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Modeling Financial Networks: a feedback approach

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Abstract

We study cascade of failures in multilayer financial networks incorporating contagion feedback effects among different economic agents. We develop a flexible framework that allows for the evaluation of systemic risk in financial networks and demonstrate that the model converges to a unique fixed point. We design a financial accelerator engine to model the feedback effect between the real and the financial sectors of the economy by using contagion transmission channels such as loan defaults, bank credit crunches, deposit withdrawals, and deposit defaults. We illustrate the model using data on Brazilian bank-bank and bank-firm loans. We show that the contagion feedback effect—which accounts for second and higher-order rounds of stress propagation and is overlooked by the existing literature—is economically significant. This finding suggests that models that were developed up to date may be severely underestimating systemic risk.

Keywords: systemic risk, feedback, financial accelerator, financial network, transmission channel, contagion.
JEL Classification: G01, G21, G28, C63.

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1 Introduction

One of the main lessons that can be drawn from the financial crisis of 2008–2009 is that we live in an interconnected world in which financial shocks can trigger large disruptions in the economic environment (Allen and Gale [2004]; Gai et al. [2011]). Modern financial networks are intrinsically complex as economic agents are entangled in a diversity of markets through nontrivial financial operations, which in turn intercommunicate via contagion transmission channels. The financial sector plays a very special role in the propagation of contagion through a variety of transmission channels. For instance, a shock to the economy can affect the financial sector, which can then feedback into the real economy amplifying its initial effects.

According to the International Monetary Fund, the Financial Stability Board, and the G20 (IMF et al. [2009]), there are three key criteria that are helpful in identifying the systemic importance of markets and financial institutions to the stability of the financial system: size, substitutability, and interconnectedness.[1] Therefore, understanding how the intercommunication or feedback mechanisms that exist among transmission channels and how they impair the financial and real sectors of the economy is critically important when assessing financial stability and systemic risk. Though of great practical importance to policymakers and the scientific community, the literature is silent in providing general frameworks to model feedback mechanisms in multiple contagion transmission channels.

To date, there are several papers that deal with the nature and causes of systemic risk in single contagion channels, normally in interbank networks. The speed that shocks propagate in financial networks has intimate relation to their topological characteristics (Silva and Zhao (2012a,b)). Allen and Gale (2000) and Freixas et al. (2000) were pioneers in showing how interconnectedness of financial institutions influences the resilience of the interbank market. They argue that more densely interconnected structures are more robust than incomplete or sparse topologies. In contrast, Blume et al. (2013) model the interbank contagion as an epidemic process and find that the likelihood of a systemic collapse increases as the number of bank counterparties grows.

In a paper that unifies these conflicting views, Acemoglu et al. (2015b) show that financial contagion exhibits a form of phase transition. In this respect, more densely connected financial networks enhance financial stability as long as the magnitude of negative shocks is sufficiently small. However, beyond a certain critical point, dense interconnections serve as a mechanism that favors propagation of shocks, leading to more fragile financial systems. Acemoglu et al. (2015a) build on that idea and propose a general eco-

[1] Size refers to the volume of financial services provided by the individual component of the financial system. Substitutability measures the extent to which other components of the system can provide the same services in the event of a failure. Finally, interconnectedness accounts for the linkages with other components of the system and how shocks transmit between different markets or financial institutions.

nomic framework that explains the network topology as an amplifying driver of small-magnitude shocks. In a similar paper, Elliott et al. (2014) study cascades of failures in financial networks and find that the effects of increasing dependence on counterparties (integration) and more counterparties per organization (diversification) have different and nonmonotonic effects on the extent of financial contagion.

Related to this literature, we develop a general framework to estimate systemic risk that accounts for feedback effects that arise between different contagion transmission channels. To the best of our knowledge, this is the first work that recognizes and quantifies the importance of feedback effects in contagion models.[2] We show that these feedback effects are economically significant using micro-level supervisory data on the Brazilian financial and real sectors. We show that the framework is flexible and has strong theoretical properties, such as the existence of a unique fixed point regardless of the magnitude of initial shocks.

We represent the systemic risk framework as a nonlinear dynamical system whose evolution is coupled to a multilayer financial network. In this network, economic agents of the same nature compose the same network layer, while economic agents of different types reside in distinct network layers.

Economic agents from different network layers intercommunicate through financial operations. The establishment of these financial operations potentially creates contagion transmission channels between different network layers. In terms of contagion, the literature so far has designed stress tests that essentially evaluate how shocks in one financial layer influence the economic agents' conditions in other layers. However, these models do not take into account the feedback mechanism that naturally arises between economic agents in different financial layers. In these models, shocks propagate from one layer, normally the one that receives the initial shock of interest, to other layers. Once there, these shocks never go back to the initial layer nor propagate between different layers in a real feedback mechanism. In this work, we propose a framework in which shocks can propagate through different financial layers via feedback mechanisms. To the best of our knowledge, this is the first work that provides a systematic and extensible way to model feedback mechanisms between economic agents that lie in a multilayer financial network.

To illustrate, we model bidirectional contagion transmission channels between banks and firms. In this respect, we consider two financial layers: the financial and the real sector layers, whose economic agents we represent by banks and firms, respectively. While the first layer models the interbank lending between banks, the second layer expresses

[2] Battiston et al. (2016) highlight that concepts of feedback, networks, and contagion have recently entered the financial and regulatory lexicon, but are in their early stages. Our work also contributes to this direction as we propose a general model that not only realizes a dynamical contagion process in financial networks but also deals with the important issue of modeling the impact feedback mechanism between economic agents.

the credit lines that firms establish between their peers.[3] In addition, banks finance or provide loans to firms that in turn use these resources to fund their projects. Similarly, firms maintain deposits in custody of banks that in turn direct them toward their financial services.

The interconnections between banks and firms potentially give rise to two major contagion transmission channels: 1) contagion from banks to firms and 2) contagion from firms to banks. In this situation, the feedback mechanism we model relates closely to the financial accelerator idea, which has been extensively studied from the macroeconomic viewpoint (Bernanke [1983]; Bernanke and Gertler [1989]; Bernanke et al. [1996]; Kiyotaki and Moore [1997]; Krishnamurthy [2010]). In this way, this work also contributes to the contagion and network literatures by providing an implementation of the financial accelerator model in financial networks.

We can rationalize the financial accelerator concept by focusing on the principal-agent problem that arises in credit markets. Borrowing and lending in credit markets is costly (agency costs) due to imperfect and asymmetric information between lenders (principals) and borrowers (agents). Principals cannot access the information on investment opportunities (project returns), characteristics (creditworthiness) nor actions (risk-taking behavior) of the agents costlessly. These agency costs characterize three conditions that give rise to a financial accelerator:

- External finance (debt) is more costly than internal finance (equity).
- The premium on external finance varies inversely with the borrower's net worth, which signals ability to repay.
- A fall in borrower's net worth reduces the base for internal finance and raises the need for external finance at the same time raising its cost.

The idea of a financial accelerator originates from the fact that the borrower amplifies an initial negative shock by further decreasing its investment and production activities. We model the financial accelerator in networks using the concept of financial stress through reductions on the capacity of economic agents to absorb losses rather than their payment ability. The motivation comes from the fact that financial stress enables us to quantify how far from insolvency economic agents are. Thus, it gives us a sense of a continuum between solvency and insolvency. In contrast, payment ability is a binary measure: either or not the economic agent can repay certain liability. Therefore, we lose

[3] Though contagion analysis in the financial sector is long studied by the literature, only now researches are recognizing the importance of the real sector in amplifying shocks in the financial system (di Giovanni et al. [2014]).

the notion of how far from insolvency the economic agents are.[4]

In our financial accelerator model, when firms cannot fulfill their obligations to the banks they have borrowed from, they generate stress in a bank or a set of banks due to assets write-offs that are absorbed by their capital buffers. The reduction on banks' capital buffers potentially places upper bounds on bank assets and thereby on bank lending on account of regulatory capital constraints. Thus, the increase in the stress levels of banks feedbacks to the real economy through a credit crunch, which then exacerbates the initial shock on firms. Closing the amplifying cycle, firms are further stressed due to the credit availability constraints imposed by banks, leading them into reduced levels of investment and consumption. This negative effect on firms' production levels causes a potential decrease in profit, which is then transmitted back to banks in the form of loan defaults and deposit withdrawals.

Two conditions are necessary for such amplified effects of our financial accelerator model to hold: the inability of banks to fully insulate their supply of lending in response to such shocks and borrowers to be dependent on banks for credit. The first one normally applies because of banks' finite capital buffers and regulatory capital constraints. The second one also holds because firms often have incentives to establish loan operations with banks in detriment to other economic agents due to better contractual conditions.

The credit crunch imposed by banks affect firms in different ways. Firms that are largely dependent on bank financing are more prone to contagion coming from the financial sector. For instance, Holmstrom and Tirole (1997) and Iyer et al. (2014) conduct a study on bank dependency of European firms and find that credit crunches hit small, collateral-poor firms the hardest. Larger firms are less affected as they can either renegotiate their loans or go directly to the commercial paper or bond markets. We account for this heterogeneity in our model by setting a firm-level upper limit for firms' stress levels that depends on their proportions of internal and external financing. The larger the proportion of internal financing that firms receive—such as of shareholders, bondholders, or past profits—the more insulated they are from suffering stress due to shocks coming from the bank contagion channel.

We evaluate the systemic risk of economic agents to the financial system by first applying an initial stress scenario on one or more economic agents. We then verify the additional stress that they cause in all of the network layers. Economic agents that inflict larger additional stress levels on the system are declared to be systemically important.

[4] To exemplify, suppose a financial institution suffers a loss amounting to 95% of its capital buffer. In the Eisenberg and Noe (2001)'s algorithm that models default cascades using the payment ability approach, the financial institution is still solvent and does not propagate any losses to its direct neighbors. In contrast, when we employ the concept of financial stress, the bank is considered as almost insolvent and therefore it propagates losses by honoring only 5% of its liabilities towards its neighbors. We observe that measures that use financial stress compute potential rather than real losses. Potential losses in turn can fully or proportionally materialize or even do not occur at all.

We proxy the economic agents' stress levels by the amount of available capital buffer they have to face additional losses. Therefore, stress levels indicate how close economic agents are from insolvency.

To evaluate how stress levels of economic agents evolve, we propose a nonlinear dynamical system that works in general multilayer networks with feedback mechanisms between network layers. The stress propagation process respects the network topology that is delineated by inter- and intra-layer financial operations between economic agents. In the dynamical process, we propagate stress differentials rather than stress levels to avoid double counting. Essentially, once an economic agent suffers an increase in its stress levels, it only propagates forward the stress increment to its direct neighbors. In turn, these direct neighbors absorb losses using their capital buffers to the extent of their vulnerability to the economic agent that has diffused financial stress. Thus, neighbors that are more exposed have larger increments in their stress levels. We recursively apply this local propagation rule until we obtain convergence of all of the economic agents' stress levels.

We show that the dynamical system has two phases that we term the transient and persistent phases. The transient phase is marked with the presence of defaults of economic agents. We show that defaults naturally drive the dynamical system in less unstable dynamics. When the dynamical system eventually achieves stability, the system enters the persistent phase in which no more defaults occur.

Using intuitive concepts of network theory and general topology maps, we show that our model converges to a unique fixed point, under mild conditions, once it reaches the persistent phase. We first demonstrate the results for the two-layered network composed of banks and firms with the built-in financial accelerator feedback engine. Then, we generalize the proof of unique fixed point to multilayer networks with arbitrary number of layers and feedback rules.

In addition, we show that, when we have a single network layer composed of banks as economic agents, and hence with no feedback, our general framework reduces to the well-known DebtRank procedure originally proposed by Battiston et al. (2012b) and further enhanced by Bardoscia et al. (2015) to account for vulnerability cycles and multiple routes.

We study the model's effectiveness using Brazilian accounting and supervisory data to build the financial and real sector networks. We employ a unique database from the Central Bank of Brazil that contains detailed information on all loans made from banks to firms and between banks. We aggregate loans to the firms by economic sectors. We then simulate shocks on specific economic segments, which allows us to evaluate which sectors contribute more to systemic risk.

We study those sectors that are more risky for banks. We find that, though the oil and gas sector takes by far the highest amounts of loans from banks, firms of that sector are not the riskiest to banks in our analyzed period. In contrast, the metal extracting and processing, tertiary, and food and beverage sectors take the lead as the riskiest sectors to banks, even though banks are less exposed to firms of these sectors. We attribute this finding to the "network effect" in which the network topology can either attenuate or amplify shocks and thus plays a major role in contagion processes.

We also analyze the sensitiveness of bank control types to sector riskiness and find that domestic private banks are the most susceptible banks to receiving shocks from firms of any economic sector. In contrast, we show that government-owned banks present a tendency of becoming more resilient to shocks coming from firms in the real sector and that foreign private banks display an oscillating pattern in their susceptibility for receiving impacts.

We investigate the factors that determine sector riskiness to banks using an empirical panel-data estimation process. We find that sector riskiness is positively related to the amount of loans that banks provide to that specific sector. In addition, we find empirical evidence to support the claim that more diversified portfolios of banks contribute to higher sector riskiness levels. In this line of research, Stomper (2004) and Acharya et al. (2006) show that portfolio concentration brings benefits to banks in view of sectorial expertise, less competition with other banks, and lower monitoring costs. In this way, portfolio concentration may bring positive consequences to financial stability while also being beneficial to banks in terms of cost.[5] We also find that sector riskiness increases as firms of that sector connect to banks that are sources of stress diffusion, which is a measure that directly relates to the network topology. Therefore, sharing the same conclusions of Acemoglu et al. (2015b) and Elliott et al. (2014), we also find that network structure plays a crucial role in establishing systemic risk levels in financial networks.

We also study the role of the feedback mechanism in amplifying systemic risk in the Brazilian bank-bank and bank-firm networks. In this respect, we elaborate on counterfactual scenarios in which we compute systemic risk levels of banks and firms with and without the feedback mechanism. We show that the second and higher-order rounds of stress propagation that occur solely due to the feedback between layers are economically and statistically significant. We find that systemic risk levels of banks are increased to values that are up to 266% higher than the version without feedback. This observation suggests that models that were developed up to date are severely underestimating systemic risk, as they do not take stress feedback between contagion channels into account.

[5]We should note that the corporate finance literature on portfolio management is controversial in this matter. For instance, Diamond (1984) and Ramakrishnan and Thakor (1984) defend that banks should diversify their portfolios among different dimensions (e.g., sectors or geographic regions). The gains of this strategy would come in the form of reduction of financial intermediation costs and less vulnerability to economic downturns. Another evidence is of the "single counterparty exposure limits" policy that Basel III preconizes (BCBS [2014b]), which is an incentive to diversification over concentration.

The remainder of the paper is organized as follows. In Section 2, we discuss the methodology and detail the feedback-based systemic risk model in multilayer networks. We show that, given an initial shock scenario, there always exists a fixed point equilibrium and that it is unique. Section 3 presents results using interbank and firm data from Brazil and discusses the empirical relevance of the model that we develop. Finally, Section 4 concludes the paper.

2 Methodology

In this section, we discuss the underpinnings of the systemic risk model. In this respect, we provide an elegant framework that relies on a mathematical apparatus to estimate systemic risk while accounting for the existence of heterogenous economic agents and multiple contagion transmission channels. While relegated by the contagion literature, we are the first paper to recognize, quantify, and model the relevance of the feedback effect in a stress contagion process. We show evidences corroborating the fact that the feedback mechanism is economically significant in real-world data.

We start by describing how the model fits into a two-layered financial network that corresponds to the real and financial sectors. We design a financial accelerator engine in networks to account for the feedback mechanism between the potential contagion channels interlinking the financial and real sector networks. Then, we take a step further and generalize the model by providing an abstract framework of systemic risk evaluation using multiple contagion transmission channels and arbitrary feedback rules among economic agents. In both cases, we also demonstrate that our model converges to a unique fixed point, irrespective to the initial conditions.

2.1 Systemic risk estimation using stress feedback between the financial and real sector networks

We focus here on the contagion model in the real and financial sectors while accounting for the feedback effects between them. We first present the intuition behind the heart of the model. Then, we explore its mathematical particularities and finally demonstrate its theoretical properties.

2.2 Intuition of the model

The multilayer financial network layer comprises the financial and real sector layers. The financial sector layer represents the interbank borrowing and lending relationships among banks and the real sector layer denotes credit lines and accounts payable to suppliers that firms establish among their peers. We term these links as intralayer links, because the economic agents of both endpoints live in the same layer. Connecting these two layers, there is the bank-firm network that is necessarily a bivariate graph. We denominate these special links as interlayer links, which act as a contagion transmission channel among firms and banks. In addition, we term the network as bivariate because there are two sub-groups of vertices, namely the firms and banks, such that no firm is connected to other firms and no bank is connected to other banks. That is, we do not see links between a bank to a bank and a firm to a firm in bivariate networks. To contrast, note that the bank-bank and firm-firm networks are necessarily univariate since there is a single type of economic agent in these layers.

Figure 1 exhibits a two-layered financial network, in which banks are located at the upper layer and firms are placed at the bottom layer. Links in the bank layer indicate an exposure from one bank to another. For instance, bank 1 is exposed to 3 due to a lending operation. To simplify our example, we consider that the firm layer has no links, i.e., firms do not lend nor borrow between themselves. Firms borrow from banks to finance their projects and hence increase their profit levels. In the figure, for example, bank 1 is exposed to firm A due to a lending operation. In the theoretical model, we also address deposits that firms hold in custody of banks. However, for didactic purposes in this example, we also assume firms do not have deposits in the interbank system.

