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A Joint Model of Nominal and Real Yield Curves*

Daniela Kubudi[†] José Vicente[‡]

Abstract

In this paper we propose a joint model of the nominal and real yield curves. The model is arbitrage free and embodies incompleteness between the nominal and real bond markets. We estimate the model using the Kalman filter with Brazilian data. First we consider that only the yields are observed with errors. Although the model fits these yields well, it presents poor inflation forecasting performance. Then, besides nominal and real yields, we estimate the model assuming that one-year ahead survey-based inflation forecasting is also observed with errors. The results show that the model using survey information does a much better inflation forecasting job. In addition, using the model estimated with survey inflation data, we show that monetary policy has significant effects on the inflation expectation and risk premium.

Keywords: Inflation risk premium, break-even inflation rate, affine models.
JEL Code: E31, E43, E44

*The views expressed are those of the authors and do not necessarily reflect the views of the Central Bank of Brazil.

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1 Introduction

According to the former Federal Reserve chairman, inflation-linked bonds are a valuable source of information about inflation expectations (Bernanke, 2004). The simplest way to explore this idea is to rely on the Fisher (1930) hypothesis, which associates the break-even inflation rate (the difference between nominal and real interest rates) with the expected inflation. However, observed prices are connected to the risk-neutral probabilities. So, the break-even inflation rate is the inflation expectation in the risk-neutral world. Therefore, to compute inflation expectation in the physical world using nominal and real bond yields, one must disentangle the components of market beliefs and risk aversion embedded in prices.

In order to extract the physical probabilities from observed prices, one should model the time evolution of the sources of risk of the economy under the pricing measure and specify the stochastic discount factor dynamics satisfying the no-arbitrage conditions. A traditional way to do this is through a parametric model which can be estimated from historical prices. This procedure allows obtaining the market beliefs (physical probabilities) and then computing the expected value of economic variables.

In this paper we adopt the affine model as the parametric framework to capture the joint dynamics of nominal and real yields in the risk-neutral and physical worlds. Affine models are nowadays the workhorse of yield curve modeling and derivatives pricing in fixed income studies. Since its inception by Duffie and Kan (1996), a collection of empirical papers has addressed various aspects and nice properties of affine models. For example, Dai and Singleton (2002) document that affine models are able to match many empirical features of the yield curve reported in the literature, including the findings of Fama and Bliss (1987) and Campbell and Shiller (1991). Another interesting example of the affine model applicability is the work of Ang and Piazzesi (2003). They propose an affine model with a joint dynamics of latent factors and macroeconomics variables (inflation and GDP) driving the nominal yield curve. They show that the inclusion of macro factors improves model forecasting performance.

Within the class of affine model, we elect the Gaussian version as our working tool. The Gaussian affine model presents a constant diffusion term which results in log-normality of factor dynamics. This specification provides easy calculation of the model's implied bond yields and inflation expectation. Besides parsimony, we have two more reasons to adopt the Gaussian version of the affine model. First, basic statistical tests reveal that our data set of yields used to estimate the model is roughly normal and homoscedastic. Second, Gaussian models present the best forecasting performance and fitting of empirical features of the yield curve among all affine models (see, for instance, Dai and Singleton, 2002 and Duffee, 2002).

Besides the Gaussian assumption, we also introduce another restriction in order to model incompleteness between the inflation-linked and traditional bond markets. Although nominal and real rates are strongly correlated, regressions between these yields show there is information in one of them which is not present in the other. To embody this empirical finding, we propose a four-factor Gaussian affine model in which two of these factors are specifically associated with one of the curves. In other words, among the four factors, two jointly affect the nominal and real yield curves, while the other two are specific sources of uncertainty connected with only one bond market. Note that this approach provides three factors driving each curve, which is in line with the results of Litterman and Scheinkman (1991).

We use data on the Brazilian fixed income market between April 2005 and February 2016. The Brazilian Treasury offers two types of nominal bonds: the *Letras do Tesouro Nacional* (LTN), which are zero-coupon bonds, and the *Notas do Tesouro Nacional F* (NTN-F), which pay semiannual coupons. The Brazilian Treasury also issues bonds indexed to IPCA, the so-called *Notas do Tesouro Nacional B* (NTN-B). IPCA is the main Brazilian consumer price index, and it is used as reference for the inflation targeting regime by the Central Bank of Brazil. From these bonds, we construct the nominal and real yield curves using the Svensson (1994) parametric model.

The inflation-linked security market has grown quickly over the past three decades. Many industrialized countries have supplemented their government bond issue programs with inflation-linked bonds. The linked bond market has also grown sharply in the emerging world. Of course, we could conduct this study using data from developed economies. However we decided to analyze the Brazilian case for several reasons. First, according to Bekaert and Wang (2010) there are differences in the operation of inflation risk hedging between emerging and developed markets. Thus, it is interesting to study emerging markets. However, the academic literature on inflation risk premium focuses mainly on developed countries, giving little attention to non-mature markets. Brazil is one of the most important emerging economies with a successful inflation-targeting regime. Moreover, its inflation-indexed bond market is the fifth largest in the world and represents a large proportion of the emerging market inflation-linked bonds. Second, as pointed out by Vicente and Graminho (2015), the NTN-B liquidity premium is insignificant. Thus, using Brazilian data we avoid the tricky procedure of estimating the liquidity premium in order to gauge the inflation premium. Third, the indexation lag of Brazilian linked bonds is negligible. Note that inflation indexes are not calculated on a daily basis, but only monthly and with a time lag. The consumer price index for a specific month can be published until three months later. As a result, yields extracted from linked bonds are not perfectly real yields. The smaller the lag, the better the approximation. NTN-Bs have a lag of only 15 days while TIPS and Gilts have a lag of 3 months. Therefore, we can associate NTN-B rates with real rates.[1] Finally, NTN-Bs are not protected against deflation - different from TIPS - which implies there is no embedded option in these bonds. These last two features are advantages because they eliminate the need to clean the noise regarding the indexation lag and deflation option premium.

We estimate the model using the Kalman filter. First, we assume that only a set of nominal and real yields are observed with errors. The results show that the model fits these yields well. However, it presents poor inflation forecasting performance. Clearly, these input variables are

[1] Yield data from TIPS or Gilts must be treated to obtain real yields. Evans (1998) proposes a method to do this.

not sufficient to allow the model to capture the inflation dynamics. Thus, we need to include information about inflation. One possible solution is to follow Ang and Piazzesi (2003) and identify a factor of the model as a price index. However, this idea increases the parameter space, making the process of estimation more difficult. So we decided to attack the problem in a simpler way, following the proposal of Joyce et al. (2010). Besides nominal and real yields, we estimate the model assuming that one-year ahead survey-based inflation forecasting is also observed with errors. The inflation expectations are read directly from "Focus", a survey conducted by the Central Bank of Brazil. The results of this second version provide interesting findings. The fits of nominal yields are slightly worse than in the first version. On the other hand, the second version does a much better inflation forecasting job. Although we only use one-year ahead Focus inflation expectation in the estimation, the second version of the model beats this inflation survey for the horizons of two, three and four years ahead. This result is promising since there is a strand of the literature that reports the difficulty of economic models to outperform survey-based inflation forecasts (see, for instance, Ang et al., 2007).

The inflation risk premium exhibits some interesting features. On average, it is less than 1%. However during stress periods, the compensation for inflation risk can be greater than 2%. Moreover, during economic exuberance periods, we observe a negative inflation risk premium. Although the economic theory does not rule out this pattern, it can be hard to explain. Some possible reasons are a strong decay in the risk aversion or mispricing of linked bonds. We also verify that the inflation risk premium is positively correlated with the nominal short-term rate. Therefore, when the Central Bank raises the prime interest rate in order to curb inflation, the risk premium also increases, which can be seen as a cost of monetary policy based on the inflation targeting regime.

