = 0$, firm k is independent of bank loans and finance its activities by (i) issuing equities to shareholders or retaining current profits or using past profits and/or (ii) issuing private debt securities, such as bonds, to external investors. Profits tend to be scarce in times of stress, which makes firms more reliant on bank credit.

By adding the bank dependency of firms in the underlying process of estimating of the funding risk, we capture the following evidences found in the literature. Smaller firms often do not have access to equity financing, bank credit, nor to the financial market because they do not have enough collateral or do not have projects with high probabilities of success. As firms develop in their sizes, they gain access, normally in this order, to the equity financing market, banking credit, and finally to the financial market (Holmstrom and Tirole (1997); Iyer et al. (2014)). While equity is the most expensive funding choice because it is not exempt of taxes, the cost of bank credit is normally higher than issuing bonds in financial markets because banks embed the monitoring costs they perform on the borrower firms. As a result, larger firms are less affected as they can either renegotiate their loans still with good contractual terms or go directly to the commercial paper or bond markets (Diamond (1991)). Equation (25) captures this effect as larger firms will normally prefer to issue bonds rather than taking bank credit, which will result in smaller λ_k and hence smaller vulnerability to credit crunches that originate from the financial sector.

Given the current time T, we evaluate firm k's relationship lending history with bank j as follows:

[23] Equation (25) treats bank loans indistinctively with regard to collateral. In this matter, however, Holmstrom and Tirole (1997) empirically show that credit crunches affect more collateral-poor firms than those with good amounts of collateral. We can also bring this observation into our model by decomposing bank_k as the sum of two terms $\text{bank}_k^{(\text{non-collateral})} + \kappa \text{bank}_k^{(\text{collateral})}$, in which $\kappa \in (0,1)$ is a collateral-dependent variable that attenuates the effect of credit crunches on firms with collateralized bank loans.

$$\mathbf{RL}_{kj}(T) = \frac{\sum_{t \in \mathscr{T}} e^{-(T-t)} \mathbf{A}_{jk}^{(\text{bank-firm})}(T-t)}{\sum_{j \in \mathscr{B}} \sum_{t \in \mathscr{T}} e^{-(T-t)} \mathbf{A}_{jk}^{(\text{bank-firm})}(T-t)}, \tag{26}$$

in which $\mathscr{T} = \{T, T-1, T-2, \ldots, 0\}$ is a set containing the previous time references and $\mathbf{A}_{jk}^{(\text{bank-firm})}(T-t)$ stands for the amount bank j lent to firm k at the instant $T-t$. Note that we sum over all the lending history between bank j and firm k. However, we give preference to more recent lending, so we apply the discount factor $e^{-(T-t)}$. Observe that the more we distance from the current time T, the smaller becomes the contribution of $\mathbf{A}_{jk}^{(\text{bank-firm})}(T-t)$. The intuition is that the more firm k borrows from bank j in the past periods, the more difficult it would become to substitute that bank.

We expect that firms with highly concentrated funding portfolios will have strong relationship lending histories with a few bank counterparties. In view of (24), our framework will amplify the funding vulnerability of these firms when they significantly depend on bank credit to finance themselves. Therefore, those sectors of the economy that concentrate more their financial operations will be more susceptible to credit crunches.

Finally, we consider the liquidity component of firms and banks. First, the amount that banks reduce credit directly relates to their current liquidity positions. The more liquid banks are, the less the credit reduction will be in case of an external negative event. At the other endpoint, the impact of this credit reduction on firms' activities also relates to their liquidity positions. Firms that are highly liquid may withstand this credit reduction more easily than those at the verge of becoming illiquid. Firms that are not able to sustain these credit crunches will have to firesale illiquid assets to cover up the liquidity shortfall and hence honor their short-term debts, thus incurring in losses. We compute the illiquidity ratio of economic agent k as follows:

$$\phi_k = \max\left(0, \frac{\text{liabilities}_k^{(\text{short-term})}}{\text{assets}_k^{(\text{liquid})}} - 1\right), \tag{27}$$

in which $\text{liabilities}_k^{(\text{short-term})}$ and $\text{assets}_k^{(\text{liquid})}$ represent the short-term liabilities and liquid assets of economic agent k. The more illiquid k is, the larger ϕ_k will become. If k is liquid enough to withstand the short-term cash outflow, then $\frac{\text{liabilities}_k^{(\text{short-term})}}{\text{assets}_k^{(\text{liquid})}} < 1$, yielding $\phi_k = 0$. In this case, $\alpha_{kj} = 0, \forall j$, in view of (23). If k does not have enough cash, then $\alpha_{kj} > 0$, for some j, making it susceptible to losses due to credit crunches.

Equation (27) also captures to some extent the riskiness of firms in terms of their short-term leverage. In this sense, banks penalize more those firms that maintain low liquidity levels during their daily operations. The larger this short-term leverage is, the more the firm nears illiquidity and therefore the larger the funding vulnerability will be. Considering that different industries have distinct operating levels of liquidity, our framework reasons that industries that are more liquid will be less affected by changes on the lending behavior of the financial sector. For instance, our framework reduces the funding vulnerability of industries that have much larger maturity levels on their asset than their liability sides, such as those that have very long-term and expensive investments, normally in the oil and gas and electrical sectors. In contrast, industries with very short maturity in their asset sides, such as the tertiary sector, would be more affected.

The feedback or financial accelerator between banks and firms: The vulnerability matrices $\mathbf{V}^{(\text{bank-firm})}$ in (17) and $\mathbf{V}^{(\text{firm-bank})}$ in (18) that connect banks and firms give rise to the stress feedback between the real and the financial sectors, which is one of the main contributions of this work. To see this effect, bank i can receive stress from firms at iteration t, which will bounce back to firms at iteration $t+1$, which in turn will regress to bank i at $t+2$ and so forth. Therefore, this feedback mechanism between banks and firms creates a micro-level financial accelerator in which banks and firms are able to individually amplify shocks they receive from counterparties.

2.4 Theoretical analysis

In this section, we show that the model converges to a fixed point regardless of the network topology and shock magnitude. Let $B = |\mathscr{B}|$ and $F = |\mathscr{F}|$ be the number of banks and firms, respectively.

We first transform the model to a standard format of state-space dynamic system. This step is useful as it facilitates the understanding of convergence issues of the system.

Proposition 1. *The systemic risk framework governed by (17) and (18) can be cast into the following state-space system:*

$$\Delta\mathbf{s}(t) = \min\left[1, \mathbf{V}\Delta\mathbf{s}(t-1)\right], \tag{28}$$

in which $\Delta\mathbf{s}(t) = \mathbf{s}(t) - \mathbf{s}(t-1) \in [0,1]^{B+F}$ *is the state of the dynamic system, in which* $\mathbf{s}(t) = [\mathbf{b}(t)\ \mathbf{f}(t)]^T$ *is a column vector with* $B+F$ *entries that compounds the stress levels of banks and firms stacked in that order and* T *is the transpose operator.*

The update or transition matrix $\mathbf{V}$ *of such system is:*

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}^{(bank\text{-}bank)} & \mathbf{V}^{(bank\text{-}firm)} \\ \mathbf{V}^{(firm\text{-}bank)} & \mathbf{V}^{(firm\text{-}firm)} \end{pmatrix}, \tag{29}$$

in which $\mathbf{V}^{(bank\text{-}bank)} \in \mathscr{B} \times \mathscr{B}$*,* $\mathbf{V}^{(bank\text{-}firm)} \in \mathscr{B} \times \mathscr{F}$*,* $\mathbf{V}^{(firm\text{-}bank)} \in \mathscr{F} \times \mathscr{B}$*, and* $\mathbf{V}^{(firm\text{-}firm)} \in \mathscr{F} \times \mathscr{F}$*. In this way, the update matrix* $\mathbf{V}$ *has dimensions of* $(B+F) \times (B+F)$*.*