The financial accelerator feedback mechanism works in-between layers, i.e., the bank-firm network. Firms can stress banks through loan defaults and banks can also stress firms through credit restrictions.[6] Concurrently, banks can also stress other banks through the interbank (bank-bank) network. Since the network topology in the firm-firm network is empty, firms cannot directly stress each other. They can, however, indirectly affect each other through the financial accelerator mechanism. Figure 2 portrays a schematic of this process that explicitly shows the bidirectional feedback mechanism exerted by the financial accelerator in the bank-firm network. The network topology of this schematic is identical to that of Fig. 1 but now we stand out the interlayer feedback mechanism.

We now explain the mechanics of the feedback effect when computing the stress of banks and firms. Suppose a shock occurs in one of the firms in Fig. 2. To exemplify, say that firm A defaults. Due to the default, firm A may not honor in full its liabilities towards its creditors, which we represent by banks 1 and 2 in Fig. 2. Upon not receiving the due payments, these two banks become distressed as well. The increase in the stress levels of banks 1 and 2 in turn causes two direct undesirable effects:

- First, in relation to the financial sector, banks 1 and 2 propagate financial stress

[6] Had firms deposits in custody of banks in this example, firms would also stress banks through deposits withdrawals and banks would stress firms through defaults on these deposits. We account for these features in the theoretical model.

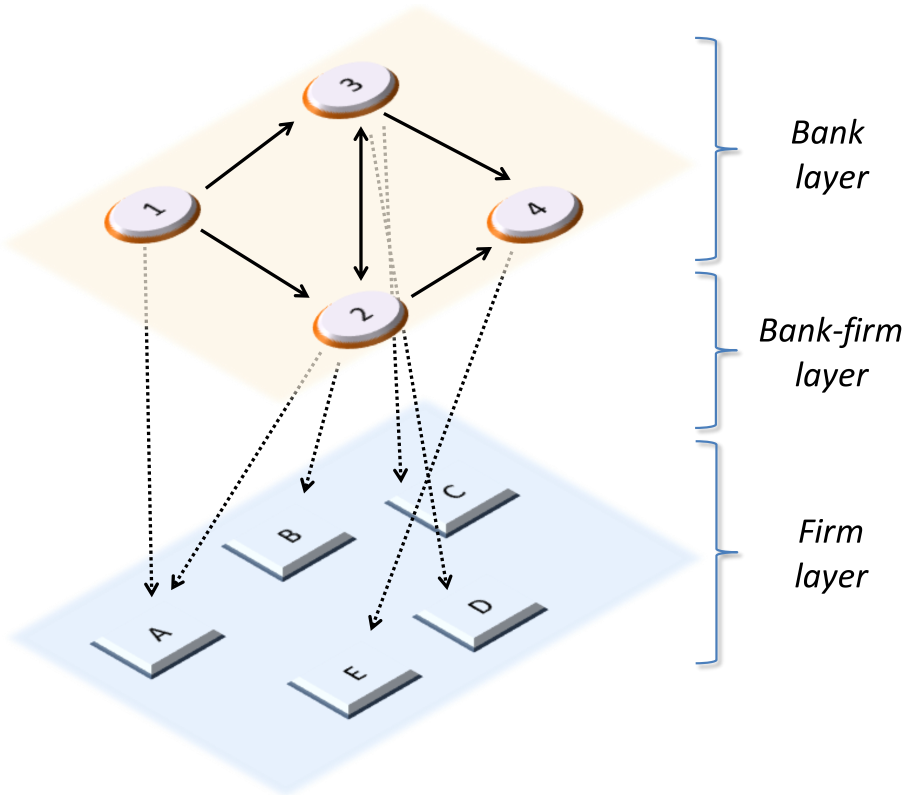


Figure 1: *A multilayer financial network with two layers: bank (circle) and firm (square) layers. Banks interconnect in the interbank market through borrowing and lending relationships. For didactic purposes in this example, we consider that the interfirm network is an empty graph, meaning that firms do not hold financial operations with each other. We also assume firms hold no deposits in custody of banks. We represent intralayer links with continuous arrowed lines and interlayer links that engine the financial accelerator with dashed arrowed lines.*

through the interbank network via potential interbank loan defaults. Upon receiving this impact, creditors of banks 1 and 2 also become distressed due to this contagion. Particularly in Fig. 2, we see that bank 3 gets impacted on account of the lending operation it has against the distressed bank 2. In addition, beside the financial stress it receives from firm A, bank 1 additionally receive another impact that originates due to its exposure to bank 2.

- Second, in relation to the real sector, banks 1 and 2 are less willing to provide further loans to other firms because of their increased financial distress as a result of assets write-offs due to the default of firm A. Consequently, they naturally restrain more and more credit to firms and banks in the market, mainly to those with low creditworthiness. This behavior in turn causes the financial stress to travel back to the real sector, in a real stress feedback mechanism. For instance in Fig. 2, firm B becomes becomes distressed on account of the reduced bank financing it receives that results from the credit crunch performed by bank 2.[7]

[7]Though firm A also receives stress feedback coming from banks, it is already in default and hence it

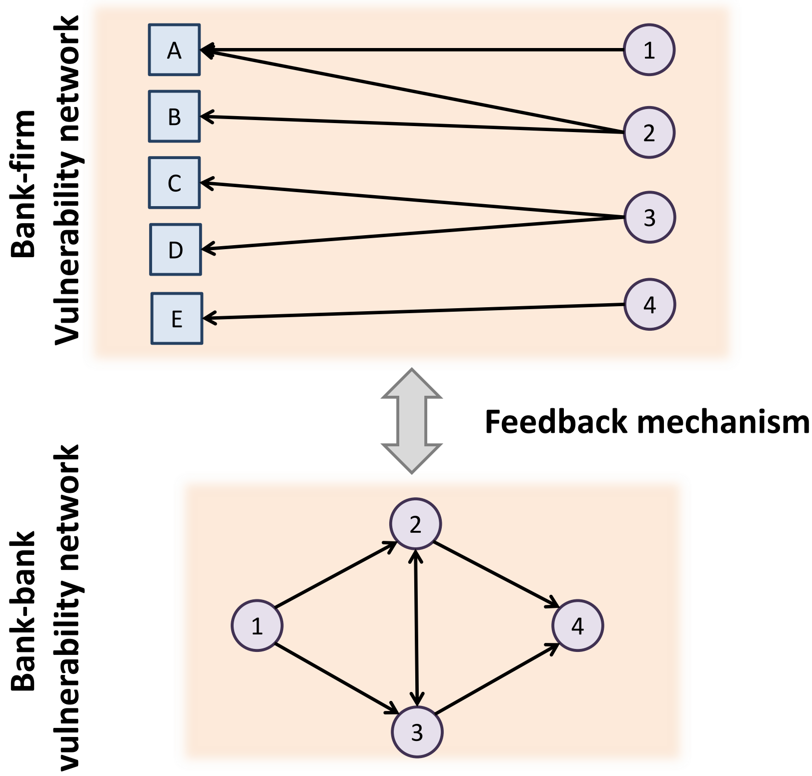


Figure 2: *Schematic of the two-way feedback mechanism for stress between the bivariate network composed of banks and firms and the univariate network composed of banks (interbank network). Squares denote firms and circles represent banks. Arrows symbolize pairwise exposures.*

The aforementioned consequences are only the direct ones. When firm B becomes distressed, it receives lower funding availability from banks, which in turn may have some impact on its revenues due to lower production levels. Consequently, the stress level of firm B increases, causing that effect to bounce back to bank 2 through loan defaults. Moreover, due to the stress that bank 3 receives in the interbank network, it affects firms C and D and also the neighboring banks in the interbank market. Thus, the shock again goes to the financial sector. This feedback/feedforward mechanism goes on until the contagion transmission system converges.

2.3 Model definition

We now define the mathematical underpinnings of the model we propose to estimate systemic risk in the real and financial sectors. We design a financial accelerator engine in a network environment that serves to feedback financial stress between economic agents. We represent the problem as a nonlinear dynamical system, whose evolution depends on the structure of the underlying two-layered financial network. Given an initial condition for the financial system, which we translate as the initial shock scenario, we are interest

does not propagate any further stress.

in the attained stress levels of firms and banks after the convergence of the dynamical system. These stress levels together proxy how harmful to the financial system that initial shock scenario would be.

We denote the financial stress levels of firms and banks at time t as $\mathbf{f}_k(t) \in [0,1]$ and $\mathbf{h}_i(t) \in [0,1]$, respectively. The stress level numerically conveys the notion of how close one economic agent is to insolvency. When $\mathbf{f}_k(t) = 1$ or $\mathbf{h}_i(t) = 1$, firms or banks default because potential losses completely deplete their available resources. When $\mathbf{f}_k(t) = 0$ or $\mathbf{h}_i(t) = 0$, firms or banks are undistressed because their loss absorbing capabilities are intact. In-between values lead to partial stress of bank i or firm k.

We update the stress levels of firms and banks in the dynamical system, which is coupled to the two-layered financial network structure, as follows:

$$\mathbf{h}_i(t) = \min\left(1, \mathbf{h}_i(t-1) + \sum_{j \in \mathscr{B}} \mathbf{V}_{ij}^{(\text{bank}-\text{bank})} \Delta\mathbf{h}_j(t-1) + \sum_{u \in \mathscr{F}} \mathbf{V}_{iu}^{(\text{bank}-\text{firm})} \Delta\mathbf{f}_u(t-1)\right), \quad (1)$$

$$\mathbf{f}_k(t) = \min\left(1, \mathbf{f}_k(t-1) + \sum_{u \in \mathscr{F}} \mathbf{V}_{ku}^{(\text{firm}-\text{firm})} \Delta\mathbf{f}_u(t-1) + \sum_{j \in \mathscr{B}} \mathbf{V}_{kj}^{(\text{firm}-\text{bank})} \Delta\mathbf{h}_j(t-1)\right), \quad (2)$$

$\forall i \in \mathscr{B}$ and $k \in \mathscr{F}$, in which $\mathscr{B}$ and $\mathscr{F}$ are the sets of banks and firms in the financial system. Note we accumulate over time the stress levels of economic agents. In the current iteration, banks and firms receive stress differentials $\Delta\mathbf{h}(t-1) = \mathbf{h}(t-1) - \mathbf{h}(t-2)$ and $\Delta\mathbf{f}(t-1) = \mathbf{f}(t-1) - \mathbf{f}(t-2)$, respectively, from those neighbors to which they are directly exposed. To circumvent the problem of stress double-counting in the diffusion process, the stress propagation relies on differentials or innovations rather than full stress levels. Financial stress can also impact economic agents that do not maintain direct exposures to each other through vulnerability routes in the network. In any case, we modulate these stress differentials in accordance with banks' or firms' sensitiveness to their direct neighborhoods.

The vulnerability matrix $\mathbf{V}^{(\text{bank}-\text{bank})}$ models the stress absorbing sensitiveness of pairs of banks. For instance, $\mathbf{V}_{ij}^{(\text{bank}-\text{bank})}$ represents the sensitiveness of bank i toward bank j in case the latter propagates stress. The larger is the sensitiveness, the larger the susceptibility of stress absorption of bank i toward bank j is. The same reasoning applies for the vulnerability matrices $\mathbf{V}^{(\text{bank}-\text{firm})}$, $\mathbf{V}^{(\text{firm}-\text{firm})}$, and $\mathbf{V}^{(\text{firm}-\text{bank})}$ that account for the sensitiveness among banks to firms, firms to firms, and firms to banks, respectively. These four vulnerability matrices tie the dynamical system evolution to the structure of the two-layered financial network. We define each of these four vulnerability matrices in the following paragraphs.

We start by introducing the vulnerability matrices that work within a network layer. We define the vulnerability matrix of the bank-bank network $\mathbf{V}^{(\text{bank}-\text{bank})} \in \mathscr{B} \times \mathscr{B}$ as follows:

$$\mathbf{V}_{ij}^{(\text{bank}-\text{bank})} = \frac{\mathbf{A}_{ij}^{(\text{bank}-\text{bank})}}{\mathbf{e}_i}, \tag{3}$$

$\forall i, j \in \mathscr{B}$ and $\mathbf{V}_{ij}^{(\text{bank}-\text{bank})} \in [0,1]$. The entry $\mathbf{A}_{ij}^{(\text{bank}-\text{bank})}$ denotes the exposure of creditor bank i toward the debtor bank j in the interbank network and e_i indicates the available resources or loss absorbing capability of bank i. All else equal, the vulnerability between bank i to j increases as the corresponding value of the lending operation increases. This is consistent with the fact that banks become more susceptible to shocks or market variations of those counterparties to which they heavily maintain investments. We take the ratio of the lending operation value to the loss absorbing capability of the creditor bank to transform the exposure into a relative measure of how representative that lending operation is to the current loss absorbing capability of the bank. As we are dealing with potential losses, Equation (3) models the fact that more distressed banks are more prone of defaulting on their interbank liabilities and hence of disseminating financial stress to their exposed neighbors.

We define the vulnerability matrix of the firm-firm network $\mathbf{V}^{(\text{firm}-\text{firm})} \in \mathscr{F} \times \mathscr{F}$ as follows:

$$\mathbf{V}_{ku}^{(\text{firm}-\text{firm})} = \frac{\mathbf{A}_{ku}^{(\text{firm}-\text{firm})}}{\mathbf{e}_k}, \tag{4}$$

$\forall k, u \in \mathscr{F}$ and $\mathbf{V}_{ku}^{(\text{firm}-\text{firm})} \in [0,1]$. The term $\mathbf{A}_{ku}^{(\text{firm}-\text{firm})}$ denotes the amount of money creditor firm $k \in \mathscr{F}$ lends to the debtor firm $u \in \mathscr{F}$. In addition, $\mathbf{e}_k$ indicates readily available resources of firm k that can be used to absorb losses. Again, all else equal, the stress sensitiveness of the creditor firm increases as the exposure to the debtor firm increases. From the economic viewpoint, Equation (4) accounts for the observation that more distressed firms are more likely to default on their interfirm liabilities and thus to generate financial stress in the financial system.

Now we present the interlayer vulnerability matrices that link economic agents of different natures. We define the vulnerability matrix of banks to firms $\mathbf{V}^{(\text{bank}-\text{firm})} \in \mathscr{B} \times \mathscr{F}$ essentially as a sum of two terms:

$$\mathbf{V}_{iu}^{(\text{bank}-\text{firm})} = \mathbf{V}_{iu}^{(\text{bank}-\text{firm})}(\text{AS}) + \mathbf{V}_{iu}^{(\text{bank}-\text{firm})}(\text{LS}), \tag{5}$$

$\forall i \in \mathscr{B}, u \in \mathscr{F}$, and $\mathbf{V}_{iu}^{(\text{bank}-\text{firm})} \in [0,1]$. The terms $\mathbf{V}_{iu}^{(\text{bank}-\text{firm})}(\text{AS})$ and $\mathbf{V}_{iu}^{(\text{bank}-\text{firm})}(\text{LS})$

represent the vulnerability of bank i to firm u that arises in light of potential exposures that can impact the asset- and liability-side of bank i's balance sheet, respectively, due to stress propagation from firm u.

We use potential loan defaults as the contagion transmission mechanism that firms in the real sector can impact the asset-side of banks' balance sheets. We model such behavior using the following mathematical expression:

$$\mathbf{V}_{iu}^{(\text{bank}-\text{firm})}(\text{AS}) = \frac{\mathbf{A}_{iu}^{(\text{bank}-\text{firm})}}{\mathbf{e}_i}, \tag{6}$$

in which $\mathbf{V}_{iu}^{(\text{bank}-\text{firm})}(\text{AS}) \in [0,1]$. The entry $\mathbf{A}_{iu}^{(\text{bank}-\text{firm})}$ indicates the loan value of bank i to firm u. Note that the more bank i is exposed to firm u, the more sensitive it is to shocks coming from that firm. In our dynamical system, Equation (6) accounts for the fact that more distressed firms are more likely to default on their obligations, such as bank loans.

We employ potential deposit withdrawals as the contagion diffusion mechanism that firms can primarily impact the liability-side of banks' balance sheets. These withdrawals cause a reduction in the available resources that banks would otherwise use to absorb losses. We model this behavior using the following expression:

$$\mathbf{V}_{iu}^{(\text{bank}-\text{firm})}(\text{LS}) = \frac{\mathbf{D}_{ui}}{\mathbf{e}_i}, \tag{7}$$

in which $\mathbf{V}_{iu}^{(\text{bank}-\text{firm})}(\text{LS}) \in [0,1]$. The term $\mathbf{D}_{ui}$ characterizes the deposit amounts that firm u has in custody of bank i. Again, note that bank i is more exposed to firm u the more it holds deposits from that firm. In our dynamical system, Equation (7) models the fact that firms are likely to withdraw more of their deposits the more they are distressed. By doing so, firms replenish their liquid positions while banks get even more distressed due to firms' deposit withdrawals.

Similarly, the vulnerability matrix of firms to banks $\mathbf{V}^{(\text{firm}-\text{bank})} \in \mathscr{F} \times \mathscr{B}$ comprises two complementary terms:

$$\mathbf{V}_{kj}^{(\text{firm}-\text{bank})} = \mathbf{V}_{kj}^{(\text{firm}-\text{bank})}(\text{AS}) + \mathbf{V}_{kj}^{(\text{firm}-\text{bank})}(\text{LS}), \tag{8}$$

$\forall k \in \mathscr{F}$, $j \in \mathscr{B}$, and $\mathbf{V}_{kj}^{(\text{firm}-\text{bank})} \in [0,1]$. The terms $\mathbf{V}_{kj}^{(\text{firm}-\text{bank})}(\text{AS})$ and $\mathbf{V}_{kj}^{(\text{firm}-\text{bank})}(\text{LS})$ indicate the vulnerability of firm k to bank j in its asset- and liability-side, respectively, which can materialize on account of shocks starting from that bank.