Certainly, we are not the first to use affine models to forecast inflation rates. A number of papers have addressed this question, including among others Hördahl and Tristani (2007), D'Amico et al. (2010), Joyce et al. (2010) and Haubrich et al. (2012). Our work differs from these studies in some key ways. First, we use data from an important emerging country with a hyperinflation history that nowadays has a stable macroeconomic environment and a consolidated inflation targeting system. Second, the inflation-linked bond price database allows us to directly read real yields. Finally, our model takes into account incompleteness between the nominal and real bond markets.

In a nutshell, our results present some contributions to practitioners from academy and policy. First we provide a framework to separate risk premium and market beliefs. So, we perform an inflation forecasting empirical exercise and analyze the inflation risk premia. Explore the information on bond prices to this end has some advantages. As remarked by Söderlind (2011), prices are available in daily basis, hence we can provide high frequency update inflation expectations. Moreover, models that rely on prices are based on the beliefs of market participants and on decisions which involve gains and losses. Thus one can expect that its predictive ability are more accurate than the ones based on pure time series models, which in turn, reflect only past information. The main difficult to implement this approach is to disengage the market beliefs from the compensation for risk, which we do in this paper throughout the arbitrage-free affine model.[2]

The remainder of this paper is organized as follows. Section 2 presents the joint model of the nominal and real yield curves. Section 3 describes the data used to estimate the model. Section 4 discusses the results and Section 5 presents our final conclusions.

2 Model

In this section we describe the Gaussian affine no-arbitrage term structure model of nominal and real rates. The uncertainty in the economy is characterized by a filtered probability space $(\Omega, (\mathbb{F}_t)_{t\geq 0}, \mathbb{F}, \mathcal{P})$ where $(\mathbb{F}_t)_{t\geq 0}$ is the standard filtration generated by a N-dimensional Brownian motion $W^{\mathcal{P}} = \left(W_1^{\mathcal{P}}, \ldots, W_N^{\mathcal{P}}\right)'$ defined on $(\Omega, \mathbb{F}, \mathcal{P})$. We assume the existence of a pricing measure $\mathcal{Q}$ under which discounted security prices are martingales with respect to $(\mathbb{F}_t)_{t\geq 0}$. Let r_t^n be the instantaneous nominal rate. The time t price of a nominal zero-coupon bond paying one monetary unit at T is given by:

$$P^n(t,T) = E_t^{\mathcal{Q}}\left[\mathrm{e}^{-\int_t^T r_u^n \mathrm{d}u}\right]. \tag{1}$$

The nominal spot yield of time to maturity $\tau = T - t$ is defined by

$$y_t^n(\tau) = -\frac{\ln P^n(t,T)}{\tau}. \tag{2}$$

Let m_t be the instantaneous inflation rate and I_t a price index. Then,

$$I_T = I_t \mathrm{e}^{\int_t^T m_u \mathrm{d}u}. \tag{3}$$

A real zero-coupon bond with time to maturity $T - t$ is a security that pays I_T/I_t at T. Its price is given by

$$P^r(t,T) = E_t^{\mathcal{Q}}\left[\frac{I_T}{I_t}\mathrm{e}^{-\int_t^T r_u^n \mathrm{d}u}\right]. \tag{4}$$

Using (3) we have

$$P^r(t,T) = E_t^{\mathcal{Q}}\left[\mathrm{e}^{-\int_t^T (r_u^n - m_u) du}\right] = E_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r_u^r du}\right], \tag{5}$$

where $r_t^r = r_t^n - m_t$ is denominated instantaneous real rate. We also define the real spot yield as

$$y^r(\tau) = -\frac{\ln P^r(t,T)}{\tau}. \tag{6}$$

[2] Although imposing a no-arbitrage condition seems a good idea to improve forecasting ability, the empirical literature about this issue is controversial. See, for instance, Almeida and Vicente (2008) and Duffee (2011).

Next, we impose an affine model for the term structure of interest rates (see Duffie and Kan, 1996). In this framework, the short term rates are affine functions of dynamic factors:

$$r_t^n = \delta_{0n} + \delta_{1n} X_t, \tag{7}$$

and

$$r_t^r = \delta_{0r} + \delta_{1r} X_t, \tag{8}$$

where $X_t = (X_t^1, \ldots, X_t^N)'$ is a vector of state variables. We set $N = 4$ and for each short term rate we associate one idiosyncratic factor:

$$r_t^n = \delta_{0n} + \delta_{1n}^1 X_t^1 + \delta_{1n}^2 X_t^2 + \delta_{3n}^1 X_t^3, \tag{9}$$

and

$$r_t^r = \delta_{0r} + \delta_{1r}^2 X_t^2 + \delta_{1r}^3 X_t^3 + \delta_{1r}^4 X_t^4, \tag{10}$$

that is, $\delta_{1n} = [\delta_{1n}^1, \delta_{1n}^2, \delta_{1n}^3, 0]$ and $\delta_{1r} = [0, \delta_{1r}^2, \delta_{1r}^3, \delta_{1r}^4]$.

Note that Equations (9) and (10) mean there are two sources of common risk (X_t^2 and X_t^3), one specific factor driving nominal rates (X_t^1) and another risk factor (X_t^4) driving real rates. Therefore, we have three stochastic movements directly connected with each yield curve, which is consistent with the principal components analysis of Litterman and Scheinkman (1991). Moreover, the uncommon factors allow us to capture some incompleteness between the nominal and real bond markets.

In addition, we also assume that the factor dynamics under the risk-neutral measure (Q) is a Gaussian process governed by the following stochastic differential equation:

$$\mathrm{d}X_t = \kappa^{\mathcal{Q}}(\theta^{\mathcal{Q}} - X_t)\mathrm{d}t + \Sigma \mathrm{d}W_t^{\mathcal{Q}}, \tag{11}$$

where $W_t^{\mathcal{Q}}$ is a four dimensional standard Brownian motion under $\mathcal{Q}$, $\theta^{\mathcal{Q}}$ is a 4 x 1 vector and $\kappa^{\mathcal{Q}}$ and Σ are 4 x 4 matrices.

Under this setup, Duffie and Kan (1996) show that the price of a nominal zero-coupon bond maturing at time T is given by:

$$P^n(t, T) = \mathrm{e}^{A^n(\tau) - B^n(\tau)' X_t}, \tag{12}$$

where $A^n(\tau)$ and $B^n(\tau)$ satisfy the following ordinary differential equations (ODEs), which are completely determined by the specification of the risk-neutral dynamics of r_t^n in Equations 7 and 11:

$$\frac{\partial A^n(\tau)}{\partial \tau} = -\delta_{0n} - \theta^{\mathcal{Q}\prime} \kappa^{\mathcal{Q}\prime} B(\tau) + \frac{1}{2} \sum_{i=1}^{4} [\Sigma' B^n(\tau)]_i^2 \tag{13}$$

and

$$\frac{\partial B^n(\tau)}{\partial \tau} = \delta_{1n} - \kappa^{\mathcal{Q}\prime} B^n(\tau) \tag{14}$$

Starting from initial conditions $A(0) = 0$ and $B(0) = 0_{4\times1}$, these ODEs can be easily solved by numerical integration.

In an analogous way, the price of a real zero-coupon bond maturing at time T, $P^r(t,T)$, is given by:

$$P^r(t,T) = \mathrm{e}^{A^r(\tau)-B^r(\tau)'X_t} \tag{15}$$

where $A^r(\tau)$ and $B^r(\tau)$ satisfy ODEs similar to Equations 13 and 14 that are completely determined by the specification of the risk-neutral dynamics of r_t^r in Equations 8 and 11.[3]

Finally, we have to specify the market price of risk. The market price of risk is the element that connects the dynamics of the factors under the risk neutral (or pricing) measure $\mathcal{Q}$ and the physical measure $\mathcal{P}$. Duffee (2002) and Dai and Singleton (2002) tested the estimation of different versions of affine processes using historical data on U.S. Treasury bonds and concluded that a flexible time varying risk premium is fundamental to capture the term structure dynamics. Based on these results, we assume that the market price of risk, Λ_t, follows the popular essentially affine risk premium specification introduced in Duffee (2002):

$$\Lambda_t = \lambda_0 + \lambda_1 X_t, \tag{16}$$

where λ_0 is a 4 x 1 vector and λ_1 is a 4 x 4 diagonal matrix.

