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Why Do Vulnerability Cycles Matter in Financial Networks?

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Abstract

We compare two widely employed models that estimate systemic risk: DebtRank and Differential DebtRank. We show that not only network cyclicality but also the average vulnerability of banks are essential concepts that contribute to widening the gap in the systemic risk estimates of both approaches. We find that systemic risk estimates are the same whenever the network has no cycles. However, in case the network presents cyclicality, then we need to inspect the average vulnerability of banks to estimate the underestimation gap. We find that the gap is small regardless of the cyclicality of the network when its average vulnerability is large. In contrast, the observed gap follows a quadratic behavior when the average vulnerability is small or intermediate. We show results using an econometric exercise and draw guidelines both on artificial and real-world financial networks.

Keywords: systemic risk, financial network, DebtRank, contagion.
JEL Classification: G01, G21, G28, C63.

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1 Introduction

The global crisis of 2007-2009 has highlighted important characteristics of financial markets that have not been properly considered before by regulators. Though the literature is controversial to the causes of the crisis, regulators and academics converge to the fact that the structural complexity of modern financial networks is a key component that had little understanding of its implications during the crisis. For instance, Basel III now recognizes financial interconnectedness as a key issue when analyzing systemic risk buildup (BCBS (2015)).

Since then, network-based analysis to identify and quantify systemic risk of financial systems have gained increased attention. Several works show that classical network centrality measures are suitable for measuring the potential systemic risk of the financial system given that one or a group of banks default (Billio et al. (2012); Markose et al. (2012); Silva et al. (2015); Thurner and Poledna (2013)). However, studies show that, in practice, the interbank channel becomes relevant only when banks' balance sheets are deteriorated or when we consider other contagion transmission channels, such as those of fire sales and correlated portfolios (Caccioli et al. (2014); Martinez-Jaramillo et al. (2014); Nier et al. (2007)). In addition, classical centrality measures have no clear interpretation of the potential losses they cause to the financial system.

DebtRank is a financial-oriented centrality measure that is able to capture the banks' distress levels and can also estimate potential losses in a financial system using the concept of financial stress (Battiston et al. (2012b)). We define financial stress as the capacity of banks to absorb losses rather than their payment ability. Thus, we can get a picture of how deteriorated banks' balance sheets are and hence how far from insolvency they are. In this way, financial stress gives us a sense of a continuum between solvency and insolvency. In contrast, payment ability is a binary measure: either banks can honor or not their liabilities. Since classical network measures use this last approach, they fall short on the notion of how far from insolvency banks are.

Battiston et al. (2012b)'s DebtRank has a serious shortcoming in that it blocks second- and high-order rounds of financial stress that may arise from cycles or multiple vulnerability routes in the network. Therefore, it can largely underestimate systemic risk levels. Bardoscia et al. (2015) deal with this problem by introducing a modified version of the DebtRank that we term here as differential DebtRank, in which banks are allowed to recursively diffuse stress increments and not their current stress levels at each iteration.[1] Consequently, the procedure accounts for network cycles and multiple vulnerability routes.

[1]The allusion to differential DebtRank comes from the fact that the algorithm only allows stress increments (stress differentials) to propagate in the network, as opposed to the original DebtRank formulation, in which we propagate stress levels.

In the interval between the definition of the original and differential DebtRank formulations, DebtRank has been applied in several financial networks worldwide (Poledna et al. (2015); Thurner and Poledna (2013)). Yet, no study has been performed to understand how different components of the network topology influence on affecting the systemic risk underestimation of the original DebtRank. In this work, we provide a qualitative analysis of the role that network cyclicality and the average vulnerability between banks play in the underestimation of the systemic risk by the original DebtRank in comparison to the differential DebtRank formulation. We attribute the gap on systemic risk levels between both approaches to the existence of network cycles and multiple vulnerability routes.

We first devise a novel artificial network generation process in which we can control for the network cyclicality and the average vulnerability of banks. By analyzing how the systemic risk level gap between the original and the differential DebtRank formulations varies as a function of those two components, we draw some guidelines as to when the original DebtRank formulation can severely underestimate systemic risk levels. We show that, when there are no cycles nor multiple vulnerability routes in the network, the gap is zero. Now, given that the network presents cyclicality, then we need to be aware of the average vulnerability of banks. We find that the gap is small regardless of the cyclicality of the network when the average vulnerability of the network is large. However, the gap width assumes a quadratic behavior when the vulnerability is intermediate or small. For extreme values of the network cyclicality, that is very small or very large, the gap is small. For intermediate values of the network cyclicality, the gap becomes large. The largest possible gap tends to happen for network cyclicality values that are inversely proportional to the network vulnerability.

We verify that researches in the literature that estimate systemic risk employing the original DebtRank do not report the network cyclicality nor the average vulnerability of banks. Our finding in this paper suggests that these results may be compromised. On one side, apart from being sparse due to monitoring costs, we cannot infer much about cyclicality of financial networks.[2] On the other side, we can draw some conclusions about the average vulnerability of banks. Considering that banks often diversify investments as a form of becoming less vulnerable to economic downturns, banks will unlikely engage and concentrate financial operations on a single counterparty.[3] Thus, this strategy naturally leads to small average vulnerability of banks. According to our guidelines, the systemic risk level gap between the original and differential DebtRank therefore increases. In light

[2]Though sparsity possibly leads to fewer cycles, that is not a necessary condition. For instance, we can construct a ring and a star network using the same number of links. The first topology is cycle-free, while the second is not.

[3]Basel III seems to favor this argument. For instance, they have developed large pairwise regulation as a tool for limiting the maximum loss a bank could face in the event of a sudden counterparty failure to a level that does not endanger the bank's solvency (BCBS (2014)).

of that, it is imperative to check not only network cyclicality in real financial networks, but more importantly how large the average vulnerability of banks is before attempting to use Battiston et al. (2012b)'s DebtRank when estimating systemic risk of financial systems.

To check our assumptions on real financial networks, we use a unique supervisory data set from the Central Bank of Brazil that contains pairwise exposures between banks. We inspect the network cyclicality and the average pairwise exposure of the Brazilian interbank network and find that it presents small cyclicality and average vulnerability of banks. We evaluate the gap between the systemic risk levels produced by the original and differential DebtRank formulations and find that it is small because the combination of small pairwise vulnerability between banks and small network cyclicality leads to small gaps in the systemic risk estimates between the two DebtRank formulations. We also use an econometric exercise to confirm that our claims hold for the Brazilian financial network.

2 Review on stress-based systemic risk measures

In this section, we follow the scientific trajectory of different DebtRank formulations in the literature. We start by showing Battiston et al. (2012b)'s original DebtRank. The original DebtRank can greatly underestimate the stress in the financial system, as it blocks second- and high-order rounds of impact diffusion coming from network cycles. Bardoscia et al. (2015) deal with this problem by introducing a modified version of the DebtRank that we term here as differential DebtRank, in which banks are allowed to recursively diffuse stress increments and not their current stress levels at each iteration.[4] This is the current state-of-the-art DebtRank methodology.