We consider potential deposit defaults as the contagion engine that banks in the financial sector can shock the asset-side of firms' balance sheets. We account for this characteristic using the following expression:

$$\mathbf{V}_{kj}^{(\text{firm}-\text{bank})}(\text{AS}) = \frac{\mathbf{D}_{kj}}{\mathbf{e}_k}, \tag{9}$$

in which $\mathbf{V}_{kj}^{(\text{firm}-\text{bank})}(\text{AS}) \in [0,1]$. Again, the terms $\mathbf{D}_{kj}$ and $\mathbf{e}_k$ denote deposits from firm k in custody of bank j and the firm k's available resources to withstand losses, respectively. Equation (9) models the fact that more distressed banks are more likely to default on their deposit obligations toward the real sector.

We use potential credit crunches of banks on firms to model the way banks in the financial sector stress the liability-side of firms' balance sheets. We devise this behavior using the following expression:

$$\begin{aligned}\mathbf{V}_{kj}^{(\text{firm}-\text{bank})}(\text{LS}) &= \frac{\mathbf{A}_{jk}^{(\text{bank}-\text{firm})}}{\mathbf{e}_k} \\ &= \frac{\mathbf{A}_{jk}^{(\text{bank}-\text{firm})}}{\mathbf{e}_k^{(\text{external})} + \mathbf{e}_k^{(\text{internal})}},\end{aligned} \tag{10}$$

$\forall k \in \mathscr{F}$, $j \in \mathscr{B}$, and $\mathbf{V}_{kj}^{(\text{firm}-\text{bank})}(\text{LS}) \in [0,1]$. We now decompose firm k's available resources $\mathbf{e}_k$ to better understand the impact of bank loans on firms. We consider that it encompasses the terms $\mathbf{e}_k^{(\text{external})}$ and $\mathbf{e}_k^{(\text{internal})}$, which are proxies for the total external and internal financing, respectively, of firm k. By the accounting principle, together they represent the firms' total assets. In relation to internal financing, firms can finance themselves from shareholders, as they can issue equities, or from past profits. The last financing source tends to be scarce in times of stress, which makes firms more reliant on bank credit. Bonds and bank loans are examples of external financing sources.

We can decompose firms' total financing in bank loans and other financing sources. The contagion transmission channel from firms to banks also acts in the amount of bank loans firms receive. We can compute the total bank loans that firms gather from banks directly from the multilayer network using the measurement called in-strength $\mathbf{s}_k^{(\text{in})}$,[8] whose expression is:

$$\mathbf{s}_k^{(\text{in})} = \sum_{j \in \mathscr{B}} \mathbf{A}_{jk}^{(\text{bank}-\text{firm})}. \tag{11}$$

[8]The terminology in-strength is borrowed from the complex network literature. Confer Silva and Zhao (2016) for a thorough review on complex network measures.

In this way, we can rewrite (10) as:

$$\mathbf{V}_{kj}^{(\text{firm}-\text{bank})} = \frac{\mathbf{A}_{jk}^{(\text{bank}-\text{firm})}}{\mathbf{s}_k^{(\text{in})} + \mathbf{e}_k^{(\text{other})}}, \tag{12}$$

in which $\mathbf{e}_k^{(\text{other})} = \mathbf{e}_k^{(\text{external})} + \mathbf{e}_k^{(\text{internal})} - \mathbf{s}_k^{(\text{in})}$ represents all of resource origins that firm k receives other than bank loans. Equation (12) models the fact that banks will restrain credit lines to firms the more they are distressed. Consequently, firms are expected to become even more distressed.

The more one firm relies on a specific bank to get financed, the more it will get distressed when that bank gets distressed as well. However, we modulate the stress increase of firms according to how dependent they are in terms of getting financed by banks. If firm k strongly depends on bank financing, then $\mathbf{e}_k^{(\text{other})}$ is small in relation to $\mathbf{s}_k^{(\text{in})}$. According to (12), the sensitiveness terms $\mathbf{V}_{kj}^{(\text{firm}-\text{bank})}$, $j \in \mathscr{B}$, tend to be large. Therefore, firm k becomes largely susceptible to stress coming from the bank transmission channel. In contrast, if firm k does not depend much on bank financing, then $\mathbf{e}_k^{(\text{other})}$ is large in relation to $\mathbf{s}_k^{(\text{in})}$. Therefore, in view of (12), firm k becomes more resilient against stress level increments coming from banks.

To better see the relation that the amount of bank loan and other sources of financing plays in the contagion process, suppose $\mathbf{e}_k^{(\text{other})} = \mathbf{s}_k^{(\text{in})}$. In this situation, firm k is financed half from financial institutions and the other half from external agents. In light of (2), that firm cannot default solely due to bank contagion. The contribution of this contagion channel to firm k's stress level would be at most $\mathbf{f}_k(t) = 0.5$. Now if $\mathbf{e}_k^{(\text{other})} = 0$, then firm k can default due to bank contagion and therefore its maximum stress level is $\mathbf{f}_k(t) = 1$. In the other extreme, if $\mathbf{e}_k^{(\text{other})} \gg \mathbf{s}_k^{(\text{in})}$, then firm k is sterilized from bank contagion, in a way that its maximum attainable stress level is $\mathbf{f}_k(t) \approx 0$.

The dynamical system in (1) and (2) models the stress propagation procedure in the real and financial sector networks using multiple contagion transmission channels and feedback effects between economic agents. We model the feedback effect using the idea behind the financial accelerator concept. To see that, suppose a bank becomes distressed due to an external shock from the network. According to the dynamics of the model, both firms and banks that have exposures toward that bank absorb that impact. Once they absorb the impact, they become more distressed and further propagate the stress differential back to that same bank. This back-and-forth stress dissemination process is in line with the idea of the financial accelerator, in which borrowers amplify an initial negative shock by further decreasing its investment and production activities. Consequently, they amplify that negative shock and transmit it back to the economic agent that originally propagated it.

Programmatically, we run the dynamical system that represents this financial system until both stress levels of banks and firms converge. Note that the dynamical system requires an initial shock in order to process the stress propagation. The initial shock can be arbitrary: an idiosyncratic shock on banks or firms, a sectorial shock due to adverse conditions on the economy, or generalized shocks on banks due to monetary policy conducted by the policy maker.

Given an initial shock, we set the initial stress conditions of banks, $\mathbf{h}(0)$, and of firms, $\mathbf{f}(0)$, accordingly. Suppose also that $\mathbf{h}(t) = \mathbf{f}(t) = 0, \forall t < 0$. The economic agents that receive shocks begin the dynamical process with positive stress levels. Say that the dynamical system converges for a sufficiently large number of steps $t_{\text{converge}} \gg 1$.[9] Then, the systemic risk of an initial shock scenario is given by the additional stress that shock causes on the multilayer financial network as follows:

$$SR(\mathbf{h}(0), \mathbf{f}(0)) = SR^{(\text{financial})}(\mathbf{h}(0)) + SR^{(\text{real})}(\mathbf{f}(0)), \tag{13}$$

in which $SR^{(\text{financial})}$ and $SR^{(\text{real})}$ stand for the systemic risk caused on the financial and real sector due to shocks $\mathbf{h}(0)$ and $\mathbf{f}(0)$, respectively. In this application, we represent these sectors as the banks and firms, respectively. We compute these measures as follows:

$$SR^{(\text{financial})}(\mathbf{h}(0)) = \sum_{j \in \mathscr{B}} \left(\mathbf{h}_j(t_{\text{converge}}) - \mathbf{h}_j(0)\right) v_j, \tag{14}$$

$$SR^{(\text{real})}(\mathbf{f}(0)) = \sum_{u \in \mathscr{F}} \left(\mathbf{f}_u(t_{\text{converge}}) - \mathbf{f}_u(0)\right) v_u, \tag{15}$$

in which v_j and v_u denote the economic values of bank i and firm u, respectively. Observe that we remove the stress caused by the initial shock scenario $\mathbf{h}(0)$ and $\mathbf{f}(0)$. In this way, we only account for the additional stress that an initial shock scenario inflicts on the system.

2.3.1 Theoretical analysis

In the next propositions, we analyze the dynamical behavior and the theoretical properties of the feedback-based contagion model in the real and financial sectors.

We first transform the model, whose evolution is coupled to the two-layered network that encompasses economic agents from the real and financial sectors, to a standard format

[9] We show in the next section that the model always converges to a unique fixed point. Then, we can assume that $t_{\text{converge}} < \infty$ with no loss of generality.

of state-space dynamical system. This step is useful as it facilitates the understanding of convergence issues of the system.

Proposition 1. *The systemic risk framework with $B = |\mathscr{B}|$ banks and $F = |\mathscr{F}|$ firms, whose behavior is determined by (1) and (2), can be cast into a state-space system:*

$$\Delta\mathbf{s}(t) = \min(1, \mathbf{V}\Delta\mathbf{s}(t-1)), \tag{16}$$

in which $\Delta\mathbf{s}(t) \in [0,1]^{B+F}$ is the state of the dynamical system and represents iteration-wise stress level differentials or increments of economic agents that occur at iteration t. Mathematically,

$$\Delta\mathbf{s}(t) = \mathbf{s}(t) - \mathbf{s}(t-1), \tag{17}$$

in which $\mathbf{s}(t) = [\mathbf{h}(t)\, \mathbf{f}(t)]^T$ is a column vector with $B+F$ entries that compounds the stress levels of banks and firms stacked in that order and T is the transpose operator.

We construct the update matrix of the feedback-based systemic risk model $\mathbf{V}$ as:

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}^{(\text{bank}-\text{bank})} & \mathbf{V}^{(\text{bank}-\text{firm})} \\ \mathbf{V}^{(\text{firm}-\text{bank})} & \mathbf{V}^{(\text{firm}-\text{firm})} \end{pmatrix}, \tag{18}$$

in which $\mathbf{V}^{(\text{bank}-\text{bank})} \in \mathscr{B} \times \mathscr{B}$, $\mathbf{V}^{(\text{bank}-\text{firm})} \in \mathscr{B} \times \mathscr{F}$, $\mathbf{V}^{(\text{firm}-\text{bank})} \in \mathscr{F} \times \mathscr{B}$, and $\mathbf{V}^{(\text{firm}-\text{firm})} \in \mathscr{F} \times \mathscr{F}$. In this way, the update matrix $\mathbf{V}$ has dimensions of $(B+F) \times (B+F)$.

Remark 1. ***Monotonically non-decreasing property of stress levels**: the trajectory of $\Delta\mathbf{s}(t)$ is non-decreasing in light of the accumulative non-negative increments $\Delta\mathbf{h}(t)$ and $\Delta\mathbf{f}(t)$. That is, $\Delta\mathbf{s}(t+k) \geq \Delta\mathbf{s}(t), \forall k \in \{1,2,\ldots\}$. This inequality holds because every entry of the update matrix $\mathbf{V}$ and the initial condition $\Delta\mathbf{s}(0)$ are non-negative. Hence, inner joins between $\mathbf{V}$ and $\Delta\mathbf{s}$ cannot yield negative values.*

Proposition 1 describes a mathematical setup that is inspired by the financial accelerator model. To date, most of the research on systemic risk has neglected the feedback between the real and financial sectors. Our model incorporates this feature that is relevant to effectively capture how micro events, such as idiosyncratic shocks on economic agents, can unfold into macro events, such as systemic risk buildup.

Remark 2. ***Monotonically non-decreasing property of stress levels**: the trajectory of $\Delta s(t)$ is non-decreasing in light of the accumulative non-negative increments $\Delta h(t)$ and $\Delta f(t)$. That is, $\Delta s(t+k) \geq \Delta s(t), \forall k \in \{1,2,\ldots\}$. This inequality holds because every* *entry of the update matrix* $\mathbf{S}$ *and the initial condition* $\Delta s(0)$ *are non-negative. Hence, inner joins between* $\mathbf{S}$ *and* X *cannot yield negative values.*

Remark 3. ***Stability of the system****: the* $\min(.)$ *operator in (16) guarantees the stability of the dynamical system. For instance, if we remove the* $\min(.)$ *operator, then the system becomes unstable once an entry of* $X(0)$ *becomes greater or equal to one. To exemplify, consider that* $X_i(t) \geq 1$ *and that the i-th column of* $\mathbf{S}$ *is full of ones. In this situation, there would be no dampening of the i-th column of* $\mathbf{S}$ *anymore. Using this fact with the non-decreasing behavior of* $X(0)$ *results in unboundedness of* $X(0)$ *and hence unstableness of the dynamical system.*

The update matrix $\mathbf{V}$ is time-invariant, meaning that its spectrum is constant over time. However, as banks or firms default, the stress diffusion potentiality of the system is reduced, since there are fewer active players. In this way, the system drives itself to a more stable dynamics. In mathematical terms, the spectrum of matrix $\mathbf{V}$ tends to decrease over time. For the purposes of stability analysis, we can make $\mathbf{V}$ time-variant in a way to have better estimates of its true potential spectrum over time. The next lemma provides a possible rearrangement of $\mathbf{V}$ that alters its spectrum while maintaining identical the behavior of the dynamical system.

Lemma 1. *The dynamical system in (16) can be rewritten as:*

$$\Delta\mathbf{s}(t) = \min(1, \mathbf{V}(t)\Delta\mathbf{s}(t-1)), \tag{19}$$

i.e., $\mathbf{V}(t)$ *is now a time-varying matrix whose entries are given by:*

$$\mathbf{V}_{ij}(t) = \begin{cases} \mathbf{V}_{ij}(t-1), & \text{if } i \text{ has not defaulted up to time } t-1. \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

Proof. Suppose the ith economic agent defaults at time t. Due to the upper limit of 1, the stress differentials of i are stacked at zero in the subsequent iterations, i.e., $\Delta\mathbf{s}_i(t+k) = 0, \forall k \in \mathbb{N}_+$.[10]

To prevent increases of economic agent i's stress levels in subsequent iterations, we can zero the ith row of the update matrix $\mathbf{V}$ as Equation (20) shows. Applying this modification, the inner products between the ith row of $\mathbf{V}(t+k)$ and $\Delta\mathbf{s}(t+k)$, $\forall k \in \mathbb{N}_+$, always result in zero. Therefore, $\Delta\mathbf{s}_i(t+k) = 0$ and hence economic agent i does not propagate further stress once in default. ■

[10] Another way to see this is by noting that the curve of $\mathbf{s}_i$ is non-decreasing due to Remark 2. Once i reaches the upper limit of 1, it never leaves that stress level.

We introduce the following theorem that is useful to bound the spectrum of a square matrix.

Theorem 1. ***Geršgorin (1931)'s Circle Theorem***: *Every eigenvalue λ of an N-dimensional square matrix $\mathbf{S}$ satisfies:*

$$|\lambda - \mathbf{S}_{ii}| \leq \sum_{j \neq i} |\mathbf{S}_{ij}|, \tag{21}$$

$\forall i \in \{1, 2, \ldots, N\}$.

Remark 4. *Geršgorin (1931)'s theorem gives estimates to bound the spectrum of a square matrix. In special, it asserts that the eigenvalues of $\mathbf{S}$ must be inside in one or more, possibly overlapping N circles centered at the main diagonal elements $\mathbf{S}_{ii}$, $\forall i \in \{1, 2, \ldots, N\}$, each of which with radius given by the sum of the elements of the respective i-th row, except for the main diagonal element.*

We now link Lemma 1 and Theorem 1 to delineate the phases through which the systemic risk framework passes as the contagion diffusion process between economic agents evolves.

Proposition 2. *The systemic risk framework passes through two distinct phases:*

1. ***Transient phase***: *marked by the presence of defaults, $\mathbf{V}(t)$ evolves to less destabilizing conditions as defaults emerge in the financial system. This phase always terminates and may or may not be present in the dynamics.*

2. ***Persistent phase***: *marked by the absence of defaults, $\mathbf{V}(t)$ is necessarily stable. This phase is always present.*

Proof. We divide the proof into two parts: definition of the transient and the persistent phases.

As defaults occur in the transient phase, some economic agents' stress levels keep increasing until they top at 1 because of the $\min(.)$ operator. Say that the economic agent i defaults at time t_{default}. Thus, the spectrum of $\mathbf{V}(t_{\text{default}} + k), \forall k \in \mathbb{N}_+$, reduces due to Lemma 1. If other defaults occur at later iterations, the spectrum $\mathbf{V}$ once again diminishes until the spectrum of $\mathbf{V}$ eventually reaches the stability zone, in which the magnitude of its largest eigenvalue becomes less than one. When no more defaults occur, the persistent phase begins and $\mathbf{V}(t)$ becomes stable from that moment onwards.

We can apply Geršgorin (1931)'s circle theorem to get a very clear intuition of the spectrum update process. We apply this theorem on the transpose of $\mathbf{V}$ with no loss of generality, since eigenvalues of a matrix are equal to its transpose. We first note that the elements of the main diagonal of $\mathbf{V}^T$ must be zero, because economic agents cannot be vulnerable to themselves. Therefore, the estimates of the eigenvalues of $\mathbf{V}^T$ are centered at circles in the origin of the plane. These circles have different radii according to the partial row sums of $\mathbf{V}^T$. Each partial row sum, which excludes the element in its main diagonal, represents an upper bound for the spectrum of $\mathbf{V}^T$. For stability and asymptotic analyses, we can only keep the largest estimate of the spectrum.

Figure 3 portrays a typical trajectory of the spectrum of $\mathbf{V}^T$. As defaults occur, the spectrum of $\mathbf{V}^T$ reduces leading to a more stabilizing financial system. Eventually, the system enters the persistent phase in which $\mathbf{V}^T$ is necessarily stable. Once it enters this area, its spectrum no longer changes and, as we will see, the dynamical process converges.