The market prices of risk allow us to relate the two Brownian motion vectors under both probability measures $\mathcal{P}$ and $\mathcal{Q}$:

$$W_t^{\mathcal{Q}} = W_t^{\mathcal{P}} + \int_0^t \Lambda_u \mathrm{d}u, \tag{17}$$

where $W_t^{\mathcal{P}}$ is a four-dimensional Brownian motion under $\mathcal{P}$.

The essentially affine risk premium specification implies that the process for the state vector dynamics under the physical measure $\mathcal{P}$ also has an affine form:

$$\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \Sigma \mathrm{d}W_t^{\mathcal{P}}, \tag{18}$$

where $\kappa = \kappa^{\mathcal{Q}} - \lambda_1$ and $\theta = \kappa^{-1}(\kappa^{\mathcal{Q}}\theta^{\mathcal{Q}} + \lambda_0)$.

The joint model of nominal and real yields provides a simple procedure to compute inflation expectations. The instantaneous inflation rate is given by

$$m_t = r_t^n - r_t^r = (\delta_{0n} - \delta_{0r}) + (\delta_{1n} - \delta_{1r})\, X_t = \delta_{0m} + \delta_{1m} X_t.$$

The inflation rate from t to $T = t + \tau$ is defined by

$$\pi_t(\tau) = \frac{1}{\tau}\ln\left(\frac{I_T}{I_t}\right) = \frac{1}{\tau}\left(\int_t^T m_u \mathrm{d}u\right). \tag{19}$$

[3]From Equations 12, 14 and 15, we can note that to ensure that the nominal and real yield curves are independent of X_4 and X_1, respectively, it is necessary to impose restrictions on the matrix $\kappa^{\mathcal{Q}'}$. We postpone this discussion to Section 4.1, when we present the identification of the model.

Therefore, the expected inflation rate is:

$$E_t^{\mathcal{P}}\left[\pi_t(\tau)\right] = \frac{1}{\tau}E_t^{\mathcal{P}}\left[\int_t^T m_u \mathrm{d}u\right] = \delta_{0m} + \frac{\delta_{1m}}{\tau}E_t^{\mathcal{P}}\left[\int_t^T X_{t+u}\mathrm{d}u\right],$$

The expectation on the right-hand side of the equation above can be easily computed. First, note that (see Karatzas and Shreve, 1991):

$$E_t^{\mathcal{P}}\left[X_{t+u}\right] = \left(I_N - \mathrm{e}^{-\kappa u}\right)\theta + \mathrm{e}^{-\kappa u}X_t,$$

where I_N is the identity matrix of order N. By Fubini's theorem

$$E_t^{\mathcal{P}}\left[\int_t^T X_{t+u}\mathrm{d}u\right] = \int_t^T E_t^{\mathcal{P}}\left[X_{t+u}\right]\mathrm{d}u = \tau\theta + \kappa^{-1}\left(\mathrm{e}^{-\kappa\tau} - I_N\right)\left(\theta - X_t\right).$$

Thus,

$$E_t^{\mathcal{P}}\left[\pi_t(\tau)\right] = \delta_{0m} + \delta_{1m}\left[\theta + \frac{\kappa^{-1}}{\tau}\left(\mathrm{e}^{-\kappa\tau} - I_N\right)\left(\theta - X_t\right)\right]. \quad (20)$$

Applying Girsanov's theorem to change measures in Equation 19, we have:

$$E_t^{\mathcal{P}}\left[\pi_t(\tau)\right] = \tfrac{1}{\tau}E_t^{\mathcal{Q}}\left[\left(\int_t^T m_u\mathrm{d}u\right)\tfrac{\mathrm{d}\mathcal{P}}{\mathrm{d}\mathcal{Q}}\right] =$$

$$\tfrac{1}{\tau}E_t^{\mathcal{Q}}\left[\int_t^T m_u\mathrm{d}u\right] + \tfrac{1}{\tau}\mathrm{cov}_t^{\mathcal{Q}}\left(\int_t^T m_u\mathrm{d}u, \tfrac{\mathrm{d}\mathcal{P}}{\mathrm{d}\mathcal{Q}}\right),$$

where $\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}\mathcal{Q}}$ is the Radon-Nikodym derivative. Therefore,

$$E_t^{\mathcal{P}}\left[\pi_t(\tau)\right] = E_t^{\mathcal{Q}}\left[\pi_t(\tau)\right] + \gamma_t(\tau), \quad (21)$$

where $E_t^{\mathcal{Q}}\left[\pi_t(\tau)\right]$ is the expected inflation rate in the risk-neutral world, which has similar expression to (20) with κ and θ replaced by $\kappa^{\mathcal{Q}}$ and $\theta^{\mathcal{Q}}$, respectively. The difference between the expected inflation rate in the physical and risk-neutral measures is the inflation risk premium, $\gamma_t(\tau) = \frac{1}{\tau}\mathrm{cov}_t^{\mathcal{Q}}\left(\int_t^T m_u\mathrm{d}u, \frac{\mathrm{d}\mathcal{P}}{\mathrm{d}\mathcal{Q}}\right)$. The inflation risk premium is the reward required by investors to hedge against unexpected changes in the price index

It is usual to define the inflation risk premium as the difference between the expected inflation rate and the break-even inflation rate,

$$\gamma_t(\tau) = E_t^{\mathcal{P}}\left[\pi_t(\tau)\right] - BEIR_t(\tau), \quad (22)$$

where $BEIR_t(\tau) = y_t^n(\tau) - y_t^r(\tau)$ is the break-even inflation rate. Equation 22 is a convenient way to express the inflation risk premium since the break-even inflation rate is directly read from the bond market prices. Due to the convexity correction, the break-even inflation is not the risk neutral expected inflation. So, the two definitions of inflation premium are not the same. However, the convexity correction is very small (see, for instance, Ang et al. (2008) and Graminho and Vicente (2015)), then both approaches provide approximately the same risk premium.[4] Since the definition given by (22) is easier to compute and more frequently employed in the finance literature, we opt to use it in this work.

3 Data

To estimate the model presented in Section 2, we use monthly data on nominal Brazilian yields with maturities of 1 month, 1, 3, 6 and 10 years and on real Brazilian yields with maturities of 2, 5 and 10 years. The database covers the period from April 2005 to February 2016. Although the sample size is smaller than others found in studies that use only nominal yields, it is compatible with the sample size of works that jointly model the nominal and real yields.[5]

In the Brazilian market, the *Letras do Tesouro Nacional* (LTN) and the *Notas do Tesouro Nacional F* (NTN-F) are the Treasury securities which allow investors to negotiate the nominal rate. LTN is a zero-coupon bond with face value equal to R$ 1,000.00.[6] NTN-F also has a face value of R$ 1,000.00, but pays semiannual coupons. The Brazilian Treasury also issues inflation-linked bonds which are called *Notas do Tesouro Nacional B* (NTN-B). There are two types of NTN-Bs. The first, known as NTN-B *Principal*, is a zero-coupon bond. The second one pays semiannual coupons. Both NTN-Bs are linked to the national consumer price index (IPCA).[7] From these bonds, we obtain the nominal and real yield curves using the Svensson (1994) parametric model with time-varying loadings.[8]

The Brazilian inflation-linked securities market is one of the largest in the world. In January 2016 it amounted over US$ 210 billion of NTN-B bonds outstanding, with average time to maturity of seven years. There are about US$ 160 billion and US$ 90 billion of LTN and NTN-F bonds outstanding, with average time to maturity of 16 and 52 months, respectively.