Proof. Moving $\mathbf{b}(t-1)$ and $\mathbf{f}(t-1)$ from the RHS to the LHS of (17) and (18) and defining the system's state accordingly, we retrieve the update matrix in (29) by inspection. ■

Remark 1. ***Stress levels are monotonically non-decreasing****: Since* $\Delta\mathbf{s}(t) \geq 0, \forall t \geq 1$*, and* $\mathbf{V}_{ij} \geq 0, \forall i, j \in \mathscr{S}$*, then the increments in (17) and (18) are non-negative. Hence,* $\Delta\mathbf{s}(t)$ *must be non-decreasing.*

The update matrix $\mathbf{V}$ is time-invariant, meaning that its spectrum is constant over time. However, as banks or firms default, the ability of the financial system to amplify shocks reduces, as they are fewer active economic agents. Thus, defaults drive the system to a more stable dynamic in terms of shock amplification. In mathematical terms, the effective spectrum of the matrix $\mathbf{V}$ decreases as defaults occur. For the purposes of stability analysis, it is important to keep track of the effective spectrum of $\mathbf{V}$. The next Lemma alters $\mathbf{V}$'s spectrum while maintaining identical the behavior of the dynamic system.

Lemma 1. *The dynamic system in (28) can be rewritten as:*

$$\Delta\mathbf{s}(t) = \min\left[1, \mathbf{V}(t)\Delta\mathbf{s}(t-1)\right], \tag{30}$$

i.e., $\mathbf{V}(t)$ *is now a time-varying matrix whose entries are given by:*

$$\mathbf{V}_{ij}(t) = \begin{cases} \mathbf{V}_{ij}(t-1), & \textit{if } i \textit{ has not defaulted up to time } t-1. \\ 0, & \textit{otherwise.} \end{cases} \tag{31}$$

Proof. Suppose the ith economic agent defaults at time t. Due to the upper limit of 1, the stress differentials of i are stacked at zero in the subsequent iterations, i.e., $\Delta\mathbf{s}_i(t+k) = 0, \forall k \in \mathbb{N}_+$.[24]

To prevent increases of economic agent i's stress levels in subsequent iterations, we can zero the ith row of the update matrix $\mathbf{V}$ as Equation (31) shows. Applying this modification, the inner products between the ith row of $\mathbf{V}(t+k)$ and $\Delta\mathbf{s}(t+k)$, $\forall k \in \mathbb{N}_+$, always result in zero. Therefore, $\Delta\mathbf{s}_i(t+k) = 0$ and hence economic agent i does not propagate further stress once in default. ■

We now delineate the phases through which the systemic risk framework passes as the contagion diffusion process between economic agents evolves.

[24] Another way to see this is by noting that the curve of $\mathbf{s}_i$ is non-decreasing due to Remark 1. Once i reaches the upper limit of 1, it never leaves that stress level.

Proposition 2. *The systemic risk framework can pass through two phases:*

1. ***Transient phase**: marked by the presence of defaults, $\mathbf{V}(t)$ evolves to less destabilizing conditions as defaults emerge in the financial system. This phase always terminates and may or may not be present in the dynamics.*

2. ***Persistent phase**: marked by the absence of defaults, $\mathbf{V}(t)$ is necessarily stable. This phase is always present.*

Proof. We divide the proof into two parts: definition of the transient and the persistent phases.

As defaults occur in the transient phase, some economic agents' stress levels keep increasing until they top at 1 because of the $\min[.]$ operator. Say that the economic agent i defaults at time t_{default}. Thus, the spectrum of $\mathbf{V}(t_{\text{default}}+k), \forall k \in \mathbb{N}_+$, reduces due to Lemma 1. If other defaults occur at later iterations, the spectrum $\mathbf{V}$ once again diminishes until the spectrum of $\mathbf{V}$ eventually reaches the stability zone, in which the magnitude of its largest eigenvalue becomes less than one. When no more defaults occur, the persistent phase begins and $\mathbf{V}(t)$ becomes stable from that moment onwards.

We can apply Geršgorin (1931)'s circle theorem to get a very clear intuition of the spectrum update process.[25] We apply this theorem on the transpose of $\mathbf{V}$ with no loss of generality, since eigenvalues of a matrix are equal to its transpose. We first note that the elements of the main diagonal of $\mathbf{V}^T$ must be zero, because economic agents cannot be vulnerable to themselves. Therefore, the estimates of the eigenvalues of $\mathbf{V}^T$ are centered at circles in the origin of the plane. These circles have different radii according to the partial row sums of $\mathbf{V}^T$. Each partial row sum, which excludes the element in its main diagonal, represents an upper bound for the spectrum of $\mathbf{V}^T$. For stability and asymptotic analyses, we can only keep the largest estimate of the spectrum.

Figure 3 portrays a typical trajectory of the spectrum of $\mathbf{V}^T$. As defaults occur, the spectrum of $\mathbf{V}^T$ reduces and the financial system is driven towards a less unstable state. Eventually, the system enters the persistent phase in which $\mathbf{V}^T$ is necessarily stable. Once it enters this area, its spectrum no longer changes and the dynamic process converges. The system can either reach a persistent phase with a positive eigenvalue, i.e., $0 < \lambda < 1$, in which case the system settles down with some economic agents not defaulting; or with a zero eigenvalue, representing the equilibrium in which all economic agents default.

Observe also that the transient phase is not required in this process: if $\mathbf{V}$ is stable upfront, then the dynamic process immediately enters the persistent phase. ■

[25] Geršgorin (1931)'s circle theorem gives estimates to bound the spectrum of an $N \times N$ square matrix by stating that eigenvalues of $\mathbf{V}$ must be inside in one or more, possibly overlapping N circles centered at the main diagonal elements $\mathbf{V}_{ii}$, $\forall i \in \{1,2,\ldots,N\}$, each of which with radius given by the sum of the elements of the respective ith row, except for the main diagonal element.

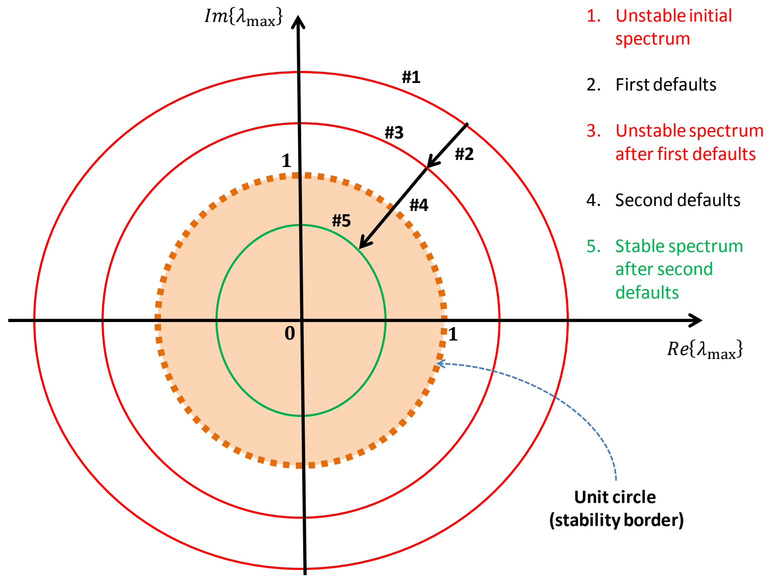


Figure 3: *Evolution of the estimated spectrum of* $\mathbf{V}^T(t)$ *as defaults of economic agents occur. In the schematic, the orange-shaded area denotes the stability zone, which is within the unit circle. The red spectra denote transient phases, while the green spectrum represents the persistent phase. The y- and the x-axis indicate the imaginary and real part of the largest partial (excluding the main diagonal element) row sum of* $\mathbf{V}^T(t)$*. As defaults occur, the spectrum reduces until it necessarily reaches the stability zone. At that point, it stays there until convergence to a unique fixed point.*

According to Proposition 2, defaults are a way that the dynamic and financial system use to expel those weak economic agents, i.e., those that cannot cope and absorb losses due to contagion. As weak economic agents are removed, the system becomes more robust and its resilience increases as the surviving economic agents are stronger.