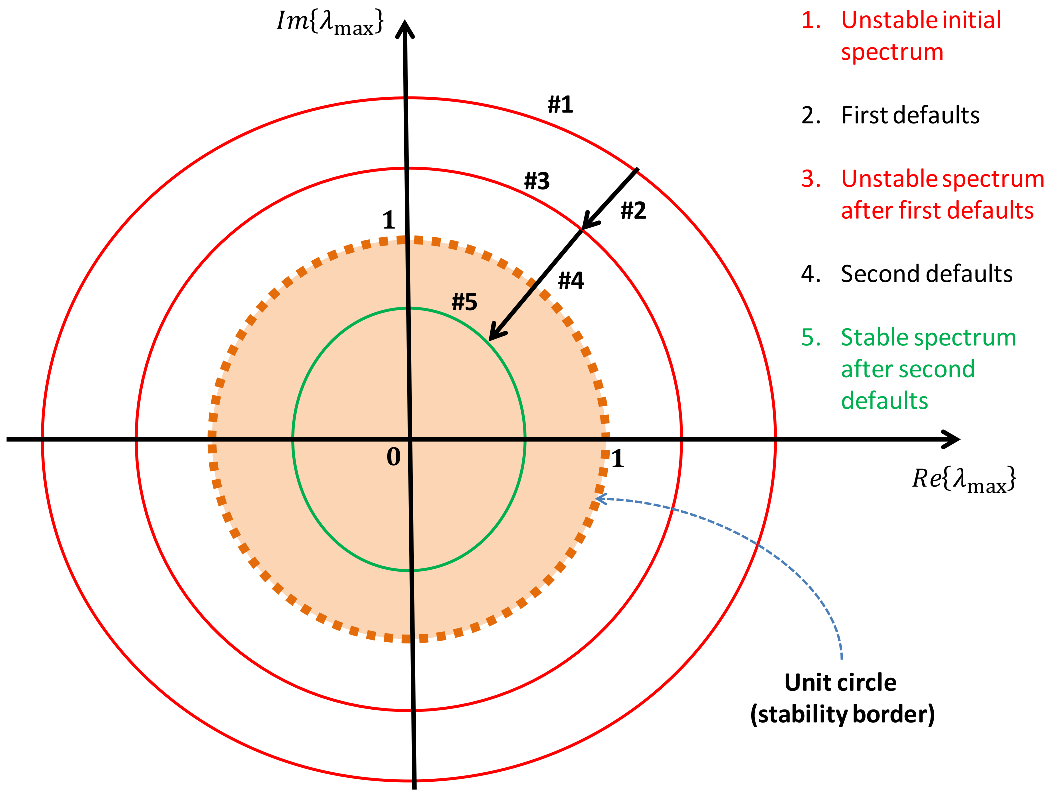


Figure 3: *Evolution of the estimated spectrum of $\mathbf{V}^T(t)$ as defaults of economic agents occur. In the schematic, the orange-shaded area denotes the stability zone, which is within the unit circle. The red spectra denote transient phases, while the green spectrum represents the persistent phase. The y- and the x-axis indicate the imaginary and real part of the largest partial (excluding the main diagonal element) row sum of $\mathbf{V}^T(t)$. As defaults occur, the spectrum reduces until it necessarily reaches the stability zone. At that point, it stays there until convergence to a unique fixed point.*

Observe also that the transient phase is not required in this process: if $\mathbf{V}$ is stable upfront, then the dynamical process immediately enters the persistent phase. ■

As economic agents default, Proposition 2 reveals that there are less potential contagion sources, allowing the financial system to drive itself to less unstable states. The feedback effect slows down the convergence to equilibrium, but does not affect the asymptotic features of the model. We still need to prove that there is unique fixed point in the contagion process. Before showing that, the following Lemma provides an important characteristic of the dynamical system while it is in the persistent phase.

Lemma 2. *Once the system enters the persistent state, the update rule of the dynamical system becomes a contraction mapping $\mathbf{g}: [0,1]^{B+F} \mapsto [0,1]^{B+F}$, in which $\mathbf{g}$ is a vectorial function that maps stress levels from $t-1$ to t.*

Proof. According to Proposition 2, the persistent phase is marked by the absence of defaults. In this way, we can remove the $\min(.)$ operator from the update rule of the dynamical system, because stress levels never reach 1. Using this observation in (19), we get:

$$\Delta \mathbf{s}(t) = \mathbf{V}_{\text{stable}} \Delta \mathbf{s}(t-1), \tag{22}$$

in which $\mathbf{V}_{\text{stable}}$ represents the stable update matrix $\mathbf{V}$ in the persistent phase. As no defaults occur in the phase, the matrix is time-invariant and therefore its spectrum does not change. In this way, we can omit the time index with no loss of generality. In addition, the dynamical system becomes linear.

Given an initial condition $\Delta \mathbf{s}(0)$, we can explicitly compute $\Delta \mathbf{s}(t)$ in terms of $\Delta \mathbf{s}(0)$ as:

$$\Delta \mathbf{s}(t) = \mathbf{V}^t_{\text{stable}} \Delta \mathbf{s}(0). \tag{23}$$

Noting that matrix $\mathbf{V}_{\text{stable}}$ possesses eigenvalues inside the unit circle, then Equation (23) represents a contraction mapping that is performed by the stable update matrix $\mathbf{V}_{\text{stable}}$ on the states $\Delta \mathbf{s}(t)$. ■

We also present Banach (1922)'s fixed-point theorem, also known as the contraction mapping theorem or contraction mapping principle, which is an important tool in the theory of metric spaces. It provides an elegant way to guarantee the existence and uniqueness of fixed points in contraction maps defined in metric spaces, while also providing a constructive method to find those fixed points.

Theorem 2. ***Banach (1922)'s fixed-point theorem**: Let $(\mathscr{X}, d)$ be a non-empty complete metric space with a contraction mapping $g: \mathscr{X} \mapsto \mathscr{X}$ with distance metric d. Then g admits a unique fixed-point $x^* \in \mathscr{X}$, i.e., $g(x^*) = x^*$. Furthermore, x^* can be found as follows: start with an arbitrary element x_0 in $\mathscr{X}$ and define a sequence $\{x_n\}$ by $x_n = g(x_{n-1})$, then $x_n \to x^*$.*

We now show the main result of the theoretical analysis: the convergence of the contagion process to a unique fixed point.

Proposition 3. *Irrespective to the initial conditions, the feedback-based systemic risk in multilayer networks has a unique fixed-point* $\mathbf{s}^*$*:*

$$\mathbf{s}^* = (\mathbf{I} - \mathbf{V}_{\text{stable}})^{-1}\varepsilon, \tag{24}$$

in which $\mathbf{V}_{\text{stable}}$ *and* ε *denote the update matrix and stress levels of economic agents, respectively, when the dynamical system enters the persistent phase.*

Proof. To show the existence of the fixed point, we show that the systemic risk dynamical system enjoys the pre-requisites of the Banach (1922)'s fixed-point theorem. First, the metric space of the map is a continuous line segment in the space $B+F$, i.e., $[0,1]^{B+F}$. Clearly, it is a traditional non-empty complete metric space with a well-defined Euclidean distance metric. In addition, the stable update matrix $\mathbf{V}_{\text{stable}}$ in (23) represents a contraction map due to Lemma 2. Putting together these facts and invoking Banach (1922)'s theorem, we conclude that the dynamical system must have a unique fixed point.

We now algebraically evaluate that fixed point $\mathbf{s}^*$. If ε represents the stress levels of economic agents when the dynamical system enters the persistent phase, then:

$$\mathbf{s}(t) = \mathbf{V}_{\text{stable}}\mathbf{s}(t-1) + \varepsilon, \tag{25}$$

$\forall t > 0$ and $\mathbf{s}(0) = \mathbf{0}$. As Equation (25) has a fixed point $\mathbf{s}^*$, then $\mathbf{s}(t) = \mathbf{s}(t-1) = \mathbf{s}^*$ and therefore:

$$\begin{aligned} & \mathbf{s}^* = \mathbf{V}_{\text{stable}}\mathbf{s}^* + \varepsilon \\ \Rightarrow \quad & (\mathbf{I} - \mathbf{V}_{\text{stable}})\mathbf{s}^* = \varepsilon, \end{aligned} \tag{26}$$

in which $\mathbf{I}$ is the identity matrix. But $\mathbf{V}_{\text{stable}}$ is stable and hence the inverse of $(\mathbf{I} - \mathbf{V}_{\text{stable}})$ exists and corresponds to a convergent geometric series.

The update matrix $\mathbf{V}_{\text{stable}}$ has the main diagonal full of zeroes. In this way, the main diagonal of $(\mathbf{I} - \mathbf{V}_{\text{stable}})$ corresponds to a vector of ones and thus it has full rank. Consequently, $(\mathbf{I} - \mathbf{V}_{\text{stable}})$ is invertible. Therefore, we can compute the equilibrium stress level of the system as:

$$\mathbf{s}^* = (\mathbf{I} - \mathbf{V}_{\text{stable}})^{-1}\varepsilon, \tag{27}$$

which retrieves (24) and the proof is complete.

■

The existence of a unique fixed point brings several interesting characteristics for the model. For instance, supervisory authorities can identify those financial institutions that are more fragile given a feasible and relevant initial shock scenario and therefore take proactive actions. This result is important for the development of financial regulation.

2.4 Systemic risk estimation using stress feedback among general multilayer financial networks

In this section, we provide the general form of framework for multilayer financial networks with arbitrary number of layers and feedback rules. We also show that the well-known DebtRank methodology is a special case of our model when there is a single network layer with no feedback.

2.4.1 Model definition

2.5 Model definition

The following proposition casts the general form of the dynamical system into a standard state-space format.

Proposition 4. *Say that there are $L > 0$ layers in a multilayer network. Suppose there are $N_i > 0$ vertices in the ith layer, $i \in \{1, \ldots, L\}$. The general formulation of the feedback-based systemic risk in multilayer networks can be cast into the following dynamical system:*

$$\Delta\mathbf{s}(t) = \min(1, \mathbf{V}\Delta\mathbf{s}(t-1)), \tag{28}$$

in which the state $\Delta\mathbf{s}(t) \in [0,1]^{N_1+N_2+\ldots+N_L}$ is given by:

$$\Delta\mathbf{s}(t) = \begin{pmatrix} \Delta\mathbf{s}_1(t) \\ \Delta\mathbf{s}_2(t) \\ \vdots \\ \Delta\mathbf{s}_L(t) \end{pmatrix} = \begin{pmatrix} \mathbf{s}_1(t) - \mathbf{s}_1(t-1) \\ \mathbf{s}_2(t) - \mathbf{s}_2(t-1) \\ \vdots \\ \mathbf{s}_L(t) - \mathbf{s}_L(t-1) \end{pmatrix}, \tag{29}$$

in which $\mathbf{s}_i(t)$ *is a vector with dimensions* $N_i \times 1$ *that symbolizes the stress levels of the economic agents placed at the ith layer,* $i \in \{1, \ldots, L\}$*. Thus,* $\Delta\mathbf{s}(t)$ *is a column vector with* $N_1 + N_2 + \ldots + N_L$ *entries.*

The update matrix of the system, $\mathbf{V}$*, is:*

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}^{(1\to 1)} & \mathbf{V}^{(1\to 2)} & \ldots & \mathbf{V}^{(1\to L)} \\ \mathbf{V}^{(2\to 1)} & \mathbf{V}^{(2\to 2)} & \ldots & \mathbf{V}^{(2\to L)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{V}^{(L\to 1)} & \mathbf{V}^{(L\to 2)} & \ldots & \mathbf{V}^{(L\to L)} \end{pmatrix}, \tag{30}$$

in which $\mathbf{V}^{(i\to j)}$ *is a matrix with dimensions* $N_i \times N_j$ *that represents a suitable vulnerability matrix that propagates stress from layer i to layer j,* $i, j \in \{1, \ldots, L\}$*. If* $i = j$*, then the vulnerability matrix propagates intralayer stress. Otherwise, it propagates interlayer stress. Note that* $\mathbf{V}$ *has dimensions* $(N_1 + N_2 + \ldots + N_L) \times (N_1 + N_2 + \ldots + N_L)$*.*

The main challenge in the feedback-based systemic risk for multilayer networks is in defining the vulnerability matrices of the dynamical process. The vulnerability matrices that model the stress diffusion inside a layer, such as $\mathbf{V}^{(\text{bank}-\text{bank})}$ in (1) and $\mathbf{V}^{(\text{firm}-\text{firm})}$ in (2), follow the same construction pattern, which reduces their design complexity. However, vulnerability matrices that model the stress diffusion process between layers in a bidirectional manner, such as $\mathbf{V}^{(\text{bank}-\text{firm})}$ in (1) and $\mathbf{V}^{(\text{firm}-\text{bank})}$ in (2), are difficult to design.

Remark 5. *We can design flexible feedback mechanisms between layers. Suppose we want:*

- *a bidirectional feedback between layers i and j, then we design* $\mathbf{V}^{(i\to j)}$ *and* $\mathbf{V}^{(j\to i)}$*, such that* $\mathbf{V}^{(i\to j)} \neq \mathbf{V}^{(j\to i)}$ *in the general case.*
- *a unidirectional "feedback" from layers i to j, then we design* $\mathbf{V}^{(i\to j)}$ *and set* $\mathbf{V}^{(j\to i)} = \mathbf{0}$*. The classical contagion models in the literature fall in this category.*
- *no feedback between layers i and j, then we set* $\mathbf{V}^{(i\to j)} = \mathbf{V}^{(j\to i)} = \mathbf{0}$*.*

2.5.1 Theoretical analysis

Proposition 5. *Suppose that the system achieves convergence after a sufficiently large number of iterations* $t_{\text{converge}} \gg 1$. *We compute the systemic risk or additional stress due to an initial shock scenario* $\mathbf{s}(0)$ *as:*

$$\mathrm{SR}(\mathbf{s}(0)) = \sum_{l=1}^{L} \sum_{t=1}^{t_{\text{converge}}} (\Delta \mathbf{s}_l(t))^T \, v_l, \tag{31}$$

in which $\mathbf{s}_l(t)$ *and* v_l *indicate the column vectors that carry the stress levels and the economic importance, respectively, of the* N_l *economic agents in the* l*th layer.*

Proof. We start by using the additional stress formula that our systemic risk methodology computes:

$$\mathrm{SR}(\mathbf{s}(0)) = \sum_{l=1}^{L} (\mathbf{s}_l(T) - \mathbf{s}_l(0))^T \, v_l, \tag{32}$$

We can evaluate $\mathbf{s}_l(t)$, $t \in \{0, 1, \ldots, t_{\text{converge}}\}$ from the dynamical system as follows:

$$\mathbf{s}_l(t) = \mathbf{s}_l(0) + \sum_{k=1}^{t_{\text{converge}}} \Delta \mathbf{s}_l(k), \tag{33}$$

in which $\Delta \mathbf{s}_l(k) = \mathbf{s}_l(k) - \mathbf{s}_l(k-1)$, as Equation (29) shows.

Substituting (33) into (32), we get:

$$\begin{aligned}
\mathrm{SR}(\mathbf{s}(0)) &= \sum_{l=1}^{L} \left(\mathbf{s}_l(0) + \sum_{t=1}^{t_{\text{converge}}} \Delta \mathbf{s}_l(t) - \mathbf{s}_l(0) \right)^T v_l, \\
&= \sum_{l=1}^{L} \sum_{t=1}^{t_{\text{converge}}} (\Delta \mathbf{s}_l(t))^T \, v_l,
\end{aligned} \tag{34}$$

which retrieves (31) and the proof is complete. ■

Proposition 6. *The dynamical system in Proposition 4 always converges to a unique fixed point, irrespective to the initial conditions.*

Proof. The mathematical apparatus used to prove the existence and uniqueness of the fixed point for a two-layered network can be extended to the general case by simply changing the state $\Delta \mathbf{s}(t)$ and update matrix $\mathbf{V}$ to those as defined in (29) and (30). ■

2.5.2 DebtRank is a special case of the proposed model

DebtRank is a well-known measurement that also evaluates additional stress inside a financial network.[11] In the next proposition, we show that the current state-of-the-art DebtRank formulation (Bardoscia et al. [2015]) is a special case of our general feedback-based systemic risk measure in multilayer networks when there is a single network layer (and hence with no feedback).

Proposition 7. *The feedback-based systemic risk model reduces to Bardoscia et al. (2015)'s DebtRank formulation when there is a single network layer composed of banks.*

Proof. Since the computation of the systemic risk of Bardoscia et al. (2015)'s DebtRank also use additional stress, we only need to check whether the stress levels of their procedure match ours. We can do that by verifying the update matrices of both dynamical processes.

By comparing Bardoscia et al. (2015)'s update rule with our general formulation in Proposition 4, it is clear that the update matrices of both processes coincide when there is a single layer composed of banks. In this case, the update matrix reduces to $\mathbf{V} = \mathbf{V}^{(\text{bank}-\text{bank})}$. ∎

3 Application: Brazilian bank-firm and bank-bank networks

In this section, we apply our feedback-based systemic risk model for the Brazilian bank-firm and bank-bank networks. We compare our systemic risk estimates against other stress network measures, such as the DebtRank by Battiston et al. (2012b) and the differential DebtRank by Bardoscia et al. (2015), that work in the univariate interbank network. We compare our method to the DebtRank formulations to elucidate the major role that the bank-firm contagion transmission channel plays in the overall stress diffusion process. In this regard, we show that systemic risk estimates largely increase when we incorporate the real sector contagion channel in addition to the classical interbank channel. We also study which economic sectors are more risky for banks by designing sectorial initial shock scenarios. Furthermore, we investigate the factors that increase or decrease sector riskiness to banks. Finally, we show how important the contribution of the feedback mechanism

[11] The literature provides more than one definition of DebtRank, each of which with improvements over the previous versions. The first version of DebtRank is due to Battiston et al. (2012b). This methodology can greatly underestimate the stress in the financial system, as it blocks second- and high-order rounds of impact diffusion coming from network cycles. Bardoscia et al. (2015) deal with this problem by introducing a modified version of the DebtRank, in which banks are allowed to recursively diffuse stress increments and not their current stress levels at each iteration. This is the current state-of-the-art DebtRank methodology. We focus on comparing our methodology to Bardoscia et al. (2015)'s DebtRank version.