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[4] Define the nominal and the real stochastic discount factors by $N_{t,T} = -\int_t^T r_u^n \mathrm{d}u$ and $R_{t,T} = -\int_t^T r_u^r \mathrm{d}u$, respectively. Note that under the assumptions of the Gaussian affine model, N and R follow normal distributions. Then $\mathrm{e}^{N_{t,T}}$ and $\mathrm{e}^{R_{t,T}}$ are log-normals. So, $\ln P^n(t,T) = E_t^{\mathbb{Q}}[N_{t,T}] + \frac{1}{2}\mathrm{var}_t^{\mathbb{Q}}(N_{t,T})$ and $\ln P^r(t,T) = E_t^{\mathbb{Q}}[R_{t,T}] + \frac{1}{2}\mathrm{var}_t^{\mathbb{Q}}(R_{t,T})$. Dividing these two equations by τ and subtracting them, we obtain $E_t^{\mathbb{Q}}[\pi_t(\tau)] - BEIR_t(\tau) = \frac{1}{2\tau}\left[\mathrm{var}_t^{\mathbb{Q}}(N_{t,T}) - \mathrm{var}_t^{\mathbb{Q}}(R_{t,T})\right]$. Using the model parameters estimated in Section 4.1, we find that the difference between the break-even inflation and risk-neutral expectation is lower than three basis points. This figure is about two hundred times less than the Brazilian inflation rate and lower than the bid-ask spread of Brazilian fixed income bonds (around five basis points).

[5] The data set of this study is provided by ANBIMA (Brazilian Financial and Capital Market Association). ANBIMA is an entity representing players operating in the financial and capital markets. ANBIMA is one of major providers of market data and statistics, producing and disclosing information on capital markets, investment funds and treasury products.

[6] The Brazilian Real/US Dollar exchange rate was around 4.0 in February 2016.

[7] IPCA is computed by the Brazilian Institute of Geography and Statistics (IBGE) and represents the price change between the first and the thirtieth day of the reference month. It is released by the fifteenth day of the next month.

[8] The Svensson (1994) model is a purely statistical procedure that does not punish deviations from no-arbitrage values. Although this could be a disadvantage, in empirical exercises the differences between market prices and bond prices generated by the model are very small. Therefore, most of the fixed income literature does not take into account any effects related to arbitrage opportunities introduced by the method used to obtain zero-coupon yields.

The NTN-B bond and its market have some interesting properties that motivated us to use Brazilian data. First, note that the yield curve of linked bonds does not match exactly the real interest curve. This occurs because of the indexation lag.[9] Evans (1998) proposed an approximation method to obtain real curves from linked bond curves. However, the NTN-B indexation lag is very short (15 days) when compared to US TIPS and British Gilts (3 months). Therefore, we can assume that the NTN-B curve is a fairly good approximation of the term structure of real interest rates. Thus, we save the cost of the implementation of Evans (1998) procedure. Second, the Brazilian inflation-linked bonds also differ from the TIPS because they are not protected against deflation which implies that we do not need to clear the noise regarding the deflation option premium. Finally, Vicente and Graminho (2015) estimate the NTN-B liquidity premium using the procedure proposed by Pflueger and Viceira (2016). The liquidity premium represents the compensation that investors require to hold real bonds that are less liquid than nominal bonds. If the liquidity premium is different from zero, we must correct Equation 22, which would complicate the analysis of the inflation premium. Vicente and Graminho fail to reject the hypothesis that the NTN-B liquidity premium is equal to zero, which means that the NTN-B liquidity risk is not priced. Although apparently puzzling, this finding can be explained by the fact that inflation-indexed bonds in Brazil are usually bought by long term investors, such as pension funds, which hold these bonds to maturity. Therefore, although linked bonds are less liquid than nominal bonds, there is no liquidity premium since NTN-B investors are not concerned about uncertainties related to the low liquidity of these bonds.

Panels A and B of Figure 5 plot monthly yields of nominal and real Treasury bonds, respectively. Observe from the panels that essentially nominal and real yields decreased steadily before the financial crisis in August 2007, when they showed a sharp peak. Next, the yields retake the gradual decay trend until the middle of 2013. After that, the Brazilian Central Bank has acted to raise the short term interest rates in an effort to manage inflationary pressures.

The summary statistics of the yield data used in our empirical analysis are reported in Table 5. The term structure of nominal yields is upward sloping while the real curve is downward sloping. Volatility is decreasing with maturities for both curves. Nominal yields have a long right tail. The 2-year real yield is slightly right-skewed while the 5- and 10-year real yields are slightly left-skewed. With the exception of the short-term nominal rate (1 month), yields do not have fat tails. The Jarque-Bera test rejects the null hypothesis of normal distribution at the 5% significance level only for the 1-month nominal yield.[10] The autocorrelations show that all yields are highly persistent.

[9]The indexation lag comes from the fact that the payment of inflation-indexed bonds refers to a lagged value of the price index. For example, the NTN-B's maturities are on the fifteenth day of the month, but they pay the IPCA updated until the previous month.

[10]The power of the Jarque-Bera test can be quite low in small samples. Thus, the evidence of normality indicated by the Jarque-Bera test should be treated cautiously. The idea here is to show that the Gaussian model of Section 2 is not a strong hypothesis. We also performed a Kolmogorov-Smirnov normality test. The results were very similar.

We also use a one-year ahead survey-based inflation forecasting to estimate the model. Inflation expectations are read from Focus, a survey carried out by the Central Bank of Brazil. Focus is the main source of information regarding expectations for the Brazilian economy. It provides expectations about several economic variables, based on responses from more than 100 market participants, most of them financial institutions. In particular, we collect Focus data of expected future inflation rate measured by IPCA one -year ahead. Some works address the performance of Focus IPCA expectation. Among them, we can cite Carvalho and Minella (2009) and Kohlscheen (2012). In short, these studies show that Focus IPCA forecasting is unbiased and weakly rational but the errors are autocorrelated. Moreover, Focus inflation forecasting has similar performance to traditional time series models. The last line of Table 5 presents statistics of Focus one-year ahead inflation forecasting. The Jarque-Bera test does not reject the hypothesis of normality at 5% significance.

Besides the Gaussian property, the model of Section 2 exhibits idiosyncratic factors driving the nominal and real yield curves. To investigate this hypothesis, we follow D'Amico et al. (2010) and run regressions in which the dependent variable is the BEIR of maturities 2, 5 and 10 years, while the independent variables are the first three principal components of the nominal yield curve:

$$BEIR_t(\tau) = \alpha^\tau + \beta_1^\tau f_t^1 + \beta_2^\tau f_t^2 + \beta_3^\tau f_t^3 + \epsilon_t^\tau, \tag{23}$$

where f_i^t is the i^{th} principal component factor of the nominal yield curve. The R^2 values of these regressions, which represent the variability of BEIR explained by nominal bond yields, are 0.23 and 0.35 and 0.61 for $\tau = 2$, 5 and 10 years, respectively. These values are much lower than those from similar regressions with nominal yields as dependent variable, which are greater than 0.97. This result suggests the existence of factors driving real yields with sources of uncertainty independent of the nominal bond market.[11]

4 Results

In this section we analyze the empirical performance of the model of Section 2. First, we address the estimation procedure. Next, we present the fit of the model to the observed yield curves. Finally, we discuss the inflation forecasting performance of the model and the inflation risk premium.

4.1 Estimation

The first step of the estimation process is to establish the identification scheme. Restrictions of the parameter space are necessary because there are transformations of the parameters and state variables of affine models that preserve the yield curve and likelihood. In this

[11] D'Amico et al. (2010) find a similar result for the US market. Moreover, they argue that the incompleteness between the nominal and real bond markets is due to the illiquidity of the TIPS.

paper, we adopt the canonical representation of our four-factor affine model by restricting the parameters in Equation 18, as described in Dai and Singleton (2000). In this setup, κ is an upper triangular matrix, θ is a vector of zeros, Σ equals the identity matrix and λ_1 is a diagonal matrix.