2.1 Original DebtRank

Though inspired by feedback centrality measures,[5] we argue that the original DebtRank is formally not a feedback centrality measure. This holds because the original DebtRank does not propagate second- and high-order rounds of impacts that come from cycles or multiple routes in the network. Due to the state mechanism that the algorithm maintains in its dynamic, banks are only allowed to propagate forward stress at the first time they receive impacts from other banks. Subsequent impacts are ignored. Thus, there is

[4]The allusion to differential DebtRank comes from the fact that the algorithm only allows stress increments (stress differentials) to propagate in the network, as opposed to the original DebtRank formulation, in which we propagate stress levels.

[5]Feedback centrality measures are those in which the centrality of a vertex recursively depends on the centrality of its neighbors. The recursiveness criterion effectively forces the centrality of each vertex to depend on the entire network structure through feedforward/feedback mechanisms.

no feedback that leads to global stress equilibrium between banks, as states act as nonlinear constraints in the diffusion process. In this respect, we can categorize the original DebtRank formulation as a nonlinear dynamical system.

The dynamical process relies on the vulnerability network of the interbank market $\mathbf{V} \in \mathscr{B} \times \mathscr{B}$, in which $\mathscr{B}$ is the set of banks, to compute the stress levels of each of the participant banks. We define such matrix as follows:

$$\mathbf{V}_{ij} = \frac{\mathbf{A}_{ij}}{E_i}, \tag{1}$$

$\forall i, j \in \mathscr{B}$ and $\mathbf{V}_{ij} \in [0, \infty)$. The entry $\mathbf{A}_{ij}$ denotes the exposure of bank i towards j in the interbank network and E_i indicates the available resources or capital buffer of bank i. Whenever $\mathbf{V}_{ij} \geq 1$, the default of financial institution j leads i into default as well. Intermediate values inside the interval $(0, 1)$ lead i into distress but not into default.

DebtRank evaluates the additional stress caused by some initial shock using a dynamical system. It maintains two dynamic variables for each bank $i \in \mathscr{B}$:

- $h_i(t) \in [0, 1]$ is the stress level of i. When $h_i(t) = 0$, i is undistressed. In contrast, when $h_i(t) = 1$, i is on default. In-between values lead to partial stress of i.
- $s_i(t) \in \{U, D, I\}$ is a categorical variable and denotes the state of i. U, D, and I stand for undistressed, distressed, and inactive, respectively.

The update rules of the dynamical system are:

$$h_i(t) = \min\left(1, h_i(t-1) + \sum_{j \in \mathscr{D}(t)} \mathbf{V}_{ij} h_j(t-1)\right), \tag{2}$$

$$s_i(t) = \begin{cases} D, & \text{if } h_i(t) > 0 \text{ and } s_i(t-1) \neq I, \\ I, & \text{if } s_i(t-1) = D, \\ s_i(t-1), & \text{otherwise.} \end{cases}, \tag{3}$$

in which $t \geq 0$ and $\mathscr{D}(t) = \{u \in \mathscr{B} \mid s_u(t-1) = D\}$. Note that the summation in (2) occurs only for those distressed banks in the previous iteration. However, once distressed, they become inactive in the next iteration due to (3). Thus, they are never able to propagate further stress. Observe that the algorithm must converge due to the min(.) operator, which places upper bounds on the banks' stress levels, and the non-decreasing property of $h_i(t)$, which derives from the non-negative entries of the vulnerability matrix as defined in (1).

For a sufficiently large number of steps $T \gg 1$, the dynamic converges. We compute the resulting DebtRank due to the initial shock scenario $h(0)$ as follows:

$$DR(h(0)) = \sum_{i \in \mathscr{B}} (h_i(T) - h_i(0))\, \varphi_i, \tag{4}$$

in which φ_i denotes the economic value of i. Observe that we remove the initial stress $h(0)$ from the DebtRank computation. Hence, it conveys the notion of additional stress given an initial shock scenario.

The great drawback of this formulation is that it prevents banks of diffusing second- and high-order rounds of stress. This means that, once a vertex propagates stress, it will never be able to propagate additional stress due to other subsequent impacts that it receives. This fact can lead to severe underestimation of the stress levels of banks.

2.2 Differential DebtRank

Battiston et al. (2012b)'s motivation for introducing states for banks is twofold. First, it prevents stress double-counting due to second- or high-order impacts through different network vulnerability routes or cycles. Second, the lagged stress level in (2) serves as an amplifying feedback mechanism, as stress levels are non-decreasing over time. These two problems arise because Battiston et al. (2012b) deal with stress levels in the update rule of the original DebtRank formulation in (2).

However, we can still account for cycles or multiple routes in the vulnerability network and therefore prevent stress double-counting by using stress differentials between one iteration and another. As a result, at each iteration, banks are only allowed to propagate the stress increment that they receive from the previous iteration. Using this mechanism, we never double-count financial stress because differentials are always innovations from one iteration to another. Again, once a bank defaults at time t, it no longer propagates financial stress during the dynamical process for $t+k$, in which k is a positive number.

We can incorporate that idea of propagating stress differentials and not stress levels by modifying (2) as follows (Bardoscia et al. (2015)):

$$\begin{aligned} h_i(t) &= \min\left(1, h_i(t-1) + \sum_{j \in \mathscr{B}} \mathbf{V}_{ij} \left[h_j(t-1) - h_j(t-2)\right]\right) \\ &= \min\left(1, h_i(t-1) + \sum_{j \in \mathscr{B}} \mathbf{V}_{ij} \Delta h_j(t-1)\right) \end{aligned} \tag{5}$$

in which $t \geq 0$, $h(0)$ denotes the initial stress scenario that the user supplies, $h(t) = 0, \forall t < 0$, and $\Delta h_j(t-1) = h_j(t-1) - h_j(t-2)$ is the stress differential of the bank j at the previous iteration $t-1$. Another important difference of the differential DebtRank to the original formulation is that the summation index in (5) runs over all of the banks. That is, we do not need to maintain states in the dynamic anymore. Therefore, Equation (5) completely characterizes the update rule of the dynamical system that describes our differential DebtRank formulation. We can then compute the DebtRank value of an initial shock scenario using (4) using the converged stress values of (5).

We compare our formulation now to that of Bardoscia et al. (2015). The authors assume that the vulnerability matrix $\mathbf{V}$ is time-dependent over time, that is, $\mathbf{V}(t)$. In special, they update $\mathbf{V}(t)$ by setting to zero the columns corresponding to those banks that default at time t. Nonetheless, we do not need to alter the vulnerability matrix over time because the differentials of banks $j \in \mathscr{B}$ that default at time t are $\Delta h_j(t+k) = 0, \forall k > 0$. Hence, we do not need to set to zero those connections of the vulnerability matrix that end up in defaulted banks. Once defaulted, they are sterilized in the dynamic process and no longer propagate stress.

In contrast to the original formulation of the DebtRank, the differential DebtRank in (5) is formally a feedback centrality measure. This is because we now have a true recursive definition of the stress levels of banks. The dynamics now reaches global equilibrium only when the direct and indirect neighborhoods of each bank are considered. In this way, the differential DebtRank takes into account multiple routes and network cycles when establishing the final stress levels of banks.