(financial accelerator) is to estimating systemic risk. For that, we compare systemic risk levels achieved by economic agents when we disable and enable the feedback mechanism. We show that the financial accelerator plays a major role in increasing the overall systemic risk.

3.1 Data

In this work, we use unique Brazilian databases with supervisory and accounting data.[12] We extract quarterly information from March 2012 through June 2015. In the next sections, we discuss how we build the bank-bank (interbank) and bank-firm networks. Due to data unavailability, we consider the network topology of the firm-firm network as an empty graph and do not account for deposits that firms hold in custody of banks.

3.1.1 Bank-bank (interbank) network

Following de Souza et al. (2015), we consider banks' loss absorbing capabilities as the parcel of the total capital (Tier 1 + Tier 2) that exceeds 8% of their risk-weighted assets. In Brazil, the capital requirement is 13% or 15% for specific types of credit unions and 11% for other financial institutions, including banks. Most financial institutions hold positive capital buffers (their regulatory capital exceeds the requirement). Financial institutions that are not compliant with this requirement are warned by the Supervision and must present a plan to recover compliance in a given period. If the plan is not credible or not feasible, the Authority intervenes. We set 8% RWA as a reference for the computation of capital buffers as we assume that if a financial institution holds less than what is recommended by the Basel Committee on Banking Supervision, i.e., 8% of its RWA, it will take longer to raise its capital to an adequate level and will likely suffer an intervention.

Although exposures among financial institutions may be related to operations in the credit, capital and foreign exchange markets, here we focus solely on unsecured debt operations of equal seniority in the money market. The money market comprises financial operations on private securities that are registered by the Cetip:[13] interfinancial deposits, debentures and repurchase agreements collateralized by debentures issued by

[12] The collection and manipulation of the data were conducted exclusively by the staff of the Central Bank of Brazil.

[13] Cetip is a depositary of mainly private fixed income, state and city public securities and other securities representing National Treasury debts. As a central securities depositary, Cetip processes the issue, redemption and custody of securities, as well as, when applicable, the payment of interest and other events related to them. The institutions eligible to participate in Cetip include commercial banks, multiple banks, savings banks, investment banks, development banks, brokerage companies, securities distribution companies, goods and future contracts brokerage companies, leasing companies, institutional investors, non-financial companies (including investment funds and private pension companies) and foreign investors.

leasing companies of the same financial conglomerate.[14] In this work, we term the last financial instrument as “repo issued by the borrower financial conglomerate.”

We use exposures among financial conglomerates and individual financial institutions that do not belong to a conglomerate. Intra-conglomerate exposures are not considered. In our sample, we only account for commercial banks, investment banks, savings banks and development banks. We classify banks according to their sizes using a simplified version of the size categories defined by the Central Bank of Brazil in the Financial Stability Report published in the second semester of 2012 (see BCB (2012)), as follows:[15] 1) we group together the micro, small, and medium banks into the “non-large” category, and 2) the official large category is maintained as is in our simplified version. Therefore, instead of four segments representing the bank sizes, we only employ two.

For each pairwise exposure between financial conglomerates or individual institutions, we remove the share that is guaranteed by the Brazilian Credit Guarantee Fund (FGC).[16] All of the financial instruments that we are using are covered by the FGC. Until May 2013, the FGC guarantees up to R$70 thousand for each deposit holder against each registered institution. After that date, due to Resolution 4222 published by the National Monetary Council, that amount increased to R$250 thousand. Say that the liability of the financial institution p to q at time t is $\mathbf{L}_{pq}(t)$. Then we adjust that liability to $\max\left[0, \mathbf{L}_{pq}(t) - \mathrm{FGC}(t)\right]$, in which:

$$\mathrm{FGC}(t) = \begin{cases} 70,000, & \text{if } t < \mathrm{May}/2013, \\ 250,000, & \text{otherwise.} \end{cases} \tag{35}$$

In order to compute our feedback-based systemic risk measure in multilayer network and also the DebtRank in its different formulations, we need a proxy for the banks' economic values. For that end, we gauge the economic value of the i-th bank, v_i, by the share of its liabilities to the total liabilities in the network, that is:

[14] Recall that repurchase agreements are technically secured operations. However, since the borrower in this type of repo guarantees the operation using collateral of a leasing company of the same financial conglomerate, the collateral bears the same credit risk of the borrower financial conglomerate. Thus, in practical terms, the financial operation turns out to be unsecured.

[15] The Financial Stability Report ranks financial institutions according to their positions in a descending list ordered by their total assets. The Report builds a cumulative distribution function (CDF) on the these total assets and classifies them depending on the region that they fall in the CDF. It considers as large financial institutions that fall in the 0% to 75% region. Similarly, medium-sized financial institutions fall in the 75% to 90% category, small-sized, in the 90% to 99% mark, and those above are micro-sized.

[16] The Credit Guarantee Fund, whose legal establishment is authorized by the Resolution 2197 emitted by the National Monetary Council, is a private institution responsible for the protection of checking/saving account holders and investors against registered financial institutions in case of intervention, liquidation or bankruptcy.

$$v_i = \frac{\sum_{j \in \mathscr{B}} \mathbf{A}_{ji}^{(\text{bank}-\text{bank})}}{\sum_{j,p \in \mathscr{B}} \mathbf{A}_{jp}^{(\text{bank}-\text{bank})}} \tag{36}$$

$\forall i \in \mathscr{B}$. In this way, $v_i \in [0,1]$ and $\sum_{j \in \mathscr{B}} v_j = 1$. Consequently, our systemic risk measure and DebtRank assumes values in the interval $[0,1]$. We can convert these indices to potential losses by simply multiplying them to the total liabilities in the network. Our option for the economic value proxy contrasts with that of Battiston et al. (2012b), who define the relative economic value of the i-th bank as its share of assets to the total assets in the network. However, we use the liabilities share because, once a banks default, the losses that other members in the network have correspond to the liabilities of that defaulted bank towards them.

3.1.2 Bank-firm network

In order to build up the bank-firm network, we first adopt a criterion to set the sample of firms. We only employ companies whose shares are traded at the Brazilian stock exchange (BOVESPA)[17] as of September 29th, 2015. Following de Castro Miranda and Tabak (2013), we choose such sample delimitation so that we are able to extract from Economatica a rich set of information, such as registration data, financial statements, and financial indicators, from these firms. We use a slight variation of the sector classification that Economatica employs. Table 1 reports the original economic sectors of Economatica. For the sake of clarity, we group some of these sectors so that our results can be better interpreted. Table 2 shows the macrosectors that we build on the top of the economic sectors of Economatica. In this clustering process, we only join sectors that have relative low feedback-based systemic risk levels, in a way that our results are not compromised. In addition, we remove the single firm representing the funds sector in Table 1 due to data inconsistency. In this process, we end up with 12 sectors.

For each of these firms registered at Economatica, we compute the loans from each financial institution in the interbank market to each firm from March 2012 through June 2015 on a quarterly basis. We extract these data from the Central Bank of Brazil's Credit Risk Bureau System (SCR)[18]. Again we consider just loans from financial conglomerates

[17]BOVESPA (São Paulo Stock Exchange), the main Brazilian stock exchange, manages the organized securities and derivatives markets, providing registration, clearing and settlement services, acting as central counterpart. The Exchange offers a wide range of products and services such as spot FX, equities and fixed-income securities trading, as well as trading in derivatives contracts based among other things on equities, financial securities, indices, rates, commodities and currencies. It lists companies and other issuers, is a securities depository, has a securities lending service and licenses software.

[18]SCR is a very thorough data set which records every single credit operation within the Brazilian financial system worth R$1,000 or above. It brings, on each operation, data as financial institution and client identification, amount, type of loan, interest rate and risk classification.

Table 1: *Firm composition by sectors.*

Economic sector	Number of firms
Electric Power	45
Finance and Insurance	37
Primary and Fabricated Metal	22
Transportation Service	20
Other	99
Agriculture and Fisheries	5
Textile	25
Food and Beverage	17
Construction	22
Trade	19
Industrial Machine	5
Electric Electron	7
Vehicle and Parts	16
Chemical	11
Telecommunication	12
Mining	6
Oil and Gas	8
Pulp and Paper	5
Nonmetallic Mineral	4
Software and Data	5
Funds	1
Total	**391**

Table 2: *Definition of macrosectors in terms of Economatica sectors.*

Macrosector	Sector Economatica	Number of firms
Industrial Machines and Goods	Electric Electron Industrial Machine Textile Vehicle and Parts	53
Metal Extraction and Processing	Mining Nonmetallic Mineral Primary and Fabricated Metal	32
Technology	Software and Data Telecommunication	17
Tertiary Sector	Finance and Insurance Trade Transportation Service	76

or independent financial institutions that do not belong to a conglomerate.

Combining the Economatica and SCR data sets, we calculate the total loans to asset ratio to each firm in a given quarter. This information is a proxy for how important financial institutions are in financing firms, i.e., a measure of the bank dependency of firms. Figure 4 depicts the bank dependency for each sector. Overall, we see that firms in our sample are not strongly dependent on bank financing, as this dependency for all sectors remains below the 25% mark. However, in addition to being small, we must observe that our sample is compound only by companies with shares traded at the stock market. Thus, those firms that cannot rely on this source of funding—and, hence, that are more bank dependent—are not being considered.

Nonetheless, in relative terms, we see that firms in the agriculture and fisheries sector are the most dependent on bank financing. Followed by that, we also verify that firms in the metal extraction and processing, pulp and paper, and food and beverage sectors are also relatively dependent on bank financing. Contrasting to that, firms in the construction, electric power, and technology sectors are practically independent of bank financing. Thus, firms in these sectors obtain financing mainly from other sources, such as of shareholders, bondholders, or past profits. Sectors with low bank dependency are less prone to bank stress and hence are more sterilized against shocks in the interbank market.

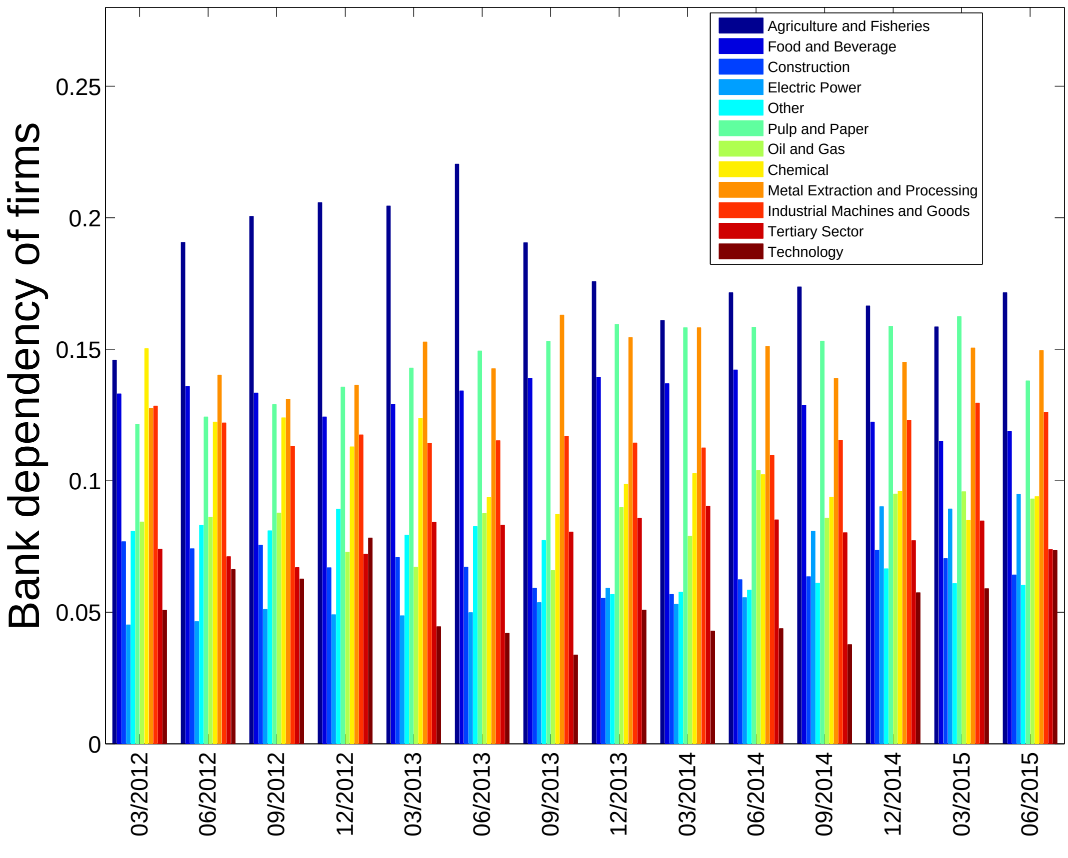


Figure 4: *Bank dependency of the economic sectors. We illustrate the evolution of these sectors on a quarterly basis according to the data gathered in Economatica.*

In order to compute our feedback-based systemic risk in multilayer networks and DebtRank of firms, we need a proxy for their economic value. We define the economic value of a firm k, v_k, as its total assets over the total assets of all of the firms in our sample, i.e.,

$$v_k = \frac{\mathbf{TA}_k}{\sum_{u \in \mathscr{F}} \mathbf{TA}_u}, \tag{37}$$

$\forall k \in \mathscr{F}$. $\mathbf{TA}_k$ stands for the total assets of firm k. In this way, $v_k \in [0,1]$ and $\sum_{u \in \mathscr{F}} v_u = 1$. Again, we can transform the results in potential losses by simply multiplying our coefficients by the total assets of firms.

3.2 How contributive is the real sector to the financial sector in terms of systemic risk?

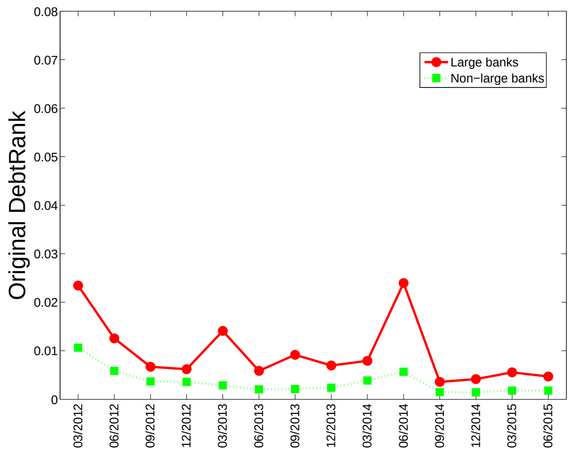
In this section, we evaluate the additional stress that firms and banks incur when we default one bank at a time. We are interested in understanding the role that the real sector plays in increasing systemic risk to the financial sector. Therefore, we compare how our feedback-based systemic risk measure, which incorporates contagion channels of the financial and the real sectors, against different DebtRank formulations, which only account for the financial sector. We attribute the systemic risk level gap between these approaches to the real sector. With regard to DebtRank, we use its original version by Battiston et al. (2012b) and the improved version by Bardoscia et al. (2015) that we term as differential DebtRank, which considers network cycles and multiple routes.

Figures 5a and 5b portray the average original and differential DebtRank, respectively. Figure 5c shows the average feedback-based systemic risk measure when we account for both the bank and firm contagion channels. We discriminate the results by bank sizes. One first perceptive characteristic is that large banks assume the largest systemic risk levels for the three approaches throughout the entire studied period. This fact happens because they are more interconnected and intermediate more financial operations by virtue of being members of the network core.[19] Using the DebtRank methodology, similar empirical studies using data from other countries also report a positive relationship between bank size and systemic risk (Aoyama et al. (2013); Battiston et al. (2013, 2012a)). However, size does not play such a key role in determining bank's systemic risk. Nonlinear relationships and different systemic risk levels for banks of same size suggest that size is not the sole determinant of systemic risk. Interconnectedness within the financial sector and with the real sector plays a very important role.

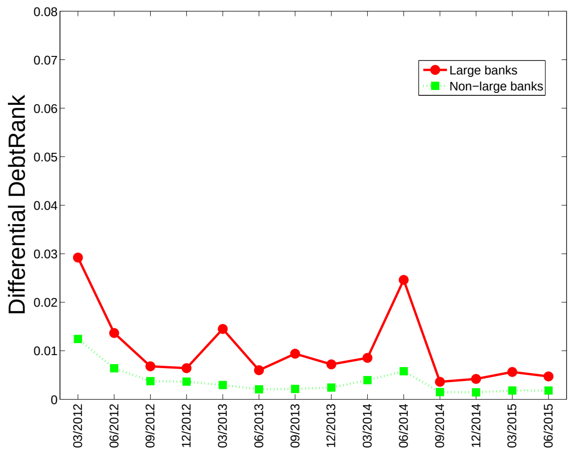
Comparing the results of the three approaches, we first see that the original DebtRank serves as lower bound for the differential DebtRank,[20] which in turn establishes

[19] Silva et al. (2016) shows that the Brazilian interbank network has a core-periphery structure in which the network core is mostly composed of large banks.