The feedback effect slows down the convergence to equilibrium, but does not affect the asymptotic features of the model. We still need to prove that there is unique fixed point in the contagion process. Before showing that, the following Lemma provides an important characteristic of the dynamic system while it is in the persistent phase.

Lemma 2. *Once the system enters the persistent state, the update rule of the dynamic system becomes a contraction mapping* $\mathbf{g} : [0,1]^{B+F} \mapsto [0,1]^{B+F}$*, in which* $\mathbf{g}$ *is a vectorial function that maps stress levels from* $t-1$ *to* t*.*

Proof. According to Proposition 2, the persistent phase is marked by the absence of defaults. In this way, we can remove the min $[.]$ operator from the update rule of the dynamic system, because stress levels never reach 1. Using this observation in (30), we get:

$$\Delta\mathbf{s}(t) = \mathbf{V}_{\text{stable}}\Delta\mathbf{s}(t-1), \tag{32}$$

in which $\mathbf{V}_{\text{stable}}$ represents the stable update matrix $\mathbf{V}$ in the persistent phase. As no defaults occur in the phase, the matrix is time-invariant and therefore its spectrum does not change. In this way, we can omit the time index with no loss of generality. In addition, the dynamic system becomes linear.

Given an initial condition $\Delta\mathbf{s}(1)$, we can explicitly compute $\Delta\mathbf{s}(t)$ in terms of $\Delta\mathbf{s}(1)$ as:

$$\Delta\mathbf{s}(t) = \mathbf{V}^{t}_{\text{stable}}\Delta\mathbf{s}(1). \tag{33}$$

Noting that matrix $\mathbf{V}_{\text{stable}}$ possesses eigenvalues inside the unit circle, then Equation (33) represents a contraction mapping that is performed by the stable update matrix $\mathbf{V}_{\text{stable}}$ on the states $\Delta\mathbf{s}(t)$. ∎

We now show the main result of the theoretical analysis: the convergence of the contagion process to a unique fixed point.

Proposition 3. *The systemic risk framework always converges to unique fixed-point* $\mathbf{s}^*$*, which depends on the network structure and the shock magnitude, with the following expression:*

$$\mathbf{s}^* = (\mathbf{I} - \mathbf{V}_{stable})^{-1}\varepsilon, \tag{34}$$

in which $\mathbf{V}_{stable}$ *(network structure) and* ε *(shock) denote the update matrix and stress levels of economic agents, respectively, when the dynamic system enters the persistent phase.*

Proof. We show that the system enjoys the pre-requisites of the Banach (1922)'s fixed-point theorem. First, the metric space of the map is a continuous line segment in the space $B+F$, i.e., $[0,1]^{B+F}$. It is then a traditional non-empty complete metric space with a well-defined Euclidean distance metric. In addition, the stable update matrix $\mathbf{V}_{\text{stable}}$ in (33) represents a contraction map due to Lemma 2. Putting together these facts and invoking Banach (1922)'s theorem, we conclude that the dynamic system must have a unique fixed point.

We now algebraically evaluate that fixed point $\mathbf{s}^*$. If ε represents the stress levels of economic agents when the dynamic system enters the persistent phase, then:

$$\mathbf{s}(t) = \mathbf{V}_{\text{stable}}\mathbf{s}(t-1) + \varepsilon, \tag{35}$$

$\forall t > 1$ and $\mathbf{s}(1) = \varepsilon$. As Equation (35) has a fixed point $\mathbf{s}^*$, then $\mathbf{s}(t) = \mathbf{s}(t-1) = \mathbf{s}^*$ and therefore:

$$
\begin{aligned}
& \mathbf{s}^* = \mathbf{V}_{\text{stable}}\mathbf{s}^* + \varepsilon \\
\Rightarrow \quad & (\mathbf{I} - \mathbf{V}_{\text{stable}})\mathbf{s}^* = \varepsilon,
\end{aligned} \tag{36}
$$

in which $\mathbf{I}$ is the identity matrix. But $\mathbf{V}_{\text{stable}}$ is stable and hence the inverse of $(\mathbf{I} - \mathbf{V}_{\text{stable}})$ exists and corresponds to a convergent geometric series.

The update matrix $\mathbf{V}_{\text{stable}}$ has the main diagonal full of zeroes. In this way, the main diagonal of $(\mathbf{I} - \mathbf{V}_{\text{stable}})$ corresponds to a vector of ones and thus it has full rank. Consequently, $(\mathbf{I} - \mathbf{V}_{\text{stable}})$ is invertible. Therefore, we can compute the equilibrium stress level of the system as:

$$
\mathbf{s}^* = (\mathbf{I} - \mathbf{V}_{\text{stable}})^{-1}\varepsilon, \tag{37}
$$

which retrieves (34) and the proof is complete. ■

3 How can microeconomic shocks develop into systemic risk?

In this section, we quantify the ability of financial networks in turning microeconomic and idiosyncratic shocks on economic agents into system-wide systemic risk. Our study sheds light on the often considered "black box" behavior of financial networks by unboxing how they amplify external shocks in view of their internal structure of interconnections.

We start this investigation by recognizing that the closed form solution of our system relates to the Leontief inverse matrix once in the persistent phase.

Definition 1. *The matrix $(\mathbf{I} - \mathbf{V}_{stable})^{-1}$ in (34) represents the Leontief inverse matrix $\mathbf{L}^{-1}$ of the financial system.*

The following proposition is useful when comparing the ability of two different financial networks in amplifying microeconomic shocks.

Proposition 4. *The ability of a financial network to develop systemic events (macroperspective) in view of shocks on microeconomic agents (microperspective) increases as the magnitude of the maximum eigenvalue of the Leontief inverse matrix $\mathbf{L}^{-1}$ grows.*

Proof. Since $\mathbf{V}_{\text{stable}}$ is stable, then the Leontief matrix $\mathbf{L}$ has the largest eigenvalue smaller than one. In this way, the smallest eigenvalue of the Leontief inverse matrix $\mathbf{L}^{-1}$ must be greater than one.[26] Therefore, the dominant eigenvalue of $\mathbf{L}^{-1}$ is larger than one. In view of this, the maximum eigenvalue of the Leontief inverse matrix dominates the growth rate of the dynamic system and hence dictates how fast the multilayer financial system amplifies shocks. ■

In the following propositions, we deal with the vector ε and the matrix $\mathbf{V}_{\text{stable}}$, which correspond to the exogenous shock applied to the financial network and the update matrix of the dynamic system in the persistent phase, respectively. If the exogenous shock ε is small enough, then it corresponds to the initial shock applied on the dynamic system. Otherwise, it corresponds to the dynamic system's state when it enters the persistent phase. In practice, we consider the initial shock as small when it does not lead any economic agents into default so that the dynamic system immediately starts out in the persistent phase.