The original DebtRank serves as a lower bound for the differential DebtRank. In the case of no multiple vulnerability routes or cycles, the differential DebtRank outputs the same results as the original DebtRank.

3 Why do vulnerability cycles matter when estimating systemic risk?

The original DebtRank formulation that we discuss in Section 2.1 does not allow for second- and high-order rounds of stress propagation. Consequently, it can severely underestimate the real systemic risk of a financial system in case the corresponding vulnerability network presents several cycles or multiple vulnerability routes with different lengths. The differential DebtRank that we introduce in Section 2.2 deals with this problem by permitting financial institutions to propagate stress differentials indefinitely. In this section, we compare the original and differential DebtRank formulations using artificial networks that we construct by controlling for the network cyclicality.

Appendix A presents a formal definition of network cyclicality. In summary, it measures to what extent a network has cyclic routes. As the network cyclicality assumes larger values, more cycles exist in the network. Theoretically, it assumes a value between 0 (acyclic graph) to $1/3$ (perfect cyclic network).

Appendix B supplies the computational details to generate the artificial networks in which we control for the network cyclicality. We generate vulnerability networks and not interbank networks, as the former are more suitable for risk-analysis. Essentially, we vary the network cyclicality and inspect how the difference of the differential and original DebtRank indices behaves.

When constructing these artificial networks, we also control for the average value of pairwise vulnerabilities $\bar{v}$ between banks, which is given by:

$$\bar{v} = \frac{1}{n} \sum_{i,j \in \mathscr{B}} \mathbf{V}_{ij}, \tag{6}$$

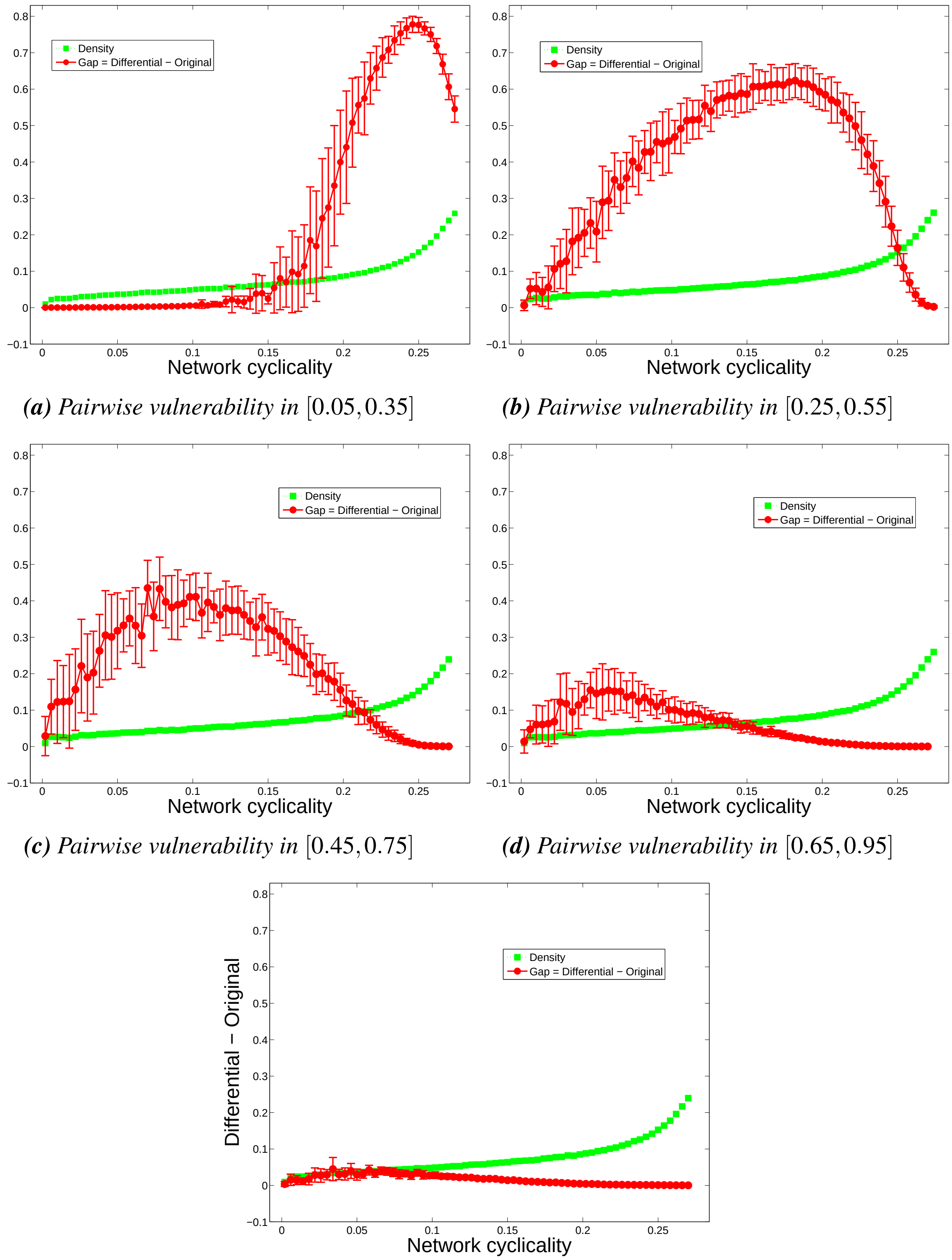
in which n denotes the number of non-zero entries of $\mathbf{V}_{ij}$.

Intuitively, we expect that smaller differences in the original and differential DebtRank formulations as the average pairwise vulnerability of banks increases. This is true because larger vulnerability values lead exposed banks into default quicker, in a way that network cycles become irrelevant in the contagion process. In real financial networks, pairwise vulnerabilities between two banks tend to be in general small, as banks often diversify their investment portfolios to minimize counterparty risks. Hence, they do not get overly exposed to a single counterpart.

Figures 1a to 1e display how the difference of the two DebtRank indices evolves as a function of the network cyclicality for five configurations of the average pairwise vulnerability between financial institutions. For example, in Fig. 1a, given that i and j are connected by a link, their vulnerability $\mathbf{V}_{ij}$ on average assumes a random value inside the interval $[0.05, 0.35]$. The larger the vulnerability index is, the more harmful is i to j. In special, when $\mathbf{V}_{ij} = 1$, the default of i directly leads j into default as well. We also report the network density to show that the generated networks are generally sparse when the network cyclicality is small. Designing artificial networks with low density better approximates our simulations to real financial networks as they normally appear as very sparse networks.[6]

Looking at the network cyclicality dimension in Figs. 1a to 1e, we see an interesting behavior. When the network cyclicality is zero, both differential and original DebtRank formulations provide similar results. This result is intuitive because if no network cycles exist, then no second- or high-order rounds of stress impact will occur. Hence, financial institutions often do not get hit more than once in the contagion process. As we increase the network cyclicality, the gap between the differential and original DebtRank rises up to

[6]See Appendix B for empirical evidences.