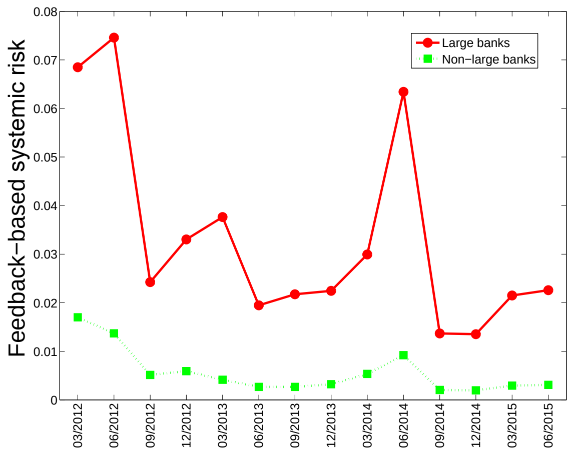
[20] The differential DebtRank is an extension of the original DebtRank that accounts for cycles and multiple routes in the vulnerability network. Therefore, it cannot be smaller than the original DebtRank by



***(a)** Original DebtRank (bank-bank)*



***(b)** Differential DebtRank (bank-bank)*



***(c)** Feedback-based systemic risk measure (bank-bank and bank-firm)*

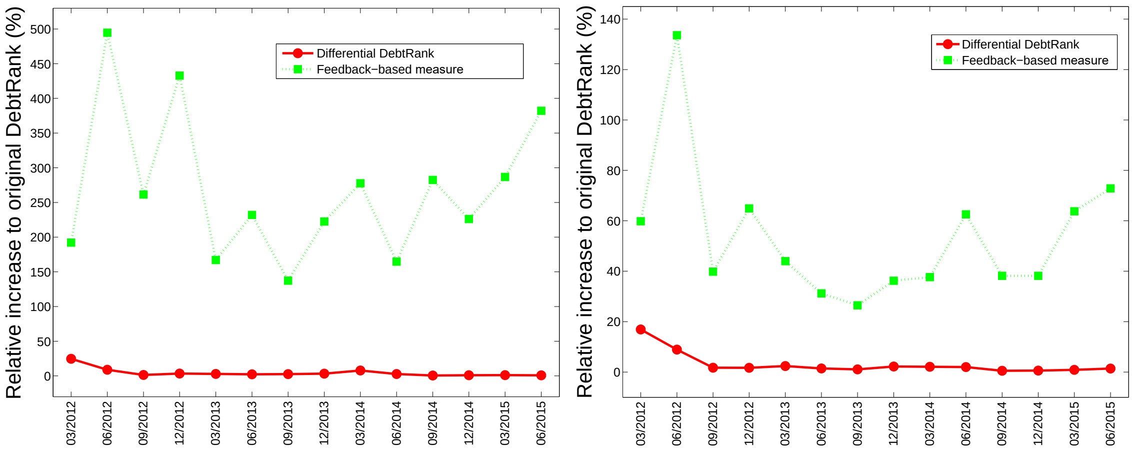
***Figure 5:** Comparison of the original and differential DebtRank methodologies and the feedback-based systemic risk measure with a financial accelerator engine between banks and firms. The initial stress scenarios consist in defaulting a single bank at a time. Each point in the trajectories correspond to average values discriminated by bank sizes.*

a lower bound for our feedback-based systemic risk measure. We plot in Figs. 6a and 6b the relative increase of the average differential DebtRank and our feedback-based systemic risk measure of large and non-large banks, respectively, in relation to the original DebtRank formulation. We verify that the differential DebtRank assumes values that are barely 20% higher in 2012 than those of the original DebtRank. After 2012, it keeps oscillating around the $[3,6]\%$ mark. Applying a right-sided Wilcoxon signed rank test on the differences of the differential and the original DebtRank for large and non-large institutions, we reject the null hypothesis that the medians of the two curves are identical at the 1% significance level.[21]

We now turn our attention to the results of our feedback-based systemic risk mea-

construction.

[21]For robustness, we also apply a right-sided paired-sample t-test and we also reject the null hypothesis that the means of the two curves are identical at the 1% significance level.



(a) Large banks *(b) Non-large*

Figure 6: *Relative increase of the average differential DebtRank and the feedback-based systemic risk values to the average original DebtRank formulation. The initial shock scenarios are identical to those of Fig. 5.*

sure that uses the financial accelerator engine to incorporate bank and firm contagion channels. Looking again at Figs. 6a and 6b, we see the feedback-based systemic risk measures supplies estimates for the additional stress that reach up to 500% of the original DebtRank value for large banks in June 2012. The average underestimation for large banks in the entire sample period in relation to the original DebtRank is 266.66% and for non-large banks is 52.82%. We see that the underestimation of the original and differential DebtRank formulations is more significant for large banks than for non-large banks. This is because large banks are strongly connected to several firms, which are financed by these banks. Each of these firms is also connected to a small quantity of other banks. When we default a large bank, all of these firms get distressed as a consequence of the credit reduction of that distressed bank. Due to the financial accelerator engine, that effect bounces back to the interbank network through banks connected to those newly distressed firms. These new distressed banks now propagate stress to the interbank lending network and also to other connected firms through further credit reductions. In contrast, non-large banks often are connected to a limited number of firms. Thus, the feedback effect on non-large banks is reduced to a large extent.

The large observed differences in the feedback-based systemic risk measure, which uses more than one contagion channel, and the DebtRank formulations, which employ a single contagion channel, corroborate Glasserman and Young (2015)'s view on the underestimation of systemic risk when we consider only single contagion channels. They show that it is relatively difficult to generate contagion solely through spillover losses using only univariate networks, such as the interbank or payments networks in isolation. They argue that additional channels, aside from pure spillover effects, are needed to generate substantial losses from contagion. Here, we verify that, by adding a new contagion channel through the bivariate bank-firm network that intercommunicates with the bank-bank network through bidirectional feedback mechanisms, the additional stress can largely increase. Thus, it is essential to consider other contagion channels if we really want to conduct robust risk analysis in financial networks.

3.3 Which sectors are more risky for financial institutions?

In this section, we analyze how different sectors inflict less or more distress in the financial system directly through the bivariate bank-firm network and indirectly through the bank-bank network. Those sectors that cause more distress in the financial system are considered to be more risky for financial institutions to maintain active financial operations.

3.3.1 Global view

In this section, we assume sectorial defaults as initial shock scenarios, i.e., the default of all firms of the same sector. We now focus on our feedback-based systemic risk measure because the original and differential DebtRank version cannot accommodate for more than one single contagion channels. The terms "feedback-based systemic risk of banks" and "sector riskiness" are employed interchangeably in this and the following sections.

Figure 7 shows the average sector riskiness to banks when we default each of the economic sectors at a time (see Tables 1 and 2). We see that the most harmful sectors for banking institutions significantly change over years. In this way, we rank each of these sectors according to their average yearly positions and report the results in Table 3.

We see that the metal extraction and processing sector stays as the most risky sector for banks in years 2012, 2014, and 2015. In 2013, the most risky sector turns out to be the tertiary sector. Consistently, the food and beverage sector remains at the second position throughout the analyzed period. The tertiary sectors also figures as one of the top 3 most risky sectors for financial institutions. The oil and gas and pulp and paper sectors seem to lose riskiness from 2012 to 2015, while the construction sector assumes more risky positions in that same period. On the other extreme, we see that the technology, agriculture and fisheries, and electric power are the least risky sectors to banks, in that order, from 2012 and 2015.

The number of firms in a sector is not a factor that determines the riskiness of that sector to the financial system. For instance, we see the two most risky sectors, metal extraction and processing and food and beverage sectors, have only 32 and 17 firms, respectively. In turn, the tertiary sector has 76 firms and remains mostly in the third position.

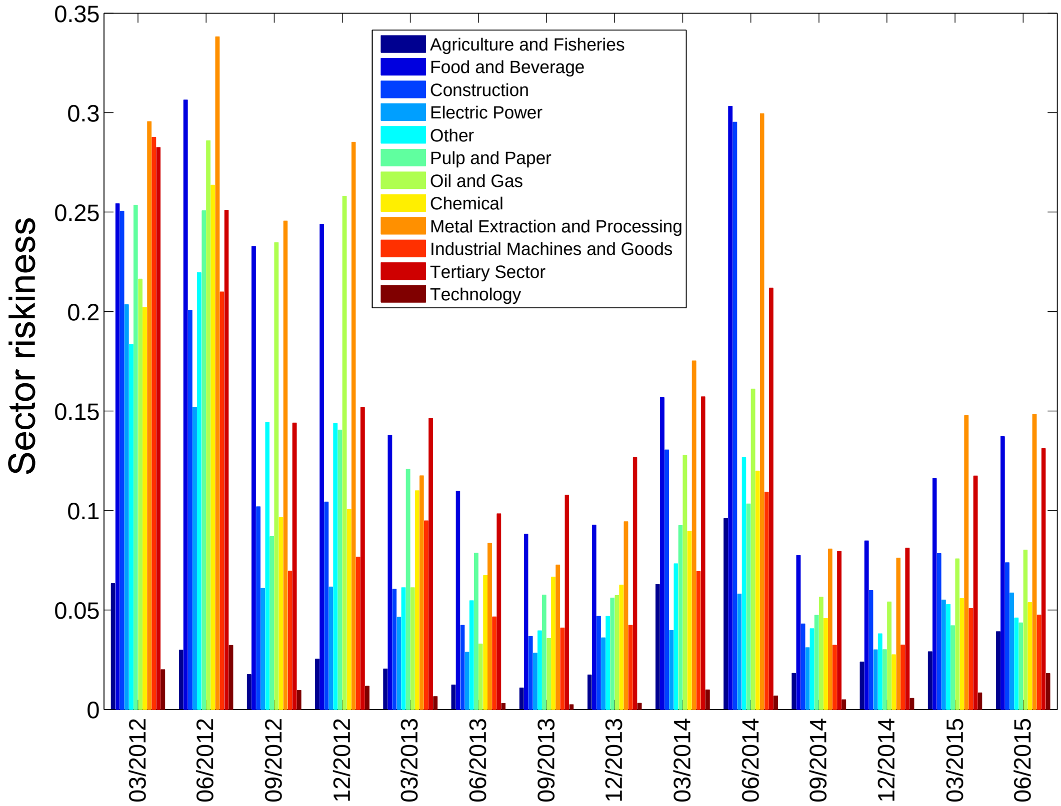


Figure 7: *Sector riskiness to banks (feedback-based systemic risk measure for banks) when we default all of the firms of the same sector.*

In addition, the "other" sector has 99 firms and still occupies modest rank positions in Table 3. The same occurs with the industrial machines and goods (53 firms) and electric power (45 firms).

Another point we tackle in the next sections is that sector riskiness is not necessarily associated to the total loans firms of the same sector receive from banks. This observation implies that the bank-bank and bank-firm networks may attenuate or amplify shocks in different ways, depending on the current network topology. Thus, understanding the "network effect" due to the bank-bank and bank-firm topologies is of utter importance if one wants to really understand the factors that increase riskiness in a financial system. We elaborate more on that in Section 3.4.

3.3.2 Bank sensitiveness to the real sector segregated by bank control types

Now we analyze how risky sectors are to banks by decomposing the bank feedback-based systemic risk due to sector defaults in terms of bank control types: government-owned, domestic private, or foreign private.[22]

Figures 8a and 8b depict the sector riskiness of large and non-large government-owned banks, respectively. We can see that large government-owned banks contribute

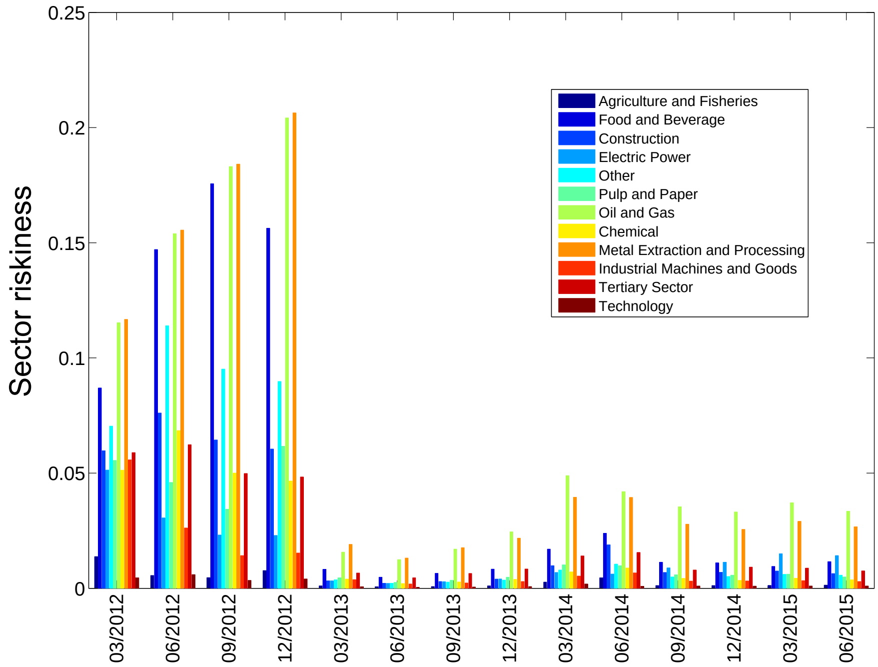
[22]Foreign private banks include those in which there is foreign control or participation.

Table 3: *Sector ranking of how harmful sectors are in terms of contributing to systemic risk in the Brazilian financial system. We provide sector rankings per each year.*

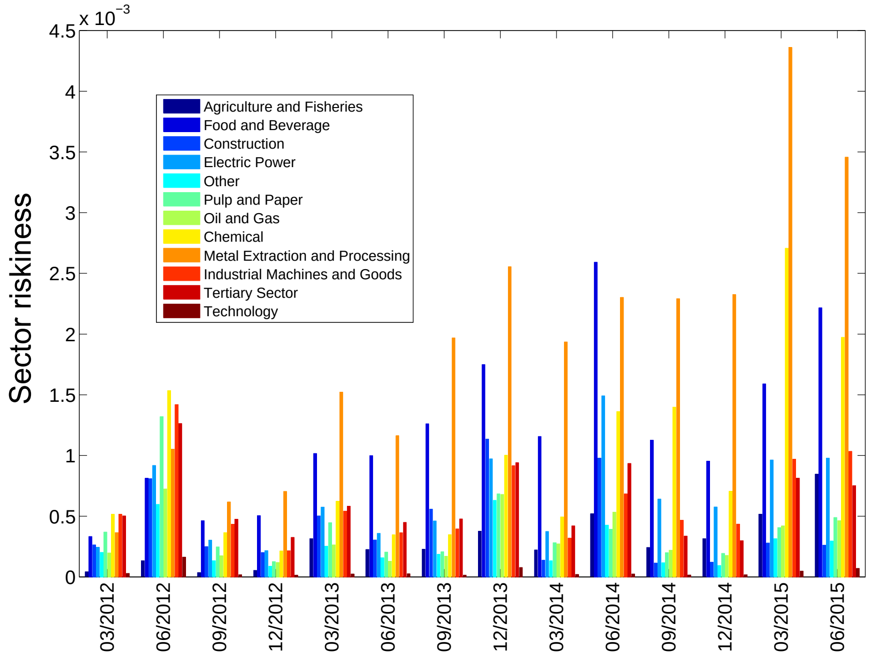
Rank	**Year 2012**	**Year 2013**	**Year 2014**	**Year 2015**
1	Metal Extraction & Processing	Tertiary Sector	Metal Extraction & Processing	Metal Extraction & Processing
2	Food & Beverage	Food & Beverage	Food & Beverage	Food & Beverage
3	Oil & Gas	Metal Extraction & Processing	Tertiary Sector	Tertiary Sector
4	Tertiary Sector	Pulp & Paper	Construction	Construction
5	Pulp & Paper	Chemical	Oil & Gas	Oil & Gas
6	Other	Other	Other	Electric Power
7	Construction	Industrial Machines & Goods	Pulp & Paper	Chemical
8	Chemical	Oil & Gas	Chemical	Other
9	Industrial Machines & Goods	Construction	Industrial Machines & Goods	Industrial Machines & Goods
10	Electric Power	Electric Power	Electric Power	Pulp & Paper
11	Agriculture & Fisheries	Agriculture & Fisheries	Agriculture & Fisheries	Agriculture & Fisheries
12	Technology	Technology	Technology	Technology

more to increasing the overall sector riskiness, specially in 2012. After that year, large and non-large government-owned banks present similar feedback-based systemic risk values. We also see that the metal extraction and processing sector is one of the riskiest sectors for both large and non-large government-owned banks. In contrast, the oil and gas sector is riskier for large than for non-large government-owned banks, suggesting that firms of the oil and gas sector take more loans from large government-owned banks. The food and beverage sector is also moderately risky for both large and non-large government-owned banks. Firms of the chemical sectors are risky for both large and non-large government-owned banks in 2012. After that period, however, the chemical sector riskiness for large government-owned bank decreases, while it increases for non-large government-owned banks. This fact suggests that firms of the chemical sectors seem to migrate their creditors to non-large government-owned banks, thus concentrating more their financing portfolios. Firms of the tertiary sectors are not risky for government-owned banks, meaning that these banks do not have large exposures to these firms.