The next Theorem gives evidence of how financial networks can turn microeconomic idiosyncratic shocks into macroeconomic events of systemic risk owing to their internal structure.

Theorem 1. *The financial network develops the systemic risk level SR due to an exogenous shock ε with the following structure:*

$$SR(\varepsilon, \nu) = \nu^T (\mathbf{L}^{-1} - \mathbf{I})\varepsilon, \tag{38}$$

in which ν is a $(B+F) \times 1$ vector that holds the economic importance of each economic agent in the financial network, $\mathbf{L}^{-1}$ is a $(B+F) \times (B+F)$ matrix that corresponds to the Leontief inverse of the economic system, i.e., $\mathbf{L}^{-1} = (\mathbf{I} - \mathbf{V}_{stable})^{-1}$, and $\mathbf{I}$ is the identity matrix with dimensions of $(B+F) \times (B+F)$.

Proof. First note that we can rewrite (16) as:

$$SR(\varepsilon, \nu) = \nu^T (\mathbf{s}^* - \varepsilon), \tag{39}$$

in which $\mathbf{s}^* = (\mathbf{I} - \mathbf{V}_{\text{stable}})^{-1}\varepsilon = \mathbf{L}^{-1}\varepsilon$ is the fixed point solution of the dynamic system as demonstrated in Proposition 3. Note that $\mathbf{s}^*$ stacks both stress levels of banks and firms, in that order. Plugging this identity into (39) results in:

$$\begin{aligned} SR(\varepsilon, \nu) &= \nu^T (\mathbf{L}^{-1}\varepsilon - \varepsilon) \\ &= \nu^T (\mathbf{L}^{-1} - \mathbf{I})\varepsilon, \end{aligned} \tag{40}$$

[26]We are using the fact that the eigenvalues of a matrix are the reciprocal of those corresponding to the inverted matrix.

which retrieves (38) and the proof is complete. ∎

Theorem 1 provides a linear expression of how financial networks develop systemic risk in terms of the Leontief inverse matrix. We now elaborate more on this finding by further decomposing the Leontief inverse matrix in terms of the centrality of economic agents in the network structure.

Proposition 5. *The amplification of the microeconomic shock ε into macroeconomic systemic risk is larger the more it targets economic agents that are central in the network structure.*

Proof. We first observe that:

$$\mathbf{L}_{ij}^{-1} = (\mathbf{I} - \mathbf{V}_{\text{stable}})^{-1} = \mathbf{I}_{ij} + (\mathbf{V}_{\text{stable}})_{ij} + (\mathbf{V}_{\text{stable}}^2)_{ij} + \dots \tag{41}$$

We can observe from (41) that $(\mathbf{V}_{\text{stable}})_{ij}^k$, $k > 0$, indicates the sum of all weighted paths exactly with length k that starts from i and end at j in the network. For instance, $(\mathbf{V}_{\text{stable}})_{ij}$ is the vulnerability matrix that indicates how vulnerable the economic agent i is to j via a direct connection. In the same spirit, $(\mathbf{V}_{\text{stable}})_{ij}^2$ expresses how economic agent i is vulnerable to j indirectly via one of its neighbors, and so on. Equation (41) reveals that the (i, j)th entry of the Leontief inverse corresponds to the sum of all existent weighted paths in the financial multilayer network that start from i and end at j with varying lengths ranging from 0 to ∞.[27]

Economic agents that have central positions in the network topology are expected to be more interconnected with other economic agents (Silva and Zhao (2016)). Therefore, they can be easily reached by the majority of the other members in the network, i.e., there are many paths that end at these vertices that take central roles in the network. Using this reasoning, if j is a central economic agent from the viewpoint of network topology, then $\mathbf{L}_{ij}^{-1}$ is relatively large for most of the other agents $i \neq j$. Therefore, we can extract the notion of centrality of j, denoted here as $\mathbf{c}_j$, by simply summing up the resultant of all weighted paths going from every economic agent $i \neq j$ as follows:

$$\mathbf{c}_j = \sum_{i \in \mathscr{S}} v_i \mathbf{L}_{ij}^{-1}, \tag{42}$$

in which $\mathscr{S} = \mathscr{B} \bigcup \mathscr{F}$ is the union set of all economic agents in our model (banks and firms). The scalar v_i denotes the economic importance of agent i. Equation (42) is a

[27] A path with length 0 would correspond to self-vulnerability by the economic agents. We do not account for this behavior in our model. An idea where self-vulnerability can arise would be due to operational risks. In contrast, paths of infinite length can arise when the network presents cycles.

weighted form of the classic Katz-Bonacich centrality measure from the complex network theory.[28] Economic agent j is more central in the network the more other economic agents are, directly or indirectly, vulnerable to it. We can obtain a direct expression of how centrality influences systemic risk by plugging (42) into (38):

$$\begin{aligned} SR(\varepsilon, \nu) &= \mathbf{c}^T \varepsilon - \nu^T \varepsilon \\ &= (\mathbf{c}^T - \nu^T)\varepsilon \\ &= \sum_{i \in \mathscr{S}} (\mathbf{c}_i - \nu_i)\varepsilon_i, \end{aligned} \tag{43}$$

in which $\mathbf{c}^T = \nu^T \mathbf{L}^{-1}$ in view of the vector format of (42). Note that the term $\nu^T \varepsilon$ corresponds to the initial stress that the system suffers and $\mathbf{c}^T \varepsilon$ corresponds to the total stress—that is the additional and the initial stress—that the system develops.

We can conclude that, whenever the initial shock vector ε contains shocks on economic agents that have large centrality in the network, the financial network will amplify more that initial shock in view of the multiplicative term between the centrality and the initial shock on each of its economic agents. ■

4 How can changes in the network structure affect systemic risk?

In the previous section, we have considered the network as fixed when analyzing the role of the network structure as a shock amplifier medium. We change the perspective in this section and study how changes in the network structure affect systemic risk. First, we present two important concepts that we use when explaining how changes on the network structure influence systemic risk.

Definition 2. ***Stress diffusion index****: the stress diffusion index of economic agent j, $\mathbf{d}_j$, is its potential ability of diffusing stress in the financial network. Mathematically, it is given by:*

$$\mathbf{d}_j(\nu) = \mathbf{c}_j(\nu) = \sum_{i \in \mathscr{S}} \nu_i \mathbf{L}_{ij}^{-1} = \nu^T \mathbf{L}_{*j}^{-1}, \tag{44}$$

in which $\mathbf{L}_{*j}^{-1}$ corresponds to the jth column of $\mathbf{L}^{-1}$.

[28] The Katz-Bonacich centrality measure is a backbone of several others feedback-based centrality measures in the complex network theory, such as the communicability index, PageRank etc. We refer the reader to Silva and Zhao (2016) for a thorough review on this topic.

We interpret the stress diffusion index $\mathbf{d}_j$ as how harmful the economic agent j is to the financial system from a systemic risk viewpoint. Members with large $\mathbf{d}_j$ can inflict more distress to others whenever the network suffers an external shock.

Definition 3. ***Stress susceptibility index***: *the stress susceptibility index of economic agent j, $\mathbf{s}_j$, is its potential susceptibility of receiving stress from an external shock ε. Mathematically, it is given by:*

$$\mathbf{s}_j(\varepsilon) = \sum_{i \in \mathscr{S}} \mathbf{L}^{-1}_{ji} \varepsilon_i = \mathbf{L}^{-1}_{j*} \varepsilon, \tag{45}$$

in which $\mathbf{L}^{-1}_{j*}$ corresponds to the jth row of $\mathbf{L}^{-1}$.