***(a)** Pairwise vulnerability in* $[0.05, 0.35]$

***(b)** Pairwise vulnerability in* $[0.25, 0.55]$

***(c)** Pairwise vulnerability in* $[0.45, 0.75]$

***(d)** Pairwise vulnerability in* $[0.65, 0.95]$

***(e)** Pairwise vulnerability in* $[0.85, 1]$

***Figure 1:** Gap between the differential and original DebtRank formulations as a function of the network cyclicality. We fix the number of vertices in our generated networks as* 300 *vertices. For each network cyclicality point, we form* 200 *artificial networks and calculate the differential and original DebtRank indices. In this process, we report the mean and standard deviation.*

a critical point, in which that gap is maximal. For larger network cyclicality values than this critical value, the gap then starts to diminish.

Now, given that the network presents cyclicality, then we need to be aware of the average vulnerability of banks. We find that the gap is small regardless of the cyclicality of the network when the average vulnerability of the network is large. However, the gap becomes large for intermediate or small values of the average vulnerability of the network. For small values of the average vulnerability, large network cyclicality tends to increase that gap. For intermediate values, in contrast, the gap tends to widen for small values of the network cyclicality.

We now give the intuition as to why the gap between the differential and original DebtRank indices decreases when the network cyclicality is large. In a perfect cyclic network, financial institutions are all interconnected with each other. When this complete vulnerability network only has connections with unitary weight, the default of one arbitrary financial institution drives all of the other financial institutions into default as well in a single iteration of the dynamical system. Thereby, network cycles are irrelevant in this process because the dynamic stops before we end up using the cyclic routes in the network. This behavior is consistent with Fig. 1e, which shows that the gap is almost nonexistent when cyclicality and the vulnerability are large. Hence, the differential and original DebtRank formulations give the same results. However, if we keep a complete network topology but decrease the average pairwise vulnerability between financial institutions, we attenuate this aggressive one-time effect. For instance, we still get a large gap between the two DebtRank formulations when the network cyclicality is large in Fig. 1a. This happens because the average pairwise vulnerability is so small that, even though every financial institution is interconnected with each other, the dynamical system takes longer to converge so that second- and high-order rounds are used in this process. As such, the differential DebtRank yields larger results than the original DebtRank.

Looking at the critical points in which the gap between the differential and original DebtRank is maximal, we see an inverse relationship of the average pairwise vulnerability to the maximum gap value and the critical network cyclicality position. To see that, note that as the average pairwise vulnerability increases, the critical network cyclicality position shifts to the left and its height gets smaller. These two facts occur because as we decrease the average pairwise vulnerability between financial institutions, the dynamical system that models the differential DebtRank requires a larger number of iterations to converge. This happens because system states evolve in a slower pace due to the large dampening factors, i.e., small entries of the vulnerability matrix. The gap between the differential and original DebtRank formulation grows as the former system takes longer to converge. The more iterations it takes to converge, more high-order rounds of stress differentials are propagated causing the difference between the results of both approaches to increase.

We can draw some practical guidelines from this investigation as follows:

- If the network has no cycles, then the differential and original DebtRank produce the same systemic risk estimates and hence the gap is zero.
- If the network has cycles, then it is imperative to inspect the average vulnerability of the network as well. In this case:
  - When the vulnerability of the network is large, the gap is small regardless of the cyclicality of the network.
  - When the vulnerability is intermediate or small, then the gap width assumes a quadratic behavior. For extreme values of the network cyclicality, that is very small or very large, the gap is small. For intermediate values of the network cyclicality, the gap becomes large. The largest possible gap tends to happen for network cyclicality values that are inversely proportional to the network vulnerability.

Assuming that financial institutions often diversify their investment portfolios so as to minimize counterpart risk, pairwise vulnerabilities are often small. In this case, we really must look at the network cyclicality before deciding which version of the DebtRank to use, as the gap between the results of the two DebtRank approaches quickly grows as the cyclicality increases in this situation. In the literature, we see several works that employ the original DebtRank methodology but they leave aside the analysis of the interbank network cyclicality.[7] Thus, we argue that these results may be compromised as they are likely to be underestimating the true systemic risk of the financial system if the vulnerability network contains cycles.

4 Application: Brazilian interbank network

In this section, we compare the systemic risk estimates using both the original version of the DebtRank by Battiston et al. (2012b) and the differential DebtRank by Bardoscia et al. (2015) in the Brazilian financial network. Consistent with our conclusions drawn on artificial financial networks in the previous section, we show that, generally, the observed systemic risk estimates of both approaches only slightly mismatch because of (i) the existence of small pairwise vulnerabilities between financial institutions and (ii) the small network cyclicality.

[7] See, for instance, Anufriev and Panchenko (2015); Aoyama (2014); Caldarelli et al. (2013); Chinazzi et al. (2013); Poledna et al. (2015).

4.1 Data

In this work, we use a unique Brazilian database with supervisory data.[8] We extract banks' accounting information and pairwise exposures from March 2012 through December 2015.

Following Souza et al. (2015), we consider the capital buffer or loss absorbing capability of financial institutions as their total capital (Tier 1 + Tier 2 capitals) that exceeds 8% of their risk-weighted assets (RWA). We set 8% RWA as a reference for the computation of capital buffers as we assume that if a financial institution holds less than what the Basel Committee on Banking Supervision (BCBS) recommends, it will take longer to raise its capital to an adequate level and will likely suffer an intervention from the national central bank.

Although exposures among financial institutions may be related to operations in the credit, capital and foreign exchange markets, here we focus solely on unsecured operations in the money market. The money market comprises financial operations on private securities that are registered by the Cetip:[9] interfinancial deposits, debentures and repurchase agreements collateralized by debentures issued by leasing companies of the same financial conglomerate.[10] In this work, we term the last financial instrument as "repo issued by the borrower financial conglomerate."

We use exposures among financial conglomerates and individual financial institutions that do not belong to a conglomerate. Intra-conglomerate exposures are not considered. In our sample, we only account for commercial banks, investment banks, savings banks and development banks. We classify banks according to their sizes using a simplified version of the size categories defined by the Central Bank of Brazil in the Financial Stability Report published in the second semester of 2012 (see BCB (2012)), as follows:[11] 1) we group together the micro, small, and medium banks into the "non-large" category,

[8]The collection and manipulation of the data were conducted exclusively by the staff of the Central Bank of Brazil.

[9]Cetip is a depositary of mainly private fixed income, state and city public securities and other securities representing National Treasury debts. As a central securities depositary, Cetip processes the issue, redemption and custody of securities, as well as, when applicable, the payment of interest and other events related to them. The institutions eligible to participate in Cetip include commercial banks, multiple banks, savings banks, investment banks, development banks, brokerage companies, securities distribution companies, goods and future contracts brokerage companies, leasing companies, institutional investors, non-financial companies (including investment funds and private pension companies) and foreign investors.

[10]Recall that repurchase agreements are technically secured operations. However, since the borrower in this type of repo guarantees the operation using collateral of a leasing company of the same financial conglomerate, the collateral bears the same credit risk of the borrower financial conglomerate. Thus, in practical terms, the financial operation turns out to be unsecured.