Figures 9a and 9b display the sector riskiness of large and non-large domestic private banks, respectively. The picture here contrasts with that of government-owned banks: non-large domestic private banks contribute more to increasing sector riskiness than large domestic private banks. In general, we see that the most risky sector for non-large domestic private banks changes over the analyzed period: in 2012, it is the pulp and paper sector; in 2013, the chemical sector; in 2013, the construction sector; and in 2014, the tertiary sector. Nonetheless, the tertiary sector consistently remains as the second most risky sector for non-large domestic private banks from 2012 to 2014. The metal extraction and processing sector also presents moderate risk for these kinds of banks in the analyzed period. In contrast, for large domestic private banks, the food and beverage sector stands as the most risky sector from 2012 to 2015. Though with small risk levels in 2012 and 2013, the oil and gas sector becomes a risky sector to large domestic private banks from 2014 onwards. This fact suggests that these banks increased their loans to these firms in



***(a)** Large government-owned banks*

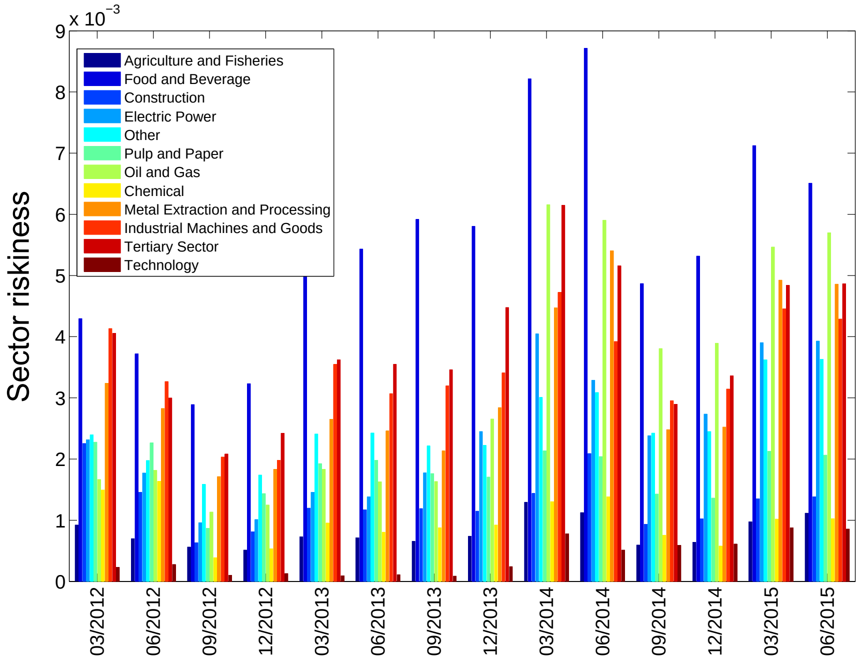


***(b)** Non-large government-owned banks*

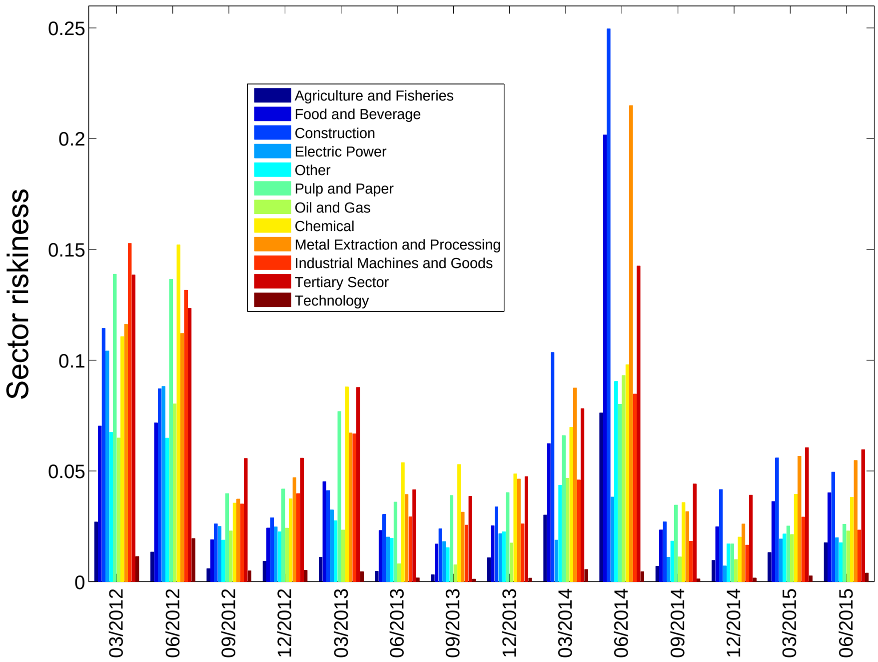
***Figure 8:** Sector riskiness to government-owned banks discriminated by size when we take as initial shock scenario the default of all of firms of a same sector.*

those years. The tertiary sector again stands as a risky sector for large domestic private banks.

Figures 10a and 10b display the sector riskiness of large and non-large foreign private banks, respectively. We see that non-large foreign private banks contribute more to



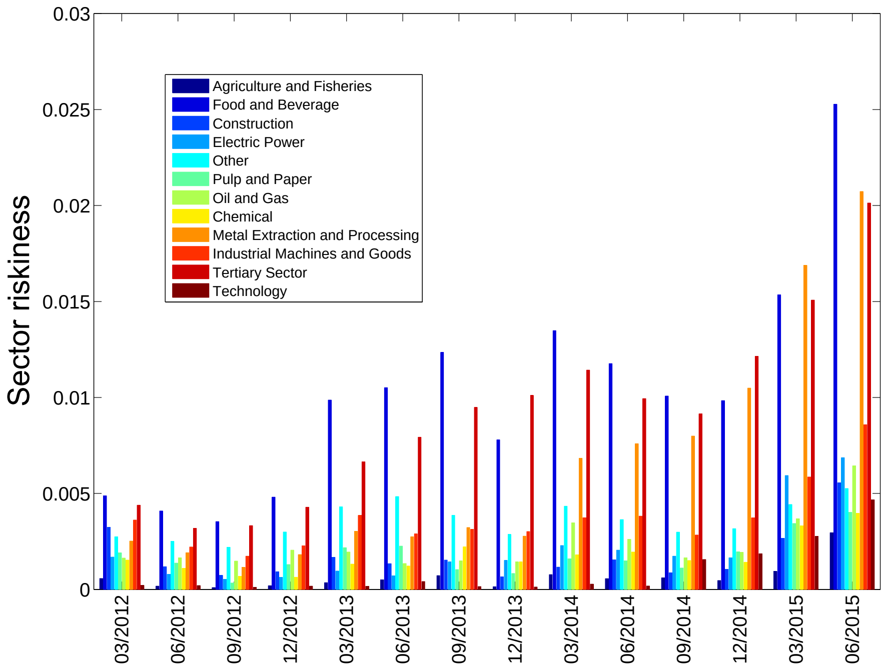
***(a)** Large domestic private banks*



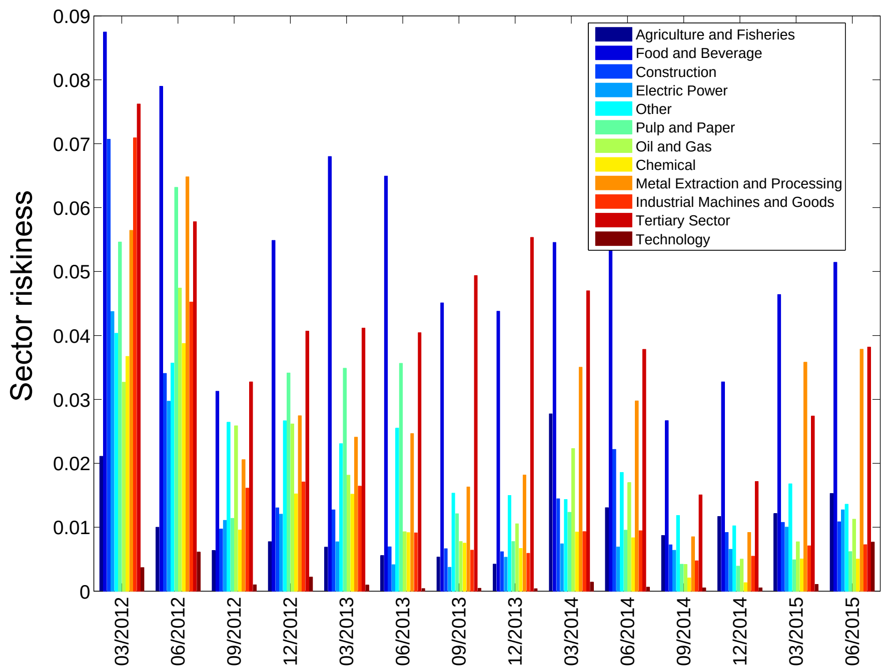
***(b)** Non-large domestic private banks*

***Figure 9:** Sector riskiness to domestic private banks discriminated by size when we take as initial shock scenario the default of all of firms of a same sector.*

increasing sector riskiness than large banks of the same control type. In addition, we see that both large and non-large banks in this control category are mainly exposed to the food and beverage sector. The tertiary sector is the second most risky sector for foreign private banks, followed by the metal extraction and processing sector.



(a) Large foreign private banks



(b) Non-large foreign private banks

Figure 10: *Sector riskiness to foreign private banks discriminated by size when we take as initial shock scenario the default of all of firms of a same sector.*

We can also decompose the average sector riskiness to banks in terms of bank control types. In this way, we get a picture of which kind of bank control type is more susceptible to shocks coming from different economic sectors. That is, banks that incorporate more sector riskiness fractions are more susceptible to impacts coming from the bank-firm channel. Figure 11 depicts the average fraction of sector riskiness for each bank control type in the analyzed period. We construct this graph as follows: 1) evaluate the average feedback-based systemic risk measure (sector riskiness) per each bank control type due to the default of each of the sectors; 2) normalize each of the three computed series.

Inspecting Fig. 11, we see that domestic private banks are the financial institutions that are the most susceptible to shocks coming from any of the economic sectors, with an exception for the second semester of 2012 in which government-owned banks turn out to be more susceptible to those external impacts. After 2013, we see that government-owned banks become more resilient against shocks coming from financed firms in any of the sectors. Foreign private banks show an oscillating pattern in their susceptibility for impacts coming from any of the sectors, with maximal relative susceptibility occurring in June 2013.

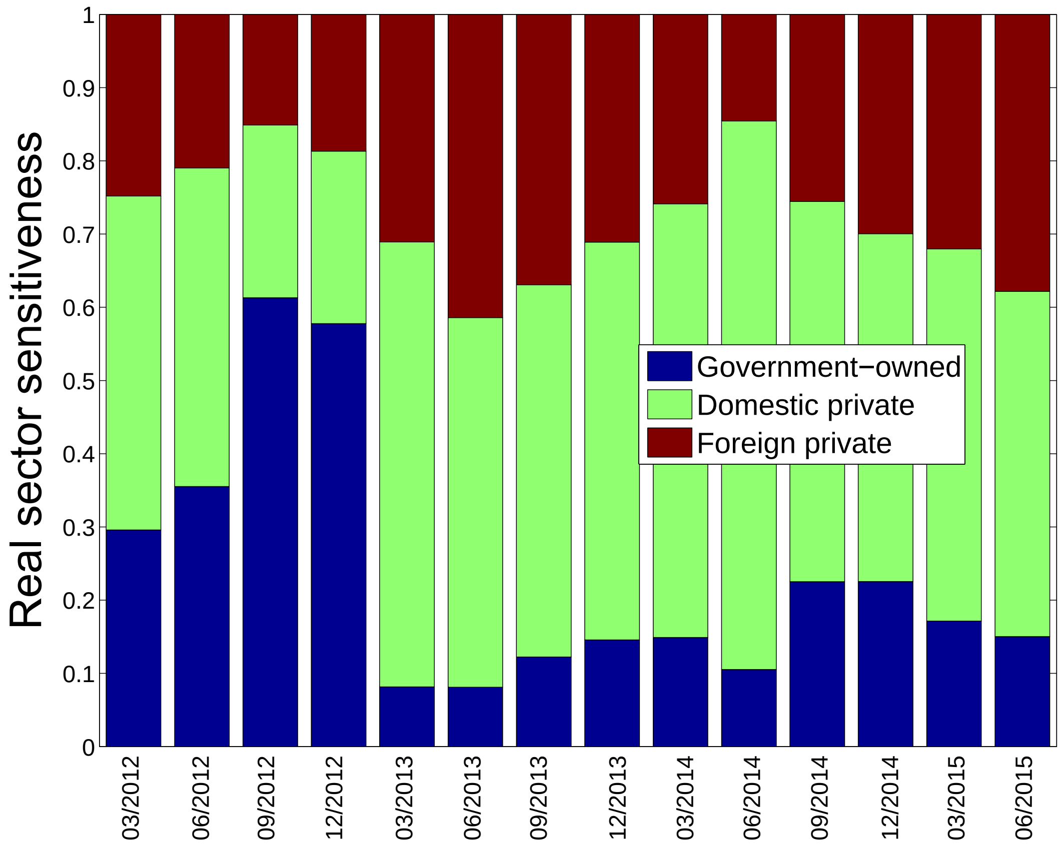


Figure 11: *Banks' susceptibility or sensitiveness of receiving impacts from the real sector discriminated by bank control types. The larger the fraction, the more susceptible, on average, is one bank control type in receiving shocks of the real economy.*

3.4 What factors increase or decrease sector riskiness to banks?

In the following sections, we define the components that we use to explain sector riskiness to banks. We also discuss the intuition behind choosing these components, as well as the expected results.

Likewise the previous section, we also term sector riskiness to banks as the feedback-based systemic risk of banks evaluated from the default of all firms of a same sector (initial shock).

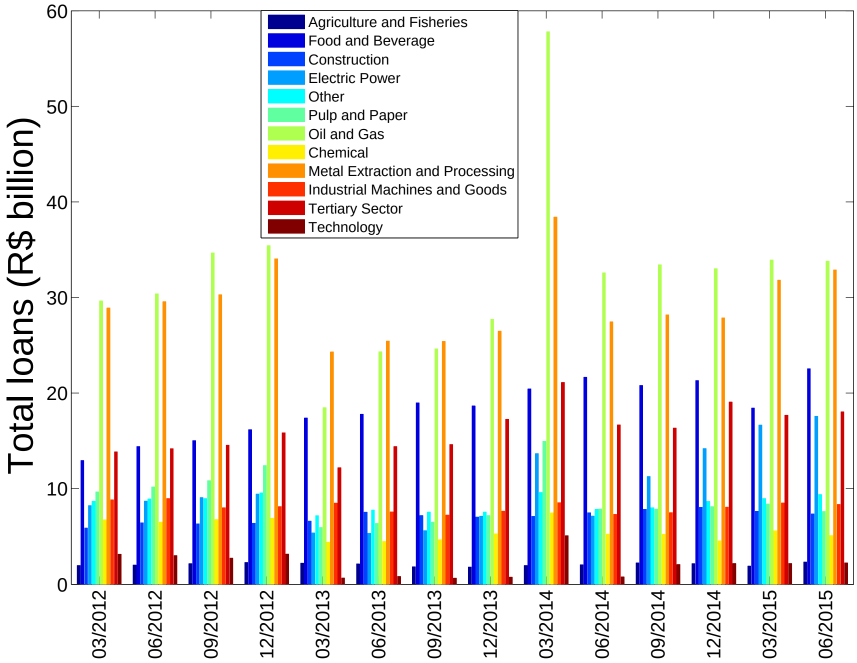
3.4.1 Total loans of banks to sectors

One natural assumption of determinants that explain sector riskiness to banks relates to the total amount of loans firms take from banks. That is, the more financial resources from banks a sector uses, the more exposed banks are to that sector. Hence, that sector is expected to be riskier for creditor banks should an external shock hit that sector. In this respect, we can formulate the following hypothesis:

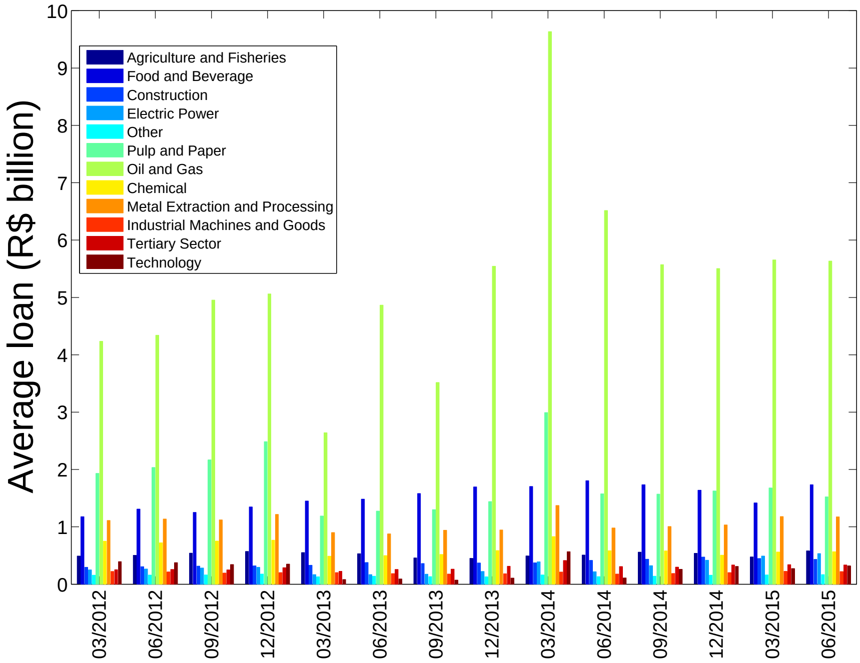
Hypothesis 1. *Sector riskiness to banks is positively related to the total amounts of loans banks provide to the sector.*

Figures 12a and 12b portray the total amount of loans and the average loan value from banks to each of the economic sectors. Though with only eight firms, we can see that the oil and gas sector takes massive amounts of loans from banks, in a way that the average loan value of the sector is large. Likewise, firms that encompass the metal extraction and processing sector receive large amounts of loans. The average value of these loans, however, is small as they are roughly dispersed among the 32 firms in that sector. The same reasoning applies for the food and beverage sector and the tertiary sector: they receive large amounts of loans that are dispersed through the large number of firms in those sectors. The pulp and paper sector exhibits large average loan values due to the small number of firms in the sample.