As discussed in Section 2, we would like to impose independence between the nominal yield curve and X_4 and between the real yield curve and X_1, so we also need to set $\kappa^{\mathbb{Q}}(1,4) = \kappa^{\mathbb{Q}}(2,4) = \kappa^{\mathbb{Q}}(3,4) = 0$. Hence, we have that the yield curves are given by:

$$y^n(\tau) = -\frac{1}{\tau}\left[A^n(\tau) - B_1^n(\tau)X_{1,t} - B_2^n(\tau)X_{2,t} - B_3^n(\tau)X_{3,t}\right],$$

and

$$y^r(\tau) = -\frac{1}{\tau}\left[A^r(\tau) - B_2^r(\tau)X_{2,t} - B_3^r(\tau)X_{3,t} - B_4^r(\tau)X_{4,t}\right].$$

Initially, we assume that only observed yields are priced with measurement errors. So filtering methods must be used and we adopt the Kalman filter to estimate the model. In order to avoid a local maximum, we employ a sequence of the Nelder-Mead simplex and genetic algorithms with different initial points randomly selected. In the estimation process, knowledge of the factor dynamics under both measures is necessary, because the pricing measure is used to fit model implied yields to the observed yields, while the physical measure is used to define the conditional probability distributions of the factors. We call this version of the model as Yields Only (YO).[12]

Although the first estimation procedure provides a great fit of nominal and real yields, it does not do a good inflation forecasting job. Thus, we also use an alternative version in which we assume that besides yields, the difference between a one-year ahead survey-based inflation forecasting and the model inflation forecasting follows a Gaussian white noise. The idea is to include information about the inflation dynamics in the estimation process. In this case, the observation equation of the Kalman filter fits yields, which are priced using the risk-neutral measure, and inflation forecasting, which is obtained through the physical dynamics of the factors, according to (20). We call the second version of the model Survey Inflation (SI).

Table 5 presents the estimated parameter values for both versions of the model, their standard deviations calculated by the Outer Product Method (BHHH), and the ratio between the absolute value of parameter estimate and the standard error, which allow the performance of standard asymptotic tests of parameter significance. Boldface ratios indicate significant parameters at a 95% confidence level.

4.2 Yield curve fitting

In order to gauge the performance of our model, Table 5 reports some goodness-of-fit statistics of the two versions of the model. This table shows the mean error (ME), the standard deviation

[12]For a detailed description of the implementation of the Kalman filter within the framework of affine term structure models, we refer to Lund (1997).

(Std), the maximum value (Max) and the root mean square error (RMSE) of the pricing errors of nominal and real yields. We define the pricing error as the difference between the observed yields and the model implied yields. The mean error of nominal and real yields across all maturities is less than 20 basis points comparable with other studies that fit nominal and real term structures (see Haubrich et al., 2012) . The fits of nominal yields are better in the YO version while the errors of the real yields are smaller in the SI version. Moreover, the errors of the nominal yields are lower than the errors of real yields for both versions. This result can be associated with the fact that we use fewer real yields to estimate the model.

Figure 5 plots the loadings of the nominal and real yield curves for both versions of the model. Note that the nominal curve loadings can be easily identified with the level, slope and curvature. On the other hand, the real curve loadings present peculiar shapes. First, the factor X_2 is almost not priced (the loadings are around 10^{-5}). So, we have only two movements of the real curve: parallel shift and slope changes. This finding is corroborated by principal component analysis, which shows that the curvature of the real curve explains only 0.18% of yield movements while the curvature of the nominal curve represents 1.25% of the yield changes.

4.3 Inflation forecasting

The model presented in Section 2 allows us to decompose the BEI rate into inflation expectation and inflation risk premium. In this section we compare the inflation expectation extracted from both versions of the model with the Focus survey for the horizons of 1, 2, 3 and 4 years.[13] Survey of professional forecaster is a perfect benchmark to test the predictive power of a model since it outperforms many forecasting methods (see, for instance, Ang et al., 2007). Table 5 presents the mean error and the RMSE of inflation forecasting for the YO and SI versions and for the Focus survey. Figure 5 shows the time evolution of the forecasting and the actual inflation rate measured by IPCA.

First, note that both versions of the model and the Focus survey have negative biases, which means they underestimate the future inflation rate. Although the mean error of the YO version is small for all horizons (lower than the Focus bias), its RMSE is very high, indicating that the YO version does not do a good inflation forecast job. The huge RMSE for the YO version comes from the fact its inflation forecasting is very volatile. When we include inflation survey information in the estimation procedure (SI version), the RMSE is greatly reduced. The SI version is superior to YO and Focus. It has the lowest RMSE for all horizons. It is important to highlight that we use only the one-year ahead Focus inflation expectation in the estimation procedure, but the SI version outperforms Focus survey for the horizons of 2, 3 and 4 years. Hence, the inclusion of Focus information has a positive effect

[13] The Focus inflation survey contains expectations for the next 18 months and for the next five calendar years. In order to obtain inflation expectation for horizons of 2, 3 and 4 years, we assume that inflation is constant through months with non-observable data and use flat-forward interpolation.

on the inflation forecasting.[14]

The ability of the Focus survey to improve the estimation of the model and the identification of inflation risk premium is consistent with other works. For instance, Joyce et al. (2010) estimate a jointly affine model of UK nominal and real interest rate term structures incorporating bi-annual Consensus survey information on expected average inflation five to ten years ahead. They find that inflation risk premia and longer-term inflation expectations fell significantly when the Bank of England was made operationally independent in 1997. Haubrich et al. (2012) estimate a model of nominal and real bond yield curves using Treasury yields, survey inflation forecasts, and inflation swap rates. They find that the model provides noteworthy empirical results such as a detailed description of the forces driving the inflation risk premium. Chernov and Mueller (2012) propose a model with a hidden factor driving the nominal yield curve. They argue that survey-based inflation expectations are macro variables that are helpful to capture the hidden factor. They show that their model outperforms a standard macro-finance model in its forecasting of inflation and yields.

4.4 Inflation risk premium

In this section we analyze the inflation risk premium. Since the SI model has the best future inflation fit, we only study the inflation risk premium extracted from this version of the model. Inflation risk premium is an important component of the management of public debt since it represents the saving of issuance linked bonds instead of traditional bonds. Moreover, from an academic point of view, risk premium is the instrument that connects the risk-neutral and physical worlds. Therefore, perfect knowledge of the risk premium is fundamental to move from prices to expectations.

Figure 5 plots the 1-, 2-, 3- and 4-year inflation risk premia. The average risk premium values over the sample are 0.66%, 0.78%, 0.84% and 0.89% for 1-, 2-, 3- and 4-year horizons, respectively. Therefore, the inflation risk premium term structure is upward-sloping. Although the average inflation risk premium is not very high, sometimes it reaches huge values. For example, during the second half of 2015 and in the beginning of 2016, the inflation risk premium is most of the time greater than 2%. This period was marked by an economic and political crisis in Brazil which led to a sharp increase of risk aversion.

Notice that between June 2006 and July 2007 and throughout the first half of 2009 the inflation risk premia for various horizons are less than zero. This means that holding nominal bonds can hedge negative surprises to consumption. So investors require a lower rate to hold nominal bonds, implying in a negative inflation risk premium. Although accommodated by economic theory, it is difficult to think of nominal bonds as a hedge instrument against inflation and consequently a negative inflation premium. Many other studies have also reported negative inflation risk premia. Some possible explanations for this apparent puzzle

[14] Although the results of the SI model are promising, they should be viewed with caution since the forecasting of SI model is in-sample while the Focus survey is out-of-sample.

have been proposed. For example, Haubrich et al. (2012) argue that liquidity differences between nominal and real bond markets are the reason for negative inflation risk premia. From the discussion of Section 3, we can conclude that this is not the case of the Brazilian market, since the NTN-B liquidity premium is almost zero. According to Grishchenko and Huang (2013), another reason for the negative inflation risk premium is the concern of investors about deflation. Once again, this probably is not the case of Brazilian market since deflation is a very rare event in Brazil. Besides this, NTN-B is not protected against deflation. In our sample we can identify that the negative risk premium is associated with expansions of the Brazilian economy. Thus, a possible explanation for the negative inflation risk premium is the decrease of risk aversion during exuberant times. In these periods, investors behave like risk-loving agents, lowering the nominal interest rate, which yields a negative inflation risk premium. Another possible reason for the negative inflation premium can be attributed to mispricing of linked bonds. Although we do not know of any study of the Brazilian market regarding this issue, there is strong empirical evidence that the TIPS market has offered arbitrage opportunities (see Fleckenstein et al., 2014).