[11]The Financial Stability Report ranks financial institutions according to their positions in a descending list ordered by their total assets. The Report builds a cumulative distribution function (CDF) on the these total assets and classifies them depending on the region that they fall in the CDF. It considers as large financial institutions that fall in the 0% to 75% region. Similarly, medium-sized financial institutions fall in the 75% to 90% category, small-sized, in the 90% to 99% mark, and those above are micro-sized.

and 2) the official large category is maintained as is in our simplified version. Therefore, instead of four segments representing the bank sizes, we only employ two.

We proxy bank i's economic value, φ_i, as its fraction of total assets with respect to the total assets of the entire financial system, that is:

$$\varphi_i = \frac{TA_i}{\sum_{j \in \mathscr{B}} TA_j}, \tag{7}$$

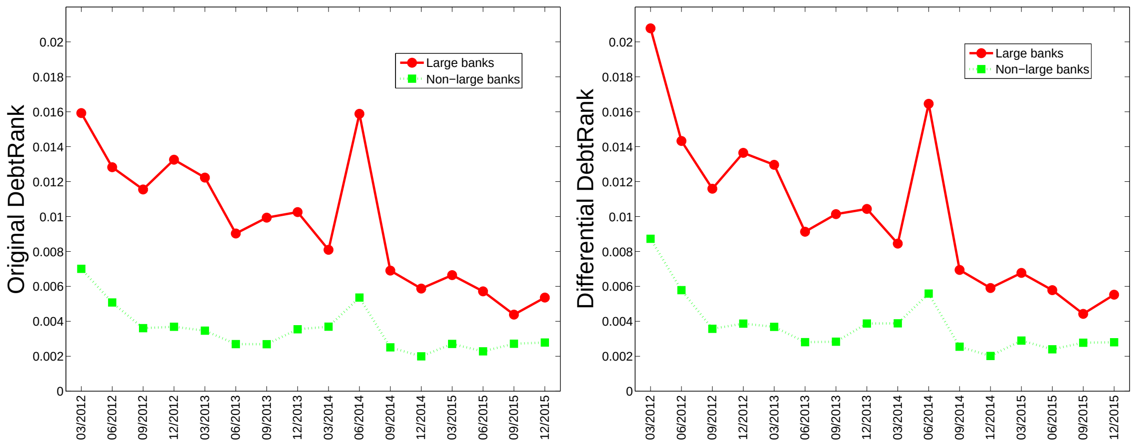
$\forall i \in \mathscr{B}$, in which TA_i represents bank i's total assets. In this setup, $\varphi_i \in [0,1]$ and $\sum_{j \in \mathscr{B}} \varphi_j = 1$. Consequently, both DebtRank formulations assume values inside the interval $[0,1]$. We can convert these indices to potential losses by simply multiplying them to the total assets of each of the participants in the financial network.

4.2 How contributive are vulnerability cycles to systemic risk buildup in the Brazilian interbank network?

Figures 2a and 2b portray the average original and differential DebtRank indices, respectively, of the Brazilian financial network. We discriminate the results by bank sizes. One first perceptive characteristic is that large banks assume the largest systemic risk levels for both indicators throughout the entire studied period. This fact happens because they are more interconnected and intermediate more financial operations by virtue of being members of the network core.[12] Using the DebtRank methodology, similar empirical studies using data from other countries also report a positive relationship between bank size and systemic risk (Aoyama et al. (2013); Battiston et al. (2013, 2012a)). However, size is not the sole determinant in establishing systemic risk levels. For instance, Silva et al. (2015) and Silva et al. (2016a) show the large heterogeneity of systemic risk levels that non-large banks potentially produce in the Brazilian financial system. Among other factors, interconnectedness and the role that banks play in the network are components that must be accounted for when estimating banks' systemic risk levels.

Comparing the results of the original and differential DebtRank formulations in Figs. 2a and 2b, we first see that the original DebtRank serves as lower bound for the differential DebtRank. We plot in Fig. 3 the relative increase of the average differential DebtRank in relation to the original DebtRank formulation for large and non-large banks, respectively. We can associate the observed systemic risk gaps in both approaches due to the existence of vulnerability cycles in the vulnerability network. We verify that the differential DebtRank assumes values that are up to 30% higher in 2012 than those of the

[12] Silva et al. (2016b) report that the Brazilian interbank network has a core-periphery structure in which the network core is mostly composed of large banks.



(a) Original DebtRank *(b) Differential DebtRank*

Figure 2: *Comparison of the original and differential DebtRank methodologies. The initial stress scenarios consist in defaulting a single bank at a time. Each point in the trajectories correspond to average values that we discriminate by bank sizes.*

original DebtRank. After 2012, it keeps oscillating around the $[0.52, 9.21]\%$ mark.

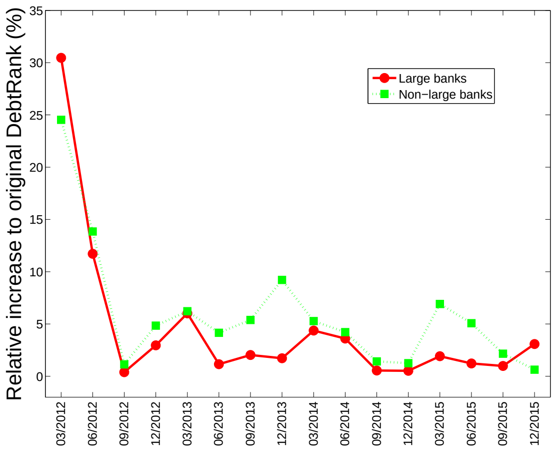
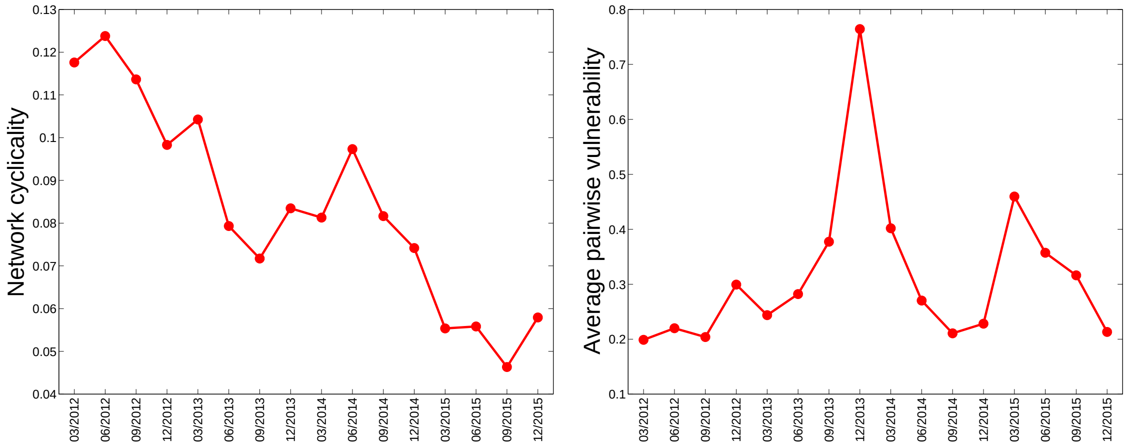


Figure 3: *Systemic risk increase using the differential and original DebtRank formulations for large and non-large banks. We assume as initial shock scenario the default of a single bank at a time. We report the results as an average of the DebtRank value of each of these banks according to their sizes.*

In order to further understand the reason of the observed gaps in both approaches, we display in Figs. 4a and 4b the network cyclicality and the average pairwise vulnerability of the Brazilian vulnerability network. We see that, overall, the vulnerability network does not present many cyclic nor multiple routes and that the average pairwise vulnerability is small. From our guidelines provided in Section 3, we see that the Brazilian financial network is a real-world case lying somewhere near Figs. 1a and 1b, or possibly a suitable linear combination of them. Inspecting Figs. 4a and 4b in 2012, it is clear that the relative cyclicality of the network is maximal and that the average pairwise vulnerability is minimal. Both characteristics contribute to widening the observed gap in the differential and original DebtRank formulations that Figure 3 reveals. After 2012, the network cyclicality tends to decrease while the average pairwise vulnerability seems to roughly oscillate. In this configuration, the original and differential DebtRank formulations produce systemic risk estimates that oscillate as well.