For clarity, Table 4 reports the rank positions of sectors per year using now the total loans that sectors take from banks. We build the rank using the data from Fig. 12a. We first see that the oil and gas sector receives by far the largest amounts of bank financing in the period, except in 2013, period in which bank loans to the metal extraction and processing sector are the majority. Besides that, the metal extraction and processing sector also consistently receives large bank financing in other years, thus maintaining the second position. Food and beverage and the tertiary sectors stay at the third and fourth positions. The pulp and paper seem to receive less and less loans from banks in the analyzed period. In contrast, the electric power sector receives an increasing amount of loans in the same period.



***(a)** Total loans*



***(b)** Average loan value*

***Figure 12:** Total and average loan values from banks to each of the economic sectors from March 2012 to June 2015.*

To get a picture of the relation between sector riskiness and total loans that sectors receive from banks, Figure 13 supplies a scatter plot of these two quantities. When building this figure, we flat out the time dimension of sector total loans and sector riskiness. We also give a linear fit using as penalty the squared distances of points (OLS) just to give a

Table 4: Sector ranking of the total amount of loans sectors take from banking institutions. We provide sector rankings per each year.

Rank	Year 2012	Year 2013	Year 2014	Year 2015
1	Oil & Gas	Metal Extraction & Processing	Oil & Gas	Oil & Gas
2	Metal Extraction & Processing	Oil & Gas	Metal Extraction & Processing	Metal Extraction & Processing
3	Food & Beverage	Food & Beverage	Food & Beverage	Food & Beverage
4	Tertiary Sector	Tertiary Sector	Tertiary Sector	Tertiary Sector
5	Pulp & Paper	Other	Pulp & Paper	Electric Power
6	Other	Industrial Machines & Goods	Electric Power	Other
7	Industrial Machines & Goods	Construction	Other	Industrial Machines & Goods
8	Electric Power	Pulp & Paper	Industrial Machines & Goods	Pulp & Paper
9	Chemical	Electric Power	Construction	Construction
10	Construction	Chemical	Chemical	Chemical
11	Technology	Agriculture & Fisheries	Agriculture & Fisheries	Agriculture & Fisheries
12	Agriculture & Fisheries	Technology	Technology	Technology

sense of the global relationship between these two quantities when we do not account for sector heterogeneities. We get a correlation coefficient of 0.45, revealing that the riskiness of the sector is somewhat correlated with the total loans that sector receives from banks. Nonetheless, the oil and gas sector is not the most risky sector even though it receives massive amounts of loans from banks. This observation suggests that the bank-bank or the bank-firm networks seem to smooth the impact of firms of this sector. We elaborate more on that in the next sections.

3.4.2 Financing portfolio concentration of firms and loan portfolio concentration of banks

Several countries possess a set of rules limiting banks' pairwise exposures to a single borrower, which is an argument in favor of the necessity of portfolio diversification. In Basel III, for instance, large pairwise regulation has been developed as a tool for limiting the maximum loss a bank could face in the event of a sudden counterparty failure to a level that does not endanger the bank's solvency (BCBS (2014b)). Basel III motivates the introduction of such regulation by historic terms.[23] In contrast, there are many banks that decide to specialize their loans activities to sectors in which they enjoy comparative advantage. In addition, maintaining few active financial operations reduces monitoring costs.

Theoretical models are also not consensual on the effects of portfolio concentration on bank performance. One strand of the literature (Diamond (1984); Ramakrishnan and Thakor (1984)) posits that banks should diversify their portfolio among different dimensions (e.g., sectors or geographic regions). The gains of this strategy would come in the form of reduction of financial intermediation costs and less vulnerability to economic downturns. On the other hand, studies grounded on the corporate finance paradigm defend

[23] Throughout history, there have been instances of banks failing due to concentrated exposures to individual counterparties. For instance, we can cite the UK crisis in 1984 and the Korean crisis in 1990.

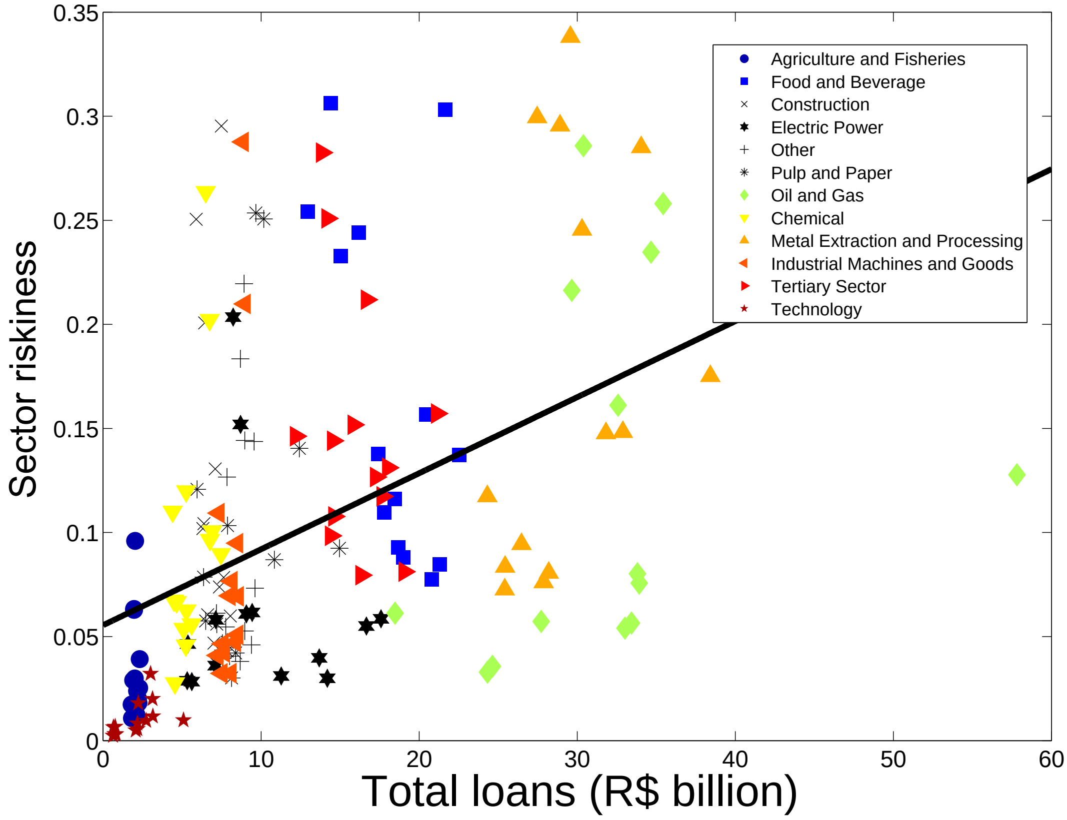


Figure 13: *Scatter plot of total loans of banks to firms vs. sector riskiness. We also plot a linear fit for the data in the black continuous line whose equation is* $\mathrm{SR} = 0.055 + 0.004\mathrm{Loan}$*, in which* SR *denotes the sector riskiness and* Loan *is the total loans that banks provide to a specific sector.*

that portfolio concentration may bring benefits to banks, as sectorial expertise (Acharya et al. (2006); Stomper (2004)) and less competition with other banks (Winton (1999)). Similarly, empirical studies report mixed results on the relationship between portfolio diversification and bank performance.[24]

The aforementioned approaches explain the phenomenon of portfolio diversification or concentration as a consequence of banks' decisions to maximize profit or minimize costs, while controlling for the assumed risks. Here, we are interested in understanding how portfolio diversification or concentration of banks and firms affects sector riskiness. In this spectrum, we argue that the more concentrated the financing portfolio of firms and the loan portfolio of banks are, the harder the stress propagation in the network will be. Consequently, sector riskiness becomes lower. That reasoning has its roots on the following peculiar observations on the Brazilian bank-bank and bank-firm networks:

1. The bank dependency of firms (debt to total assets ratio of firms) is small in our sample. In this way, firms do not become overly stressed due to stress coming from

[24] Tabak et al. (2011) present a comprehensive review on these empirical studies.

banks. Consequently, the contagion transmission channel in the "firms to banks" direction is small.

2. The vulnerability of banks to firms is often small. Though banks can run into a default, the reduced pairwise vulnerability between banks and firms attenuates the stress that is propagated to sectors. In this way, the contagion transmission channel in the "banks to firms" direction is small.

3. The pairwise vulnerability between banks is highly heterogeneous. In this way, we have large and small pairwise vulnerabilities in the interbank vulnerability network.

In essence, the intuition is that, the fewer banks get stressed, the smaller the propagation of stress in the interbank market (the stress propagation that really contributes to financial stress) will be. From a sector viewpoint, firms must connect to fewer banks as possible. From a bank viewpoint, banks must connect to fewer firms as possible, because fewer banks will be then stressed by those firms in a second round of stress propagation.

In light of these arguments, we formulate the following hypotheses:

Hypothesis 2. *Sector riskiness to banks decreases as the financing portfolio of firms becomes more concentrated.*

Hypothesis 3. *Sector riskiness to banks decreases as the loan portfolio of banks becomes more concentrated.*

We can capture the concentrations of the financing portfolio of firms and the loan portfolio of banks in a specific sector by using the Herfindahl–Hirschman Index (HHI).[25] While the first perspective gives us a sense of how firms diversify (or concentrate) their financing over different banks in the interbank market, the second perspective extracts how banks diversify (or concentrate) their loans with respect to sectors that are external to the interbank market.

The concentration measure $\mathrm{HHI}(u)$ is an individual-level index that we compute for each firm or each bank in our sample. Mathematically, we evaluate $\mathrm{HHI}(u)$ as follows:

$$\mathrm{HHI}(u) = \sum_g p^2(u,g), \tag{38}$$

in which $p(u,g)$ is the relative proportion or share of u on g, whose expression we discuss in the next paragraphs. The index g may run over two different sets:

[25] HHI assumes values in-between 0 and 1. When $HHI = 1$, there is perfect concentration on a single counterparty. As HHI assumes smaller values, economic agents become more diversified.

1. the set of banks $\mathscr{B}$ if we are evaluating the concentration of the financing portfolio of firm $u \in \mathscr{F}$; or
2. the set of sectors $\mathscr{S}$ in case we are calculating the concentration of loan portfolio of bank $u \in \mathscr{B}$ towards sectors.

We first discuss how to evaluate the concentration of financing portfolio of firms. In this case, $p(u,g) = p^{(\text{finance})}(u,g)$ denotes the relative financing share that firm $u \in \mathscr{F}$ receives from bank $g \in \mathscr{B}$, i.e.:

$$p^{(\text{finance})}(u,g) = \frac{\mathbf{A}_{gu}^{(\text{bank}-\text{firm})}}{\sum_{p \in \mathscr{B}} \mathbf{A}_{pu}^{(\text{bank}-\text{firm})}}. \tag{39}$$

We can then calculate the average financing concentration of firms in a sector $s \in \mathscr{S}$ as:

$$\text{HHI}^{(\text{finance})}(s) = \frac{1}{|\mathscr{F}_s|} \sum_{k \in \mathscr{F}_s} \text{HHI}(k), \tag{40}$$

in which $\mathscr{F}_{\text{s}}$ denotes the set of firms in sector s $\in \mathscr{S}$.

We now discuss how to evaluate the loan portfolio concentration of banks towards sectors. First, we need to aggregate the bank-firm network $\mathbf{A}^{(\text{bank}-\text{firm})}$ into a bank-sector network $\mathbf{A}^{(\text{bank}-\text{sector})}$. For that, we simply sum up the exposures banks have to firms of the same sector. Mathematically, we map $m : \mathscr{B} \times \mathscr{F} \mapsto \mathscr{B} \times \mathscr{S}$, in which m is the function that sums up exposures of banks to firms of the same sector.

Let $p(u,g) = p^{(\text{loan})}(u,g)$ account for the relative loan share that bank $u \in \mathscr{B}$ provides to sector $g \in \mathscr{S}$. We express this as:

$$p^{(\text{loan})}(u,g) = \frac{\mathbf{A}_{ug}^{(\text{bank}-\text{sector})}}{\sum_{s \in \mathscr{S}} \mathbf{A}_{us}^{(\text{bank}-\text{sector})}}. \tag{41}$$

To compose the average loan portfolio concentration of banks to sector $s \in \mathscr{S}$, we take a weighted convex linear combination of the concentration indices of banks that maintain connections with firms of sector s:

$$\text{HHI}^{(\text{loan})}(s) = \frac{1}{TL(s)} \sum_{j \in \mathscr{B}} \mathbf{A}_{js}^{(\text{bank}-\text{sector})} HHI(j), \tag{42}$$

in which $TL(s)$ characterizes the total loan that sector s receives from banks. Observe that we give more weight to those banks that provide more loans to sector s. Note that Equation (42) does not account for concentration indices of banks that do not have loan operations with sector s. To see that, consider that bank j does not have loan operations towards sector s, then $\mathbf{A}_{js}^{(\text{bank}-\text{sector})} = 0$.

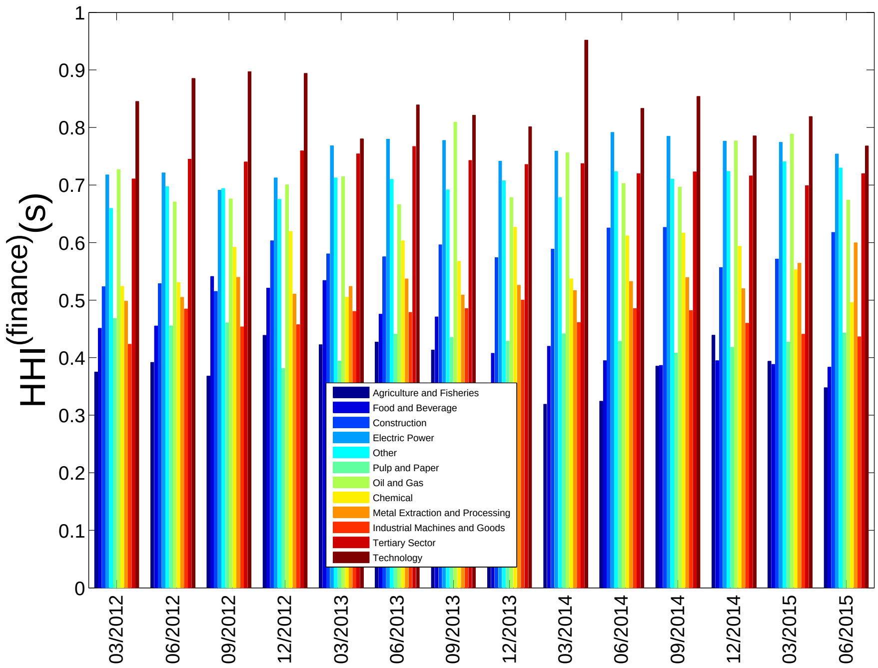
Figure 14a displays the average financing portfolio concentration of firms of the same sector, while Table 5 reports the respective sector rankings. We see that the financing of the technology sector is highly concentrated in very few banks, namely large banks. In some cases, it achieves an average HHI of more than 90%, suggesting roughly that a single large bank is financing the entire sector. Followed by that, we also see that the tertiary, electric power, oil and gas sectors are also highly concentrated in few financing banks. On the other extreme, we verify that the agriculture and fisheries and food and beverage sectors are the most diversified sectors using the firms in our sample. The pulp and paper and industrial machines and goods sectors also show diversified financing portfolios.

Table 5: *Sector ranking of the financing portfolio concentration of firms. We provide sector rankings per each year.*

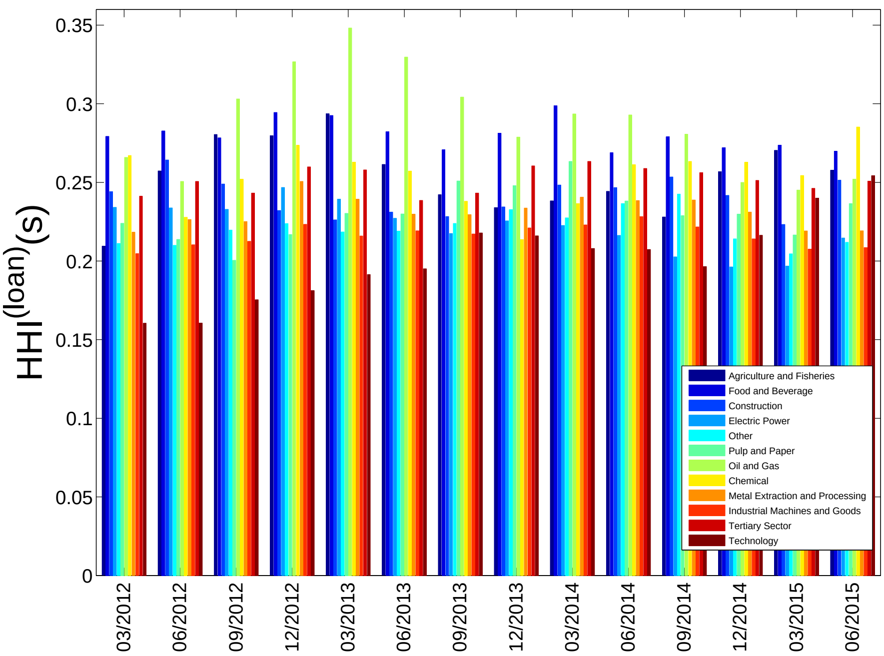
Rank	Year 2012	Year 2013	Year 2014	Year 2015
1	Technology	Technology	Technology	