In short, we find that the Brazilian inflation risk premium can be sizeable. Besides this, it clearly varies through time. Hence, the Brazilian government can lower its financing costs by issuance inflation-linked bonds instead of traditional bonds. Moreover, linked bonds have an important role for the financial markets since they provide a simple hedge instrument against inflation risk.

4.5 Monetary policy

Our model provides a very simple guide to test the efficiency of Brazilian monetary policy. To this end, we run the following regressions:

$$\pi_t^e(\tau) = \alpha_1^\tau + \eta_1^\tau ys_t + \epsilon_{1,t}^\tau, \quad \tau = 1, 2, 3, 4 \tag{24}$$

and

$$\gamma_t(\tau) = \alpha_2^\tau + \eta_2^\tau ys_t + \epsilon_{2,t}^\tau, \quad \tau = 1, 2, 3, 4, \tag{25}$$

where ys is the Selic rate (Brazilian prime interest rate). It is the main monetary instrument used by Central Bank of Brazil to ensure prices stability. The Selic target is redefined by the Monetary Policy Committee of the Central Bank of Brazil approximately every 45 days. Between two meetings, the Selic rate remains almost constant very near to the target. Therefore, although regressions 24 and 25 are contemporaneous, they also embed some lagged effects of the Selic rate.

Panel A of Table 5 presents the slope coefficients on the Selic rate. Notice that inflation expectation has an inverse relationship with the basic rate while the inflation risk premium covaries positively with it. Apart from the horizon of one year in the inflation expectation regression, these findings are significant at the 10% level. Thus, when the Central Bank raises the Selic rate, the market beliefs about future inflation decreases, which is line with the Central Bank's mission. However, the risk premium increases, which causes a higher cost to roll over the public debt and an increase of agents' risk aversion. Since the absolute values of η_1^τ and η_2^τ are decreasing with τ, these impacts of Central Bank actions are more prominent over short horizons.

A possible concern with these results is related to the fact that inflation expectation and risk premium may be correlated with relevant factors that are omitted from regressions 24 and 25. In order to address this issue, we add a set of control variables in the right-hand side of the regressions, including: (i) a measure of the domestic and foreign market connection, the log return of Brazilian Real/US Dollar exchange rate; (ii) a measure of the Brazilian stock market, the log return of the Ibovespa;[15] (iii) a proxy for the Brazilian sovereign risk, the 5-year CDS rate; (iv) a measure of global risk, the VIX; (v) a proxy for the local risk, the realized volatility of daily Ibovespa returns in the month; (vi) a measure of the overall direction of commodity sectors, the log return of the Commodity Research Bureau (CRB) index;[16] and (vii) the 12-month accumulated IPCA. Panel B of Table 5 shows the results. Note that after controlling for a wide variety of factors, the sign of the coefficients of the Selic rate remain the same and the significance levels are greater than in the first two regressions (without control factors). Therefore, the findings are robust after controlling for important factors that can also drive inflation expectation and its premium.

5 Conclusion

In this paper we propose a four-factor joint model of nominal and real yield curves. The model belongs to the class of Gaussian affine models. In order to capture incompleteness between the nominal and real bond markets, we connect one risk factor with only one of the curves. The model is estimated using the Kalman filter technique. The data is from the Brazilian market. First, we assume that only observed yields are priced with measurement errors. Although the model presents good fit of the yields, it has poor inflation predictive ability. So, we also use an alternative version in which we assume that besides yields, the one-year ahead survey-based inflation forecasting is also observed with errors. When we include information about inflation survey in the estimation procedure, the model outperforms the inflation survey not only for the short horizon but also for medium and long horizons. Therefore, the inclusion of survey information improves the inflation forecasting performance of the model. Finally, we find that the main monetary policy instrument of the Central Bank of Brazil (prime rate) has a negative relation with inflation expectation. Therefore, when the Central Bank raises the short-term rate, the market belief of future inflation decreases, which can be interpreted as Central Bank credibility in its mission to curb inflation. On the order hand, tight monetary policy increases the risk premium, representing a cost of the inflation targeting regime.

[15] Ibovespa is the main index of the Brazilian Stock Exchange (BM&FBovespa).

[16] We control for CRB index since the Brazilian export basket is highly concentrated in commodities.

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Table 1: Yields and Focus summary statistics

	Central moments				Autocorrelations			JB
	Mean	Stdev	Skew	Kurt	Lag 1	Lag 2	Lag 3	
Nominal term structure								
1m	11.86%	2.97%	0.93	3.95	0.99	0.97	0.95	0.0023
1y	12.09%	2.70%	0.48	3.11	0.98	0.95	0.91	0.0627
3y	12.65%	2.21%	0.32	3.01	0.95	0.89	0.84	0.2697
6y	12.83%	1.96%	0.39	3.08	0.92	0.84	0.77	0.1321
10y	12.89%	1.82%	0.47	2.99	0.90	0.81	0.74	0.0680
Real term structure								
2y	6.61%	2.37%	0.39	2.99	0.98	0.95	0.93	0.5520
5y	6.76%	1.74%	0.02	2.90	0.97	0.94	0.90	0.9531
10y	6.60%	1.34%	-0.10	2.94	0.97	0.92	0.88	0.8903
Focus inflation forecasts								
1y	4.98%	0.77%	-0.05	2.13	0.98	0.96	0.92	0.1178

Notes: This table reports the summary statistics of the nominal and real yield data (mean, standard deviation, skewness, kurtosis, autocorrelations with lags of 1, 2 and 3 months and the p-value of the Jarque-Bera test). Nominal yields have time to maturity of 1 month, and 1, 3, 6 and 10 years, and real yields mature in 2, 5, and 10 years. Yields are obtained from Brazilian Treasury bonds through the Svensson (1994) parametric model with time-varying loadings. The null hypothesis of the Jarque-Bera test is that the data came from an unspecified normal distribution. The last line of the table contains the statistics for Focus inflation forecasts one year ahead. Focus is a survey conducted by the Central Bank of Brazil in order to collect expectations of many economic variables. The sample period is 2005:04 to 2016:02 with monthly frequency.

Table 2: Parameters estimates

	YO			SI		
Parameter	Value	Std Error	Ratio	Value	Std Error	Ratio
δ_{0n}	**0.1044**	0.0017	62.55	**0.1254**	0.0040	31.44
δ_{1n}^1	**0.0490**	0.0012	39.48	**0.0485**	0.0046	10.61
δ_{1n}^2	**0.0000**	0.0000	7.25	**0.0217**	0.0066	3.27
δ_{1n}^3	**0.0000**	0.0000	6.22	0.0000	0.0024	0.01
δ_{0r}	**0.0885**	0.0020	43.79	**0.0716**	0.0025	28.32
δ_{1r}^2	**0.0000**	0.0000	6.92	0.0000	0.0056	0.00
δ_{1r}^3	**0.0279**	0.0022	12.61	**0.0296**	0.0036	8.27
δ_{1r}^4	**0.0794**	0.0016	49.16	**0.0689**	0.0022	31.61
$\kappa(1,1)$	**6.2371**	0.1757	35.50	**5.2739**	1.0068	5.23
$\kappa(1,2)$	**0.5767**	0.0018	313.45	**0.5793**	0.1319	4.39
$\kappa(1,3)$	**2.0989**	0.0120	174.48	**0.8172**	0.1046	7.81
$\kappa(2,2)$	**0.5630**	0.0018	309.80	**3.9269**	0.0646	60.74
$\kappa(2,3)$	**0.1093**	0.0018	59.30	-0.0798	0.1636	-0.48
$\kappa(3,3)$	**0.2923**	0.0021	140.70	**2.7823**	0.0603	46.14
$\kappa(4,4)$	**12.9546**	0.1842	70.32	**13.0561**	0.0520	251.28
λ_0^1	0.9891	0.8282	1.1942	0.45	**0.0767**	5.96
λ_0^2	**0.1979**	0.0192	10.30	-0.1969	0.1954	-1.00
λ_0^3	**2.4953**	0.3107	8.03	**4.0812**	0.2933	13.91
λ_0^4	**-0.2043**	0.0074	-27.51	**-0.2055**	0.0051	-40.13
$\lambda_1(1,1)$	**-5.1308**	0.0190	-269.40	**-5.1849**	1.0074	-5.14
$\lambda_1(2,2)$	**-0.5411**	0.0016	-330.16	**-2.7996**	0.1555	-18.00
$\lambda_1(3,3)$	**1.6036**	0.0018	880.89	**2.0078**	0.1595	12.58
$\lambda_1(4,4)$	**-12.7814**	0.0162	-789.06	**-12.9040**	0.0910	-141.86