***(a)** Network cyclicality*

***(b)** Average pairwise vulnerability*

***Figure 4:** Network cyclicality and average pairwise vulnerability of the interbank network.*

4.3 Matching our theoretical claims on the determinants of divergence of systemic risk levels

We can empirically check our conjectures of possible causes that explain differences in the systemic risk estimates arising from the differential and original DebtRank formulations by using an econometric exercise. To get more reliable estimates, we use monthly data from January 2012 to December 2015, totaling 48 points in time.

Our goal is to explain how cyclicality and average pairwise vulnerability of the network influence on the gap formation between the two DebtRank approaches. Looking at Fig. 4a, we confirm that the Brazilian financial network has cycles throughout the entire studied period, such that the gap between both DebtRank approaches is nonzero. Moreover, inspecting Fig. 4b, we check that the average pairwise vulnerability of the network is 0.33 with a standard deviation of 0.12. Therefore, we are somewhere in-between Figs. 1a and 1b at the ascending part of the curve depicting the gap width. In this region with small pairwise vulnerability and relative small cyclicality, our guideline suggests that the vulnerability has a quadratic behavior. In this way, we will use both the linear and the quadratic functional forms of the average vulnerability of the network in our econometric exercise.

Consistent with the observations that the gap amplitude achieves maximum values that are both dependent on the vulnerability and the cyclicality of the network, we also take the interaction between these two measures in the form of a linear cross-product. According to our observations, we see that, for the same network cyclicality, increases in the average pairwise vulnerability of the network shift the maximum observed gap to the left. Therefore, we expect the interaction coefficient to be negative.

Since we are at the ascending part of the curve in-between Figs. 1a and 1b, we expect a positive relation between the gap width and the cyclicality as well as the average pairwise vulnerability of the network.

While controlling for network-related components, network cyclicality and average pairwise vulnerability will determine the gap diameter between the differential and original DebtRank formulations irrespective to its previous values. Therefore, we do not expect persistence of the gap diameter, which is a positive feature that prevents bias in our estimates due to error autocorrelation.

As we are dealing with time series data, the observed financial network evolves over time, in the form of new financial operations, interruption of old ones or rearrangements between different counterparties. Consequently, its topological characteristics—which directly govern the stress propagation mechanism of both DebtRank formulations—change as well. Theoretically speaking, we can observe different gap levels for two different network topologies even in case their cyclicality and average pairwise vulnerability are similar.[13] To control for differences on the gap level in view of these changing topological characteristics of the network, we introduce network descriptors in our model that essentially capture both strictly local and global network information.

We use the following empirical specification to identify and assess the determinant factors that explain the gap in systemic risk levels of the differential and original DebtRank formulations:

$$G_t = \beta_1 C_t + \beta_2 V_t + \beta_3 V_t^2 + \beta_4 C_t V_t + \beta_5^T D_t + \alpha + \varepsilon_t, \tag{8}$$

in which T is the transpose operator, the terms β_i, $i \in \{1, \ldots, 5\}$, are our estimates, and:

- G_t denotes our dependent variable and is the gap in the systemic risk levels produced by the differential and original DebtRank formulations at time t.
- C_t is the network cyclicality at time t.
- V_t is the average pairwise vulnerability of the network at time t.

[13] We can observe this fact by simply rearranging a few edges of the network. For instance, we can change two existent edges with different pairwise vulnerabilities that connect distinct pairs of vertices. This procedure will maintain the cyclicality and the average pairwise vulnerability of the network unaltered. However, the stress propagation mechanism that underlies the DebtRank technique will produce different systemic risk estimates on account of that differential change on the network structure.

- D_t is a set of network-based controls that we evaluate using the network snapshot at time t. We use the following control variables:[14]
  - *Average degree in the network*: the degree corresponds to how many counterparties each bank connects in the financial network. Therefore, we can interpret the degree as a proxy of banks' portfolio diversification inside the financial network. Since this is a bank-level network measurement, we take the average value of all participant banks.
  - *Average strength in the network*: the strength corresponds to the volume of financial transactions each bank performs inside the financial network. Therefore, we can interpret the strength as a proxy of the level of participation of banks inside the financial network. Since this also corresponds to a bank-level network measurement, we take the average value of all participant banks.
  - *Network disassortativity*: the disassortativity quantifies the tendency of banks to link with similar counterparties in a network. Silva et al. (2016c) show that, under certain conditions, we can employ the disassortativity measure to estimate to what extent the financial network is compliant to a perfect core-periphery structure. Therefore, it is a measure that captures the global topology of the network and therefore is susceptible to link rearrangements.
- α is a constant.
- ε_t is the error term that, by hypothesis, is identically and independently distributed with zero mean and constant variance σ_ε^2, i.e., $\varepsilon_t \sim \text{IID}(0, \sigma_\varepsilon^2)$.

Table 1 reports the summary statistics of the dependent variable and the regressors we employ in our econometric exercise. We apply a log transformation on all of the independent and dependent variables in the econometric model. In this way, we can interpret the estimates in terms of elasticity.

Table 1: *Summary statistics of the dependent variable and the regressors.*

Variable	Mean	Std. Dev.	Min.	Max.
Gap	0.0003	0.0004	0.0000	0.0020
Cyclicality	0.0862	0.0243	0.0042	0.1294
Vulnerability	0.3256	0.1167	0.1865	0.7883
Degree	3.1131	0.5423	1.3800	4.2200
Strength	0.9806	0.2869	0.5700	2.3200
Disassortativity	1.2502	0.0751	0.1400	0.4800

[14]Confer Silva and Zhao (2016) for a formal introduction on the degree, strength, and disassortativity network measurements.

To verify to what extent the regressors are correlated, which can lead to increased standard errors in the econometric model, we report in Table 2 their pairwise cross-correlation. Overall, most regressors are relatively correlated to the dependent variable while non-correlated among themselves.