We interpret the stress susceptibility index $\mathbf{s}_j$ as how likely economic agent j is to receiving stress from any other economic agent due to an external shock ε. Members with large $\mathbf{s}_j$ are more prone of being distressed whenever the network suffers an external shock.

Remark 2. *The stress diffusion index corresponds to the weighted Katz-Bonacich centrality measure as defined in (42).*

Remark 3. *The stress diffusion and susceptibility indices supply orthogonal but complementary views in what concern systemic risk. While the former identifies sources of stress diffusion in the network, the latter captures the stress absorbing ability of economic agents.*

We consider that the network structure can change by link rearrangements, such as an increase/decrease in the link weight, and link removal or insertion. Therefore, we are interested in understanding how changes in the vulnerability matrix affect systemic risk. However, Equation (38) does not have an explicit relation of systemic risk in terms of the vulnerability matrix $\mathbf{V}$. Instead, it depends on the matrix $\mathbf{L}^{-1}$, which is a nonlinear combination of $\mathbf{V}$ as Equation (41) shows. Therefore, we need to first find an expression in which we explicitly bring in the differential of $\mathbf{V}$, instead $\mathbf{L}^{-1}$. The next proposition supplies this result using the concepts of stress diffusion and susceptibility.

Theorem 2. *Fix an exogenous shock ε. If the vulnerability matrix $\mathbf{V}_{stable}$ is perturbed by a differential amount $d[\mathbf{V}] > 0$, then the variation or differential of systemic risk $d[SR]$ is:*

$$d[SR(\varepsilon, \nu)] = \sum_{i,j \in \mathscr{S}} d[\mathbf{V}]_{ij} \mathbf{d}_i \mathbf{s}_j = \mathbf{d}^T d[\mathbf{V}] \mathbf{s}, \tag{46}$$

in which $d[\mathbf{V}]_{ij}$ represents the (i,j)th entry of the differential matrix $d[\mathbf{V}]$ and $\mathbf{d}$ and $\mathbf{s}$ are the stress diffusion and susceptibility column vectors.

Proof. Applying the differential operator d[.] with respect to the vulnerability matrix in (38), we obtain:

$$\begin{aligned} \mathrm{d}[\mathrm{SR}(\varepsilon,\nu)] &= \mathrm{d}\left[\nu^T(\mathbf{L}^{-1}-\mathbf{I})\varepsilon\right] \\ &= \nu^T\mathrm{d}\left[\mathbf{L}^{-1}-\mathbf{I}\right]\varepsilon \\ &= \nu^T\mathrm{d}\left[\mathbf{L}^{-1}\right]\varepsilon, \end{aligned} \tag{47}$$

in which we make use of the linearity of the differential operator in the first transition and the fact that the differential of a constant matrix is zero, i.e., $\mathrm{d}[\mathbf{I}] = 0$, in the second transition.

We can write the differential of $\mathrm{d}\left[\mathbf{L}^{-1}\right]$ with respect to the vulnerability matrix in terms of $\mathrm{d}[\mathbf{L}]$ by noting that:

$$\begin{aligned} & \mathbf{L}^{-1}\mathbf{L} = \mathbf{I} \\ \implies\quad & \mathrm{d}\left[\mathbf{L}^{-1}\mathbf{L}\right] = \mathrm{d}[\mathbf{I}] \\ \implies\quad & \mathrm{d}\left[\mathbf{L}^{-1}\right]\mathbf{L} + \mathbf{L}^{-1}\mathrm{d}[\mathbf{L}] = 0. \end{aligned} \tag{48}$$

The first equation originates from the matrix identity equation that holds on account of the non-singularity of $\mathbf{L}$. In the first transition, we apply the differential operator in both sides of the equation. Finally, we make use of implicit differentiation and the chain rule in the second transition. Equation (48) enables us to find $\mathrm{d}\left[\mathbf{L}^{-1}\right]$ in terms $\mathrm{d}[\mathbf{L}]$ as follows:

$$\mathrm{d}\left[\mathbf{L}^{-1}\right] = -\mathbf{L}^{-1}\mathrm{d}[\mathbf{L}]\,\mathbf{L}^{-1}. \tag{49}$$

Plugging (49) back into (47), we get:

$$\mathrm{d}[\mathrm{SR}(\varepsilon,\nu)] = -\nu^T\mathbf{L}^{-1}\mathrm{d}[\mathbf{L}]\,\mathbf{L}^{-1}\varepsilon. \tag{50}$$

However, recall that the Leontief matrix is $\mathbf{L} = \mathbf{I} - \mathbf{V}$, such that:

$$d[\mathbf{L}] = d[\mathbf{I} - \mathbf{V}] = -d[\mathbf{V}]. \tag{51}$$

We can finally arrive in the desired expression that relates differential modifications on the vulnerability matrix to variations on systemic risk by plugging (51) into (50):

$$\begin{aligned} d[\mathrm{SR}(\varepsilon, \nu)] &= -\nu^T \mathbf{L}^{-1}(-d[\mathbf{V}])\mathbf{L}^{-1}\varepsilon \\ &= \nu^T \mathbf{L}^{-1} d[\mathbf{V}]\, \mathbf{L}^{-1}\varepsilon. \end{aligned} \tag{52}$$

We can apply matrix decomposition from linear algebra to write the resulting matrix that involves the multiplication of the three matrices $\mathbf{L}^{-1}d[\mathbf{V}]\,\mathbf{L}^{-1}$ as a sum of outer products between the ith column of left-most matrix and the jth row of the right-most matrix weighted by the corresponding (i, j)th element of $d[\mathbf{V}]$:

$$\begin{aligned} d[\mathrm{SR}(\varepsilon, \nu)] &= \nu^T \left(\sum_{i,j \in \mathscr{S}} d[\mathbf{V}]_{ij}\, \mathbf{L}^{-1}_{*i} \mathbf{L}^{-1}_{j*} \right) \varepsilon \\ &= \sum_{i,j \in \mathscr{S}} d[\mathbf{V}]_{ij}\, \nu^T \mathbf{L}^{-1}_{*i} \mathbf{L}^{-1}_{j*} \varepsilon. \end{aligned} \tag{53}$$

But, according to (44) and (45), $\mathbf{d}_i = \nu^T \mathbf{L}^{-1}_{*i}$ and $\mathbf{s}_j = \mathbf{L}^{-1}_{j*}\varepsilon$. Substituting these two expressions of stress diffusion and susceptibility into (53), we arrive at:

$$d[SR(\varepsilon, \nu)] = \sum_{i,j \in \mathscr{S}} d[\mathbf{V}]_{ij}\, \mathbf{d}_i \mathbf{s}_j = \mathbf{d}^T d[\mathbf{V}]\, \mathbf{s}, \tag{54}$$

which proves the theorem. ∎

Example 2. *Consider the financial network with six economic agents in Fig. 4. We want to evaluate how systemic risk changes due to an increase of the exposure of economic agent* 1 *to* 2 *by* $d[\mathbf{V}]_{12} > 0$. *Then, the exact systemic risk variation is* $d[SR(\varepsilon, \nu)] = d[\mathbf{V}]_{12}\, \mathbf{d}_1 \mathbf{s}_2$ *in accordance with Theorem 2. We can interpret this variation as follows:*

1. *We first measure the stress susceptibility of the debtor economic agent* 2 *to the external shock* ε, *i.e.,* $\mathbf{s}_2$. *This step corresponds to gauging to what extent economic agent* 2 *absorbs the external shock* ε *due to the network structure.*

2. *Upon receiving these shocks, the stress level of economic agent* 2 *increases. Therefore, all neighbors that are exposed to that economic agent will receive and absorb*