Notes: This table reports the estimated parameters and standard errors for the YO and SI versions of the model. In the YO version, only zero-coupon bond yields are considered as observed variables. In the SI version, we include the Focus inflation forecasts one year ahead in the observation equation of the Kalman filter. The last column for each model reports the ratio between the absolute value of parameter estimate and the standard error. Results that are statistically significant at 0.05 are in boldface.

Table 3: Summary statistics of pricing errors

	YO				SI			
Maturity	Mean	Std	Max	RMSE	Mean	Std	Max	RMSE
Nominal yields								
1m	0.00%	0.01%	0.09%	0.01%	0.00%	0.01%	0.03%	0.01%
1y	0.24%	0.27%	0.88%	0.36%	-0.10%	0.50%	1.28%	0.51%
3y	0.00%	0.01%	0.13%	0.01%	0.00%	0.00%	0.00%	0.00%
6y	0.02%	0.03%	0.15%	0.03%	0.18%	0.38%	1.00%	0.42%
10y	-0.05%	0.14%	0.56%	0.15%	-0.10%	0.51%	1.33%	0.52%
Real Yields								
2y	-0.46%	0.78%	2.92%	0.90%	-0.21%	0.68%	2.44%	0.71%
5y	-0.02%	0.23%	0.00%	0.24%	-0.02%	0.25%	0.00%	0.25%
10y	-0.03%	0.31%	0.66%	0.32%	-0.05%	0.31%	0.64%	0.31%

Notes: This table reports the summary statistics of the pricing errors of real and nominal yields for both versions of the model. In the YO version we use only yields to estimate the model, while we include the Focus inflation forecasts one -year ahead in the observation equation of the Kalman filter. We define the pricing error as the difference between the observed yields and the model-implied yields. The column Mean is the sample mean of the pricing errors; Std is the standard deviation; Max is the maximum absolute error; and RMSE is the root mean squared errors. The sample period is 2005:04 to 2016:02, with monthly frequency.

Table 4: Inflation forecasting

	YO		SI		Focus	
Horizon	Mean	RMSE	Mean	RMSE	Mean	RMSE
1y	0.44%	2.34%	-0.69%	1.41%	-0.69%	1.41%
2y	-0.15%	2.18%	-0.47%	1.02%	-0.94%	1.26%
3y	-0.66%	2.06%	-0.41%	0.79%	-1.09%	1.24%
4y	-0.84%	1.75%	-0.35%	0.72%	-1.14%	1.30%

Notes: This table reports the mean error and RMSE of the inflation forecasts of the YO and SI versions and of the Focus survey for the horizons of 1, 2, 3 and 4 years.

Table 5: Effects of monetary policy

Panel A: regressions without control factors

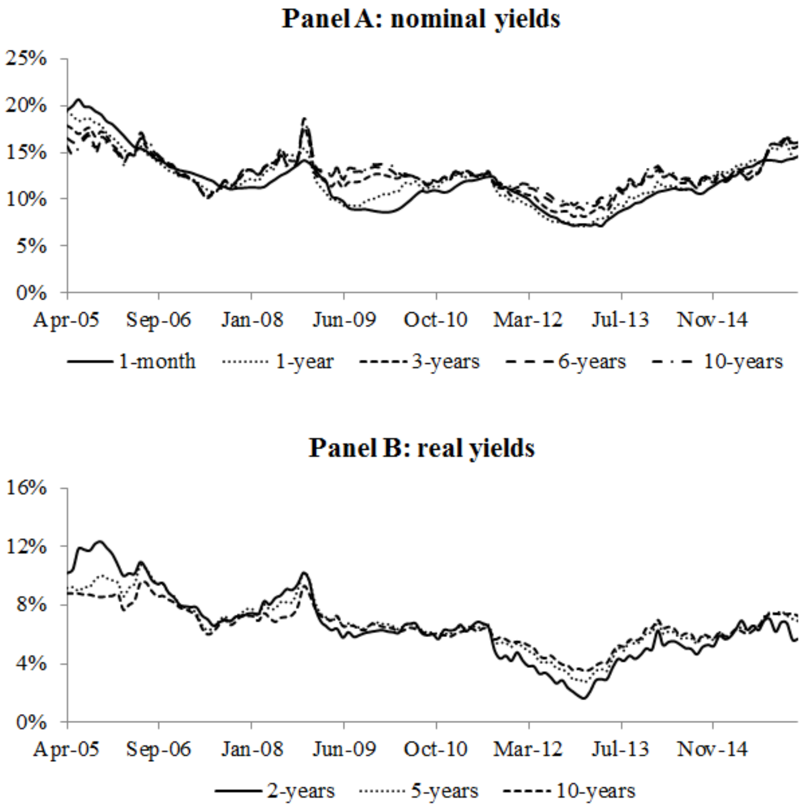
Dependent variable	1 year	2 years	3 years	4 years
Inflation expectation	-0.0362	-0.0244*	-0.0166*	-0.0124*
Risk premium	0.1534**	0.1310**	0.1151**	0.1083**

Panel B: regressions with control factors

Dependent variable	1 year	2 years	3 years	4 years
Inflation expectation	-0.0691**	-0.0422**	-0.0285**	-0.0214*
Risk premium	0.1406**	0.0940**	0.0672**	0.0554**