Table 2: *Cross-correlation between the dependent variable and the regressors we employ in our analysis.*

	Gap	**Cyclicality**	**Vulnerability**	**Degree**	**Strength**	**Disassortativity**
Gap	1.00					
Cyclicality	0.55	1.00				
Vulnerability	-0.20	-0.42	1.00			
Degree	0.60	0.33	-0.56	1.00		
Strength	0.24	-0.01	0.52	0.00	1.00	
Disassortativity	0.35	0.25	-0.12	0.36	-0.01	1.00

Table 3 reports the estimates for our panel regressions using plain OLS. We show robust standard errors for the estimates to account for possible heteroskedasticity problems. For each of the models, we perform Ramsey (1969)'s regression specification-error test for possible omitted variables that can result in biased estimates. The null hypothesis of this model is that the specification has no omitted variables. Therefore, we should not be able to reject the null hypothesis in our estimations.

The model 1 in Table 3 is the benchmark and only accounts for, apart from transformations and interactions, the two key determinants in which we are interested: the cyclicality and average pairwise vulnerability of the network. In the models 2–4, we gradually add network-based measures to serve as controls to the changing financial network during the analyzed period. Model 5 is the full specification with all network-based controls.

We first study the effects of changes on the average pairwise vulnerability of the network. Taking the derivative of (8) with respect to the vulnerability and substituting in the estimates as reported in model 1 of Table 3, we arrive at $\frac{\partial \hat{G}_t}{\partial V_t} = 0.0028 - 0.0212 C_t$.[15] If we plug in the average value of the network cyclicality regressor, i.e., $\bar{C} = 0.0862$ (see Table 1), we get $\hat{G}_t \simeq 0.0010 > 0$. Therefore, increases on the pairwise vulnerability lead to larger gaps on the systemic risk estimates of both DebtRank formulations. This fact is consistent with our claim that the Brazilian financial network is at the ascending part of the gap formation curve, as illustrated for the case of artificial financial networks in Fig. 1.

We note also that $\frac{\partial \hat{G}_t}{\partial V_t}$ has a linear form, which leads to a quadratic behavior of $\hat{G}_t$ with respect to variations of the average pairwise vulnerability. The turning point happens when the derivative achieves zero, i.e., when $C_t = 0.1321$. Thus, when the network

[15] We consider as zero the coefficient related to the quadratic form of the vulnerability as it is statistically insignificant.

Table 3: *Panel regressions on the relative importance of network components—such as the network cyclicality and average pairwise vulnerability and their interaction—in determining the gap that arises between systemic risk estimates of the differential and original DebtRank formulations.*

Gap between differential and original DebtRank formulations					
Variables	**Model 1**	**Model 2**	**Model 3**	**Model 4**	**Model 5**
Vulnerability$_t$	0.0028***	0.0035***	0.0032***	0.0027*	0.0040*
	(0.0010)	(0.0012)	(0.0010)	(0.0012)	(0.0019)
Vulnerability$_t^2$	-0.0003	-0.0004	0.0033	-0.0003	0.0080
	(0.0019)	(0.0019)	(0.0025)	(0.0020)	(0.0129)
Cyclicality$_t$	0.0176***	0.0090**	0.0153***	0.0171**	0.0136*
	(0.0059)	(0.0038)	(0.0049)	(0.0072)	(0.0071)
Cyclicality$_t$ · Vulnerability$_t$	-0.0212**	-0.0245**	-0.0392**	-0.0202	-0.0367*
	(0.0096)	(0.0103)	(0.0156)	(0.0131)	(0.0192)
Degree$_t$		0.0019**			0.0018
		(0.0008)			(0.0016)
Strength$_t$			0.0028**		0.0005
			(0.0012)		(0.002)
Disassortativity$_t$				0.0001	-0.0008
				(0.0007)	(0.0007)
Constant	-0.0014***	-0.0035***	-0.0018***	-0.0014**	-0.0036**
	(0.0005)	(0.0012)	(0.0006)	(0.0005)	(0.0018)
Observations	48	48	48	48	48
Adjusted R^2	0.312	0.393	0.389	0.296	0.372
F (p-value)	0.000	0.000	0.000	0.00	0.000
Ramsey (p-value)	0.160	0.128	0.174	0.326	0.205

Model 1: benchmark with only the network cyclicality, linear and quadratic average pairwise vulnerability, and their linear interaction. Model 2: we increment the benchmark by using the degree measure, which is a strictly local network indicator that points the extent of diversification of banks' portfolios. Model 3: we increment the benchmark by employing the strength measure, which is another strictly local network indicator that gives us a sense on the financial operation volumes that banks establish in the financial network. Model 4: we increment the benchmark by using the disassortativity measure, which is a network-level indicator that captures the network topology and serves as a proxy to measure how compliant the network is to a perfect core-periphery model. Model 5: full model with all the network measures.
Standard errors in parentheses; ***, **, * stand for 1, 5 and 10 percent significance levels respectively.

cyclicality surpasses this critical value, positive variations on the average pairwise vulnerability cause reductions on the gap between the two DebtRank approaches. This case would correspond to the descending part of the gap formation curve as portrayed in Fig. 1. Whenever the network cyclicality is below that threshold, increases on the average pairwise vulnerability cause augments on the gap width. We observe that the maximum network cyclicality of the Brazilian financial network is 0.1294, which is smaller than the critical threshold. In this way, the average pairwise vulnerability acts as an amplifying force to widen the gap between the systemic risk estimates of the two DebtRank formulations throughout the entire studied period.

Observe that the quadratic form of the average pairwise vulnerability of the network remains statistically insignificant for all specifications. We expect such outcome because the amplitude of the vulnerability is thoroughly covered by the ascending part of the curve in Fig. 1. Therefore, a linear approximation—which stays as statistically significant in all models—suffices as it is a good fit for the range that the Brazilian data covers. Had we had average pairwise vulnerability values that would encompass both the ascending and descending part of the curve, the associated quadratic coefficient would probably be statistically significant.

We can also see that the network cyclicality maintains statistically significant over all the model specifications. If we derive (8) with respect to the network cyclicality and substitute the estimates as reported in model 1 of Table 3, we obtain $\frac{\partial \hat{G}_t}{\partial C_t} = 0.0176 - 0.0212 V_t$. Knowing that the average value of the network vulnerability regressor is $\bar{V} = 0.3256$, then the average influence of changes in the network cyclicality in the gap diameter is $\hat{G}_t \simeq 0.0011 > 0$. Thus, our models reports a positive coefficient that matches our conjectures in the sense that larger network cyclicality leads to larger gaps in systemic risk estimates of the differential and original DebtRank formulations in view of the region we are at in Fig. 1. The gap is significant because more stress will cycle through in the financial network via second or higher order rounds of stress propagations. This mechanism, while captured by the differential DebtRank, is neglected by the original DebtRank formulation.

We also see that the interaction between the cyclicality and the average pairwise vulnerability of the network remains mostly significant in our model specifications. Moreover, the negative coefficient matches our conjecture that cyclicality and vulnerability are inversely related with respect to gap width that arises between the systemic risk estimates of both DebtRank approaches.

5 Conclusion

Using both artificial networks and a real-world example of interbank loans between banks in the Brazilian banking system, we show that the estimation of systemic risk can be underestimated depending on which DebtRank is used. We attribute the gap in the systemic risk estimates of the original and differential DebtRank formulations to the existence of cycles and multiple routes in the vulnerability network.