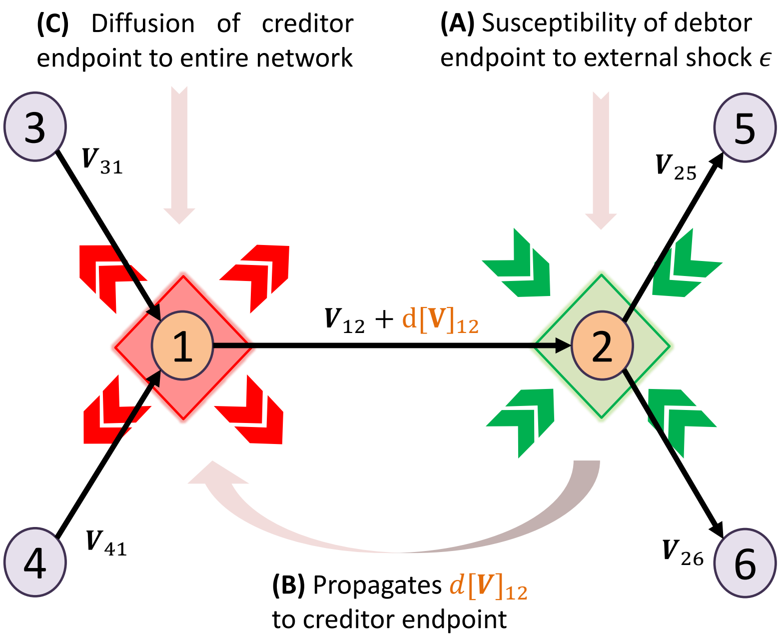


Figure 4: *A financial network with six economic agents. Fix the external shock ε that causes the systemic risk level $SR(\varepsilon,\nu)$ in the original network (before the link change). Then, suppose the financial network evolves in a way that the link from economic agent 1 to 2 increases by $d[\mathbf{V}]_{12} > 0$. Theorem 2 identifies the variation on systemic risk $d[SR(\varepsilon,\nu)]$ that this structural modification causes by (A) multiplying the susceptibility of the debtor economic agent to the external shock ε; (B) modulating this susceptibility to the exposure variation of 1 toward 2, i.e., $d[\mathbf{V}]_{12}$; then, (C) diffusing that additional stress that economic agent 1 suffers to the entire network.*

stress to the extent of their sensitivity or vulnerability to economic agent 2. Since in our example only the exposure $\mathbf{V}_{12}$ has changed to $\mathbf{V}_{12} + d[\mathbf{V}]_{12}$, then the creditor economic agent 1 absorbs the stress increase of 2—which corresponds to its susceptibility $\mathbf{s}_2$—by a sensitivity of $d[\mathbf{V}]_{12}$, i.e., $d[\mathbf{V}]_{12}\mathbf{s}_2$.

3. *When the stress level of the creditor economic agent 1 increases, it diffuses that stress level increase back to the entire network in accordance with its diffusion ability $\mathbf{d}_1$ modulated by the incoming stress $d[\mathbf{V}]_{12}\mathbf{s}_2$. Thus, in this process, systemic risk varies positively by $d[\mathbf{V}]_{12}\mathbf{s}_2\mathbf{d}_1$.*

Remark 4. *Theorem 2 exactly identifies the systemic risk variation if a single link changes. If two or more links vary, then it provides the best linear approximation for the systemic risk change. For instance, if the edge linking economic agent 4 to 1 also changes by an amount of $d[\mathbf{V}]_{41}$ in Fig. 4, then the approximate systemic risk variation is $d[\mathbf{V}]_{12}\mathbf{d}_1\mathbf{s}_2 + d[\mathbf{V}]_{41}\mathbf{d}_4\mathbf{s}_1$. The error in this approximation originates from the fact that the stress diffusion and susceptibility of economic agents may change once a network link changes. Provided that the interval is small enough, we can always decompose systemic risk variations in an exact manner.*

Theorem 2 states that systemic risk will vary more the larger the stress diffusion index of the creditor endpoint and the stress susceptibility index of the debtor endpoint are with respect to the modified links. This result has an important practical implication. The largest exposures in a financial system may not represent those that respond more to systemic risk: if one of the endpoints has low stress diffusion or susceptibility, then changes in that link will not result in large variations in systemic risk.

In sum, while Theorem 1 supplies a static view on the most harmful economic agents in the financial network, Theorem 2 provides subsidies to understanding why some connections yield more systemic risk variations while others do not. While the first component has been studied in the literature before (Acemoglu et al. (2015)), the second component is a total novel way of understanding the sources of systemic risk variations in financial networks. In regard to the second component, traditional works in the literature attempt to identify the most harmful connections in the financial network by using the leave-one-out approach (Poledna et al. (2015); Poledna and Thurner (2016)). In this approach, one evaluates the systemic risk with and without the connections of interest. The difference on the systemic risk estimates is then attributed to those connections. While this strategy can identify which connections are more harmful, it does not tell us why one link is more harmful than others. Theorem 2 explains systemic risk variations as a function of the network characteristics of the economic agents.

5 Application: systemic risk in the Brazilian financial and real sectors

In this section, we apply the systemic risk framework to the Brazilian bank-firm and bank-bank networks.

5.1 Data

We collect and match several unique Brazilian databases with supervisory and accounting data from the beginning of 2012 to the end of 2015. Due to lack of access to the data, we do not consider the trade network between firms and investments that firms have in bonds issued by banks.

Bank-bank network: To create the financial sector layer, we collect loan-level data between banks using supervisory data from the Central Bank of Brazil. In our sample, we use all banking institutions in the Brazilian jurisdiction, which encompass commercial banks, investment banks, savings banks and development banks. There are, on average, 123 active banking institutions in our sample for the analyzed period. We do not include non-banking institutions, such as credit unions, because their contribution to systemic risk is negligible.[29]

Most of the secured lending is through repurchase agreements with very short maturities that are collateralized with Brazilian federal bonds (94% of total secured lending). Most of the unsecured lending comes from interfinancial deposits (20%), financial bills and debentures (11%), repos issued by the borrower financial conglomerate (7%), and interbank credit (7%). Since the great majority of secured lending has federal bonds as collateral, which in turn are very liquid, if a debtor defaults, creditors can sell off the bonds with no losses even in the very short term. Therefore, we remove secured lending for systemic risk purposes as they do not have counterparty risk.

Bank-firm network: To compose the real sector layer, we consider all the 391 firms that trade shares in the Brazilian stock exchange (BOVESPA). We only use these firms because we need accounting variables of firms' balance sheets that are only available to participants of BOVESPA through Economatica. Though our sample of firms is very small (391 firms) in comparison to the entire universe of firms in Brazil (12,895,860 active local units as of June 2016), they respond to a large portion of payments transfers in Brazil. For instance, using the Brazilian Payments System, we observe that these firms transacted about 50% of the total amount of payment transfers of all Brazilian firms in June 2016. Most of the firms in Brazil are micro enterprises with one employee at most that are unlikely to engage in large financial transactions.

For each of these firms registered at Economatica, we compute the total outstanding loans that each bank grant to each firm on a quarterly basis. We extract these data from the Central Bank of Brazil's Credit Risk Bureau System (SCR)[30]. The main credit modalities granted to firms are working capital, fixed capital investment and rural credit. We have information on the maturity of these loans, enabling the segregation of short- and long-term loans. We consider short-term loans those that mature in less than 3 months.