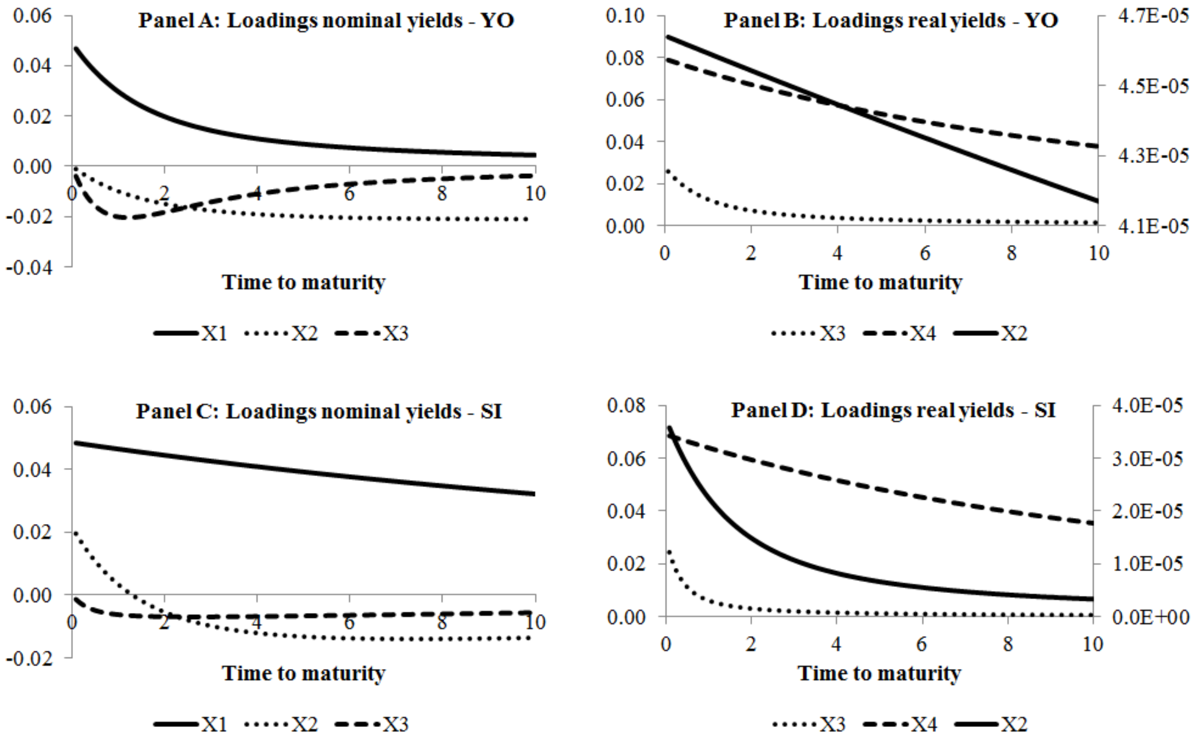
Notes: This table reports the results of the regression of inflation expectation and inflation risk premium on the Selic rate. Panel A displays the results without control factors while panel B presents the results after controlling for a wide variety of factors. * and ** denote significance at 10% and 5%, respectively.

Figure 1: Brazilian zero-coupon yields



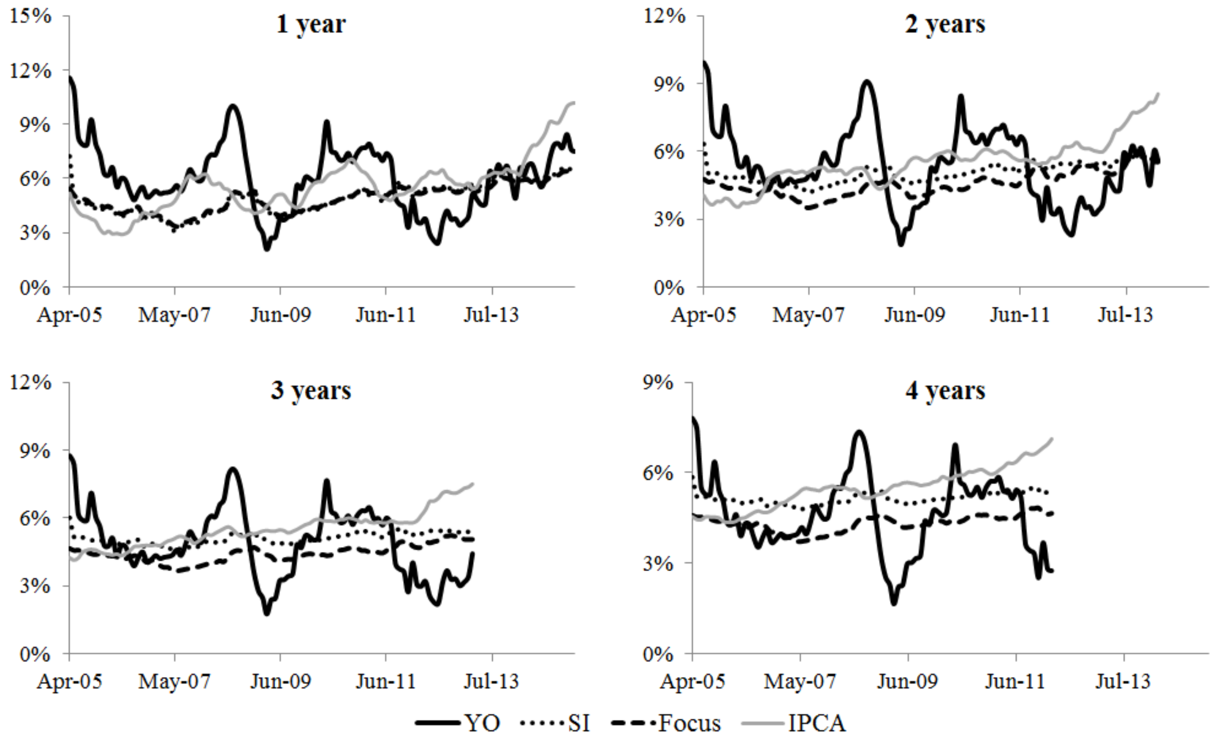
Notes: This figure plots zero-coupon nominal (Panel A) and real yields (Panel B) over the period from April 2005 to February 2016

Figure 2: Loadings of the yield curve



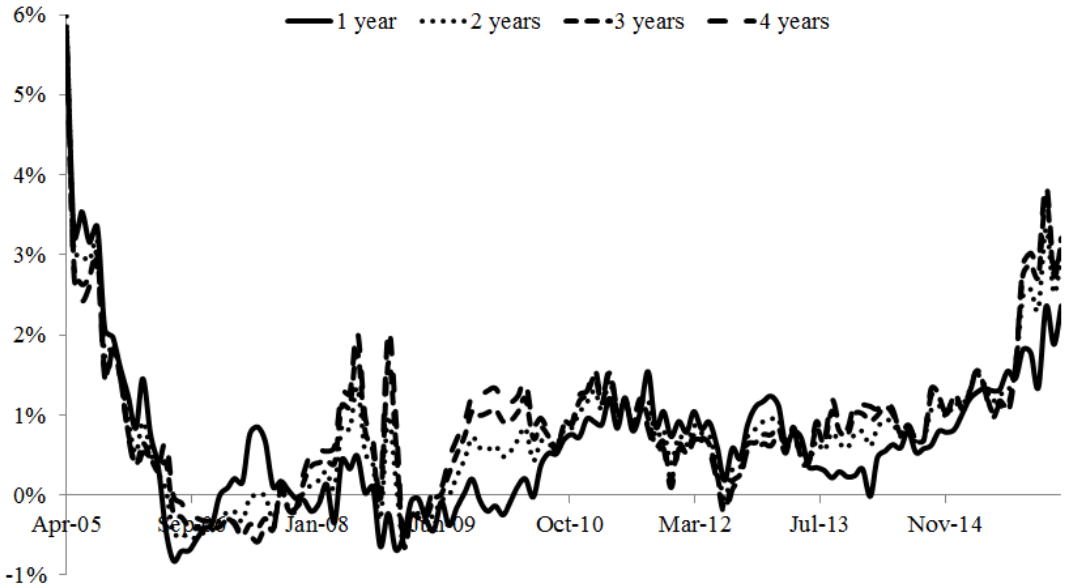
Notes: This figure shows the loadings of the nominal and real yield curves estimated for both versions of the model. In the YO version, only zero-coupon bond yields are considered as observed variables. In the SI version, we include the Focus inflation forecasts one -year ahead in the observation equation of the Kalman filter. Horizontal axes present the time to maturity in years. For the loadings of the real curve, vertical axes of X_2 factor are displayed on the right -side.

Figure 3: Inflation expectation



Notes: This figure shows the inflation expectations extracted from the YO and SI versions of the model and the Focus inflation forecasts for the horizons of 1, 2, 3 and 4-year. In the YO version, only zero-coupon bond yields are considered as observed variables. In the SI version, we include the Focus inflation forecasts one-year ahead in the observation equation of the Kalman filter. The figure also contains the actual inflation measured by the IPCA (gray line). IPCA is the main Brazilian consumer price index.

Figure 4: Inflation risk premia



Notes: This figure shows the 1-, 2-, 3- and 4-years inflation risk premia implied by the SI version of the model. The version was estimated using zero-coupon bond yields (nominal and real) and the one-year ahead Focus inflation forecasting as observed variables of the Kalman filter.