We draw some guidelines that help in understanding how wide the gap in the systemic risk estimates would be should we use both DebtRank formulations. We show that both approaches supply the same results when the network has no cycles, regardless of the average vulnerability of banks. However, given that the network presents cycles, then it becomes essential to evaluate the average vulnerability of banks. We find that the gap is small regardless of the cyclicality of the network when the average vulnerability of the network is large. However, the gap width assumes a quadratic behavior when the vulnerability is intermediate or small. For extreme values of the network cyclicality, that is very small or very large, the gap is small. For intermediate values of the network cyclicality, the gap becomes large.

Considering that banks are unlikely to become overexposed to a single counterparty, the average vulnerability of banks is expected to be small, which turns the differential DebtRank formulation into a much better candidate to estimate systemic risk in financial networks. We confirm this hypothesis using real-world on Brazilian interbank loans.

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Appendix A Network cyclicality

The network cyclicality measures to what extent a network has cyclic routes by characterizing the degree of circulation in networks by considering cycles of all orders from 3 up to infinity (Kim and Kim (2005)). We first define the concept of vertex cyclicality and then present how we compute the network cyclicality.

The cyclicality θ_i of vertex i is the average of the inverse size of the smallest cycle that connects that vertex and any of two of its neighbor vertices. Mathematically, it is calculated as follows:

$$\theta_i = \frac{2}{k_i\,(k_i-1)} \sum_{j,k \in \mathcal{N}(i)} \frac{1}{S^i_{jk}}, \tag{9}$$

in which S^i_{jk} is the smallest size of the closed shortest path that passes through vertex i and its two neighbor vertices j and k. Note that the sum goes over all of the neighbor pairs (j,k) of i. If vertices j and k are directly linked to each other, then vertices i, j, and k form a triangle. It is a cycle of order 3 and $S^i_{jk} = 3$, which is the smallest value of S^i_{jk}. If no paths exist that connect vertices j and k except for that one that crosses vertex i, then vertices i, j, and k form a tree structure. In this case, there is no closed loop passing through the three vertices i, j, and k, in a way that $S^i_{jk} = \infty$.

The network cyclicality, θ, is equal to the average value of all of the vertex cyclicality coefficients:

$$\theta = \frac{1}{V} \sum_{i \in \mathscr{B}} \theta_i. \tag{10}$$

The network cyclicality takes a value between 0 and $1/3$, in which 0 means the network has a tree structure in which no cycle can be found, and the opposite case ($\theta = 1/3$) indicates that there is a connection between all pairs of vertices.

Appendix B Artificial network formation process

We build up our artificial networks using as baseline the classical random network model of Erdös and Rényi (1959). Starting from V vertices completely disconnected (no edges in the network), we construct the network by gradually adding random created edges, in such a way that we avoid self-loops. Computationally speaking, we can perform this task by running through all of the vertex pairs and create a link between them with probability $p > 0$. Therefore, Erdös and Rényi (1959)'s model accepts as input two parameters: the number of vertices V and the link probability p.

The link probability p modulates the network density. When $p \approx 0$, we expect that few links will be established in such a way that the network density is low (sparse network). In the other extreme, when $p \approx 1$, we expect that several links will appear and thus the network density is high (dense network). Domestic interbank networks often are sparse networks,[16] as the cost of keeping several active financial operations with different market participants is high. In light of that, banks often have a small subset of participants with which they maintain relationship lending. Hence, we expect the vulnerability networks associated to those interbank networks will be even sparser. In view of that, we opt to use small values of p when generating the artificial vulnerability networks.

Erdös and Rényi (1959)'s random network model only generates binary networks, i.e., either the link is absent or present with unitary weight. To simulate the pairwise vulnerability values that lie within the interval $[0,1]$, once the random network is generated, we independently re-weight all of its links using a uniform distribution inside the unitary interval. Since we also study the influence of the average pairwise vulnerability in the network, we establish the lower and upper limits of the uniform distribution in a way to only cover line segments inside that interval.

We are left to discuss how we calibrate the network cyclicality in the generated networks. As we increase p, it is more likely that cycles will exist in the artificial network. In this way, p can be used as a proxy for fine-tuning the network cyclicality.

We employ the following steps in the artificial network formation process. First, we generate several random networks with p varying inside the interval $(0, p_{\max}]$, in which we assume a small $p_{\max}$ of 0.25 due to the sparse nature of vulnerability networks. Since it is a stochastic process, for each fixed p value, we generate $S = 200$ networks and compute the network cyclicality, the differential DebtRank, and the original DebtRank of each network realization. Once we slowly slide through the entire interval $(0, p_{\max}]$, it is expected that we will have a smooth increases of the network cyclicality in such a way that we can plot the curve of the gap between the differential and original DebtRank as a function of the network cyclicality.

Algorithm 1 provides the pseudo-code for the artificial network formation process. The procedure takes as input six parameters:

1. V: number of vertices (banks);

2. $p_{\max}$: the maximum link probability employed when constructing Erdös and Rényi (1959)'s random networks;

[16] We can find in several works empirical evidences revealing the sparseness of domestic interbank markets, such as in Anand et al. (2015); Craig and von Peter (2014); Langfield et al. (2014); Lux (2015); Silva et al. (2016b).

3. φ: economic value of the banks;
4. minVul: minimum value for all entries of the vulnerability matrix;
5. maxVul: maximum value for all entries of the vulnerability matrix.
6. S: total number of network realizations (simulations) performed for each fixed link probability p.

In our simulations, we set as equal the economic value φ of all of the banks. We supply some comments on Algorithm 1:

- The function `randomNetwork`(V, p) in Line 5 returns a random network with V vertices and link probability p.
- The function `randomMatrix`$(\text{minVul}, \text{maxVul}, V, V)$ in Line 6 returns a $V \times V$ matrix whose entries are randomly set following a uniform distribution U(minVul, maxVul).
- The operator $\bigodot$ in Line 7 denotes the Hadamard or entrywise matrix product.

Algorithm 1 Artificial network formation process.

1: **procedure** NETWORKFORMATION(V, $p_{\max}$, φ, minVul, maxVul, S)
2: intervalSegments $\leftarrow$ DISCRETIZEINTERVAL(0, $p_{\max}$)
3: **for** $p \in$ intervalSegments **do**
4: **for** $s = 1$ to S **do**
5: randomBinaryNetwork $\leftarrow$ RANDOMNETWORK(V, p)
6: randomMatrix $\leftarrow$ RANDOMMATRIX(minVul, maxVul, V, V)
7: vulnerabilityNetwork $\leftarrow$ randomBinaryNetwork $\bigodot$ randomMatrix
8: cyclicality $\leftarrow$ NETWORKCYCLICALITY(vulnerabilityNetwork)
9: originalDebtRank $\leftarrow$ BATTISTONDR(vulnerabilityNetwork, φ)
10: differentialDebtRank $\leftarrow$ DIFFERENTIALDR(vulnerabilityNetwork, φ)
11: results $\leftarrow$ STORERESULTS(cyclicality, originalDebtRank, differentialDebtRank)
12: **end for**
13: **end for**
14: **return** results
15: **end procedure**