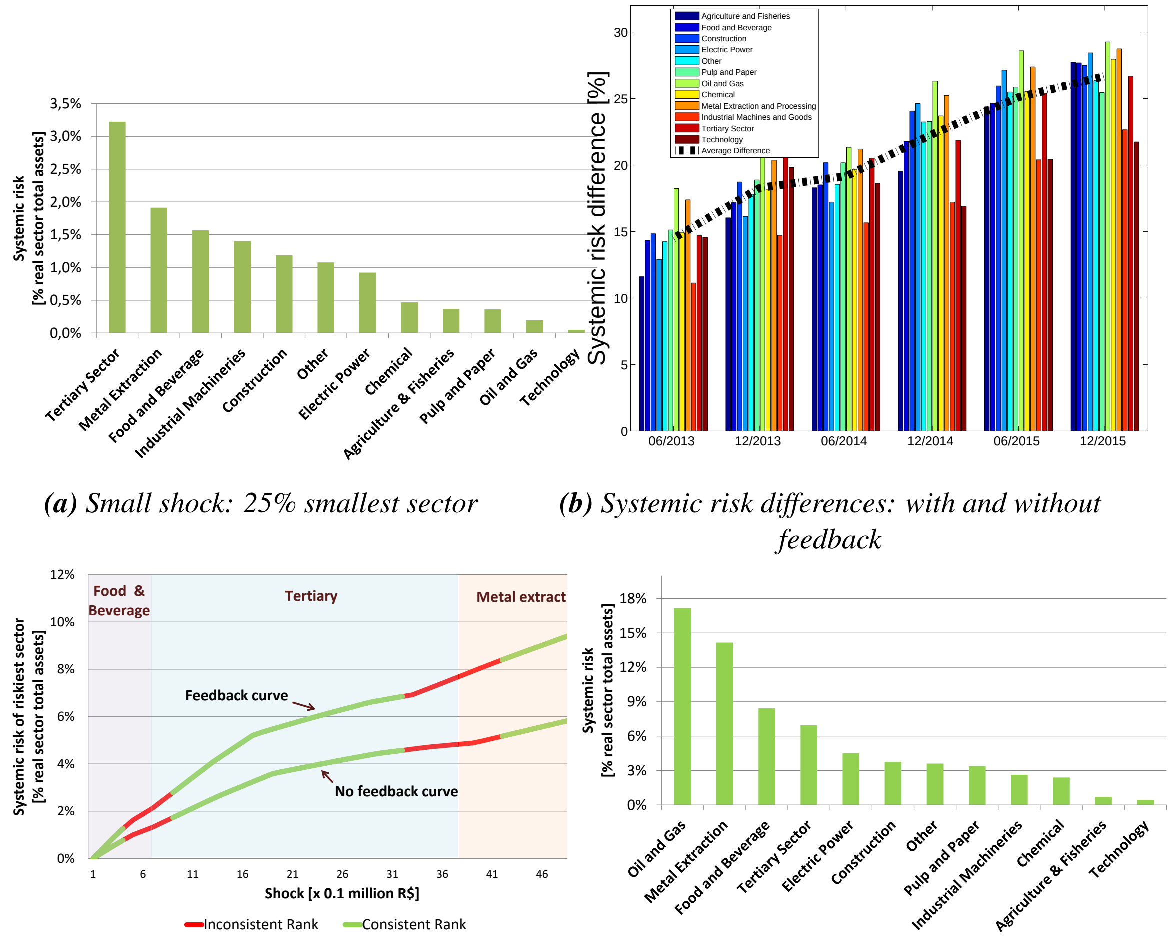
5.2 The importance of the feedback between the real and financial sectors

In this section, we provide empirical evidence that not taking into account the feedback between the real and financial sectors can lead to two undesired properties: (i) underestimation of systemic risk and (ii) rank inconsistency between the riskiest sectors of the economy.

[29]See Silva et al. (2016b) for a comprehensive analysis of the role that banking and non-banking institutions play in systemic risk buildup.

[30]The Central Bank of Brazil's Credit Risk Bureau System is a very thorough data set which records bank credit that individuals or firms take in the Brazilian financial system worth R$1,000.00 or above in the analyzed period. The data set has a broad set of information on financial institution and client identification, such as total exposure, type of loan, interest rate and risk classification. For a thorough list of the available information, see http://www.bcb.gov.br/pt-br/#!/n/SCR.

To first explore our data, Figure 5(a) shows the additional stress that each sector of the economy causes to the financial sector. Each bar corresponds to a common shock to all firms of the same sector. We apply a small shock corresponding to 25% of the total equities of the smallest sector (agriculture and fisheries). Though the oil and gas sector is the largest sector in the economy, it does not generate significant additional stress because firms of that sector are highly capitalized, in a way that they absorb the external shock. We identify firms of the tertiary sector as the riskiest ones to the financial sector for this external shock. Firms of that sector largely amplify shocks because (i) they have the largest amounts of short-term loans and (ii) they significantly depend on bank credit to finance themselves. Both factors contribute to strengthening the feedback mechanism that couples the financial and real sectors.



***(a)** Small shock: 25% smallest sector*

***(b)** Systemic risk differences: with and without feedback*

***(c)** Rank inconsistencies: with and without feedback*

***(d)** Large shock: full sectorial default*

***Figure 5:** Importance of the feedback effect in financial contagion models. (a) Systemic risk estimates for a small shock. (b) Percentage difference on systemic risk estimates due to the feedback. (c) Systemic risk of the riskiest sectors in the economy for the versions with and without feedback. (d) Systemic risk estimates for a large shock.*

To inspect the role of the feedback mechanism in amplifying shocks, we rerun the same experiment but enabling and then disabling the feedback mechanism between firms and banks in Figure 5(b). We report the systemic risk percentage differences between the two approaches with and without feedback per each sector and also the average from 2012 to 2015. We first note that the gap between the systemic risk estimates increases over time, going from 15% to 25% of systemic risk underestimation when we do not account for the feedback. This observation suggests that the importance of the feedback mechanism increases. In addition, we verify that different sectors respond differently to the feedback mechanism, which is a consequence of the firm-specific heterogeneity in our sample, such as distinct bank dependency, relationship lending profiles, and amounts of short- and long-term loan.

The heterogeneity that the feedback mechanism imposes on different sectors in terms of stress propagation can lead to rank inconsistencies when we consider the versions with and without feedback. For instance, the feedback mechanism can largely increase the riskiness of one sector but only slightly change that of the other. In this case, the positioning of these two sectors in terms of riskiness would change from the version with and without feedback.

Figure 5(c) plots the additional stress of only the riskiest sector using the approaches with and without feedback for different shocks on the firms' equities. When the curves are green, then the riskiest sector that both approaches identify matches. When the curves are red, then there is a mismatch in the riskiest sector identification. For instance, for a shock of R$ 0.1 to 0.4 million, the versions with and without feedback identify the food and beverage sector as the riskiest sector. In the region between R$ 0.4 to 0.9 million, the feedback version identifies the tertiary as the riskiest sector, while the version without feedback identifies the food and beverage. We see that there are two regions in which there are discrepancies in the riskiest sector identification for a small amplitude of initial shocks (R$ 0.1 to 5.1 million).

Figure 5(c) also highlights that the riskiest sector in the economy to the financial sector strongly depends on the initial shock magnitude. A natural question that arises is then to evaluate which sectors would be the riskiest to the financial sector in case of strong negative macroeconomic scenarios. We can model a strong collapse in a sector by jointly defaulting all firms of that sector. Figure 5(d) shows the results for this shock. We see now that the riskiest sectors become the oil and gas and the metal processing sectors, which are the largest sectors in the economy. They generate a large disruption in the financial system: the oil and gas stress almost 20% of all the financial system's total assets and the metal extraction inflicts roughly 14% of that proportion. This impairment occurs mainly because of the large amounts of loans that banks have against these firms.

We also compare the results from our model with two versions of the DebtRank algorithm: Bardoscia et al. (2015) (with cycles) and Battiston et al. (2012) (without cycles). Such algorithm also propagates financial stress in the network, but only considers the interbank network. It thus does not consider the importance of the real sector nor the feedback effect between firms and banks when estimating systemic risk. We find that these models underestimate systemic risk in several dates. The difference can be up to 250%, which is quite large. Therefore, the real sector and the feedback effect seem to be economically significant in systemic risk analysis.

Our results highlight the importance of the feedback mechanism when estimating systemic risk. Ideally, macroprudential supervision and policy should rely on the current systemic risk levels of the financial systems to act. Therefore, it becomes crucial to account for the real-financial sector linkages, as well as the negative feedback mechanism that links them. Otherwise, policymakers and regulators could be underestimating systemic risk and also not correctly identifying the largest sources of systemic risk from the real economy.

6 Conclusion

We develop a general framework to estimate systemic risk that accounts for feedback effects between the real and financial sectors. To the best of our knowledge, this is the first work that quantifies the economic importance of feedback effects in financial contagion models. We show that the model has strong theoretical properties, such as the existence of a unique fixed point. We derive theoretical properties for both the cross-sectional and time components of systemic risk with useful implications for financial system surveillance.

We demonstrate that the feedback effect gives rise to a micro-level financial accelerator, which we calibrate by incorporating several empirical evidences reported in the literature on bank-firm relationship behavior in adverse scenarios. Among the features, we consider contagion transmission channels that arise in light of deterioration of bank loan values, bank credit crunches, and early redemptions of bank bonds and firms securities. The financial accelerator models the fact that shocks can go back and forth in a negative spiral between economic agents.

We illustrate and validate our theoretical claims using a comprehensive Brazilian data set that encompasses loan-level information among banks and between banks and firms. We find that the feedback effects between the real and financial sectors are economically significant, implying that models that do not incorporate feedback effects could be significantly underestimating systemic risk. We also find evidences that the feedback effect influences economic sectors in different ways, suggesting that systemic risk models that consider and that do not consider feedback may have divergences on ranking the riskiest sectors in the economy. Our results corroborate the importance of considering other contagion transmission channels besides the traditional interbank market.

The model focuses on two sources of vulnerability that may lead to financial contagion, which is counterparty and funding risk, and abstracts away from other possible financial contagion sources, such as common or correlated exposures, illiquidity spirals due to margin calls, among others. One possible extension would be to adapt the model to account for these features that also influence systemic risk.

Appendix A Other functional forms of stress diffusion

In the main text, we consider linear functional forms of stress diffusion in view of Assumptions 1 and 2. In both assumptions, we represent variations of inside- and outside-network assets of economic agent i due to a solvent counterparty j using the functional form:

$$\begin{aligned}\mathbf{A}_{ij}(t+1) &= \mathbf{A}_{ij}(t)\frac{\mathbf{E}_j(t)}{\mathbf{E}_j(t-1)} \\ &= \alpha(t)\mathbf{A}_{ij}(t), \end{aligned} \tag{55}$$

in which $\mathbf{A}_{ij}(t)$ is some form of exposure of i to j at iteration t and $\alpha(t) = \frac{\mathbf{E}_j(t)}{\mathbf{E}_j(t-1)}$ is the dampening factor that reduces economic agent i's assets from iteration t to $t+1$ and consequently its net worth because of the deterioration of the net worth of j. In the main text, we choose a linear $\alpha(t)$, in which negative variations of the net worth of j are reflected on an $1:1$ basis to negative variations of i's assets.

In this appendix, we explore other functional forms of shock diffusion that comply with the model's microfoundations and its theoretical properties. For this matter, looking at Section 2.2, especially (6) and (9), we see that our derivations still hold for every homogenous polynomial of degree 0 with respect to the equity value at iteration t and $t-1$, i.e.:

$$\mathbf{A}_{ij}(t+1) = \alpha^n(t)\mathbf{A}_{ij}(t), \tag{56}$$

in which $n > 0$. When $n = 1$, then we have a linear shock transmission mechanism alike the one in the main text. When $0 < n < 1$, then we have sub-linear forms of shock transmission. When $n > 1$, we have super-linear forms of shock transmission. Systemic risk estimates in our model rise as we increase the exponent n.

For illustrative purposes, Figure 6 portrays how the total assets of an economic agent varies with respect to its initial value as a function of $\alpha^n(t)$, for $n \in \{0.5, 1, 2, 4\}$. We see that sub-linear forms of transmission attenuate the sensitivity of the economic agent to shocks in the network, while super-linear forms of transmission aggravate its financial condition.

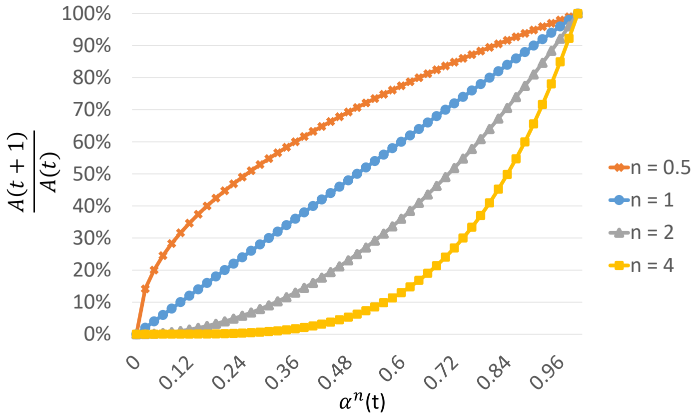


Figure 6: *Different functional forms of the shock diffusion mechanism: linear when $n = 1$ and non-linear when $n \neq 1$. When $n < 1$, we have sub-linear shock transmission and, when $n > 1$, super-linear shock transmission.*

The model also accepts individual-level values for the exponent n in (56). In this way, each bank or firm could absorb shocks in heterogeneous manners. A possible future work would be to calibrate n for economic agent i as a function of its reputation in the market. Economic agents that hold positive reputation may absorb shocks in a sub-linear way, i.e., $0 < n < 1$. Economic agents that do not a good stance in the market may absorb in a super-linear way, i.e., $n > 1$. Finally, unknown or new economic agents may absorb in a linear way. Another interesting approach would be to choose time-varying n values. Economic agents nearing insolvency would increase the exponent n to mimic a potential run on its funding portfolio.

There are several works in the literature that deal with non-linear ways of transmitting financial losses. For instance, Merton (1974)'s contingent claims approach uses a non-linear stochastic procedure to model how losses on assets can lead economic agents into default. He considers that the value of assets through time is uncertain and model its evolution using a diffusion mechanism composed of a drift and a stochastic term. He then evaluates the probability that the assets value becomes lower than a distress barrier, whose computation relies on the amount of short- and long-term liabilities of the economic agent, using a risk metric denominated distance to default. As the assets value approaches the distress barrier, the probability of default becomes higher. In this line of research, Gray and Malone (2008) also show a non-linear form of risk transmission that relies on a structural approach based on Merton (1974)'s model. However, since these works heavily rely on the stochastic behavior of assets, they need to assume several hypotheses over the data. In addition, they do not consider that economic agents are interlinked through a financial network, which can generate negative spillover effects and hence balance sheet co-movements among economic agents. As such, they would require non-trivial adjustments to make them suitable for analyzing negative externalities such as systemic risk due to financial contagion.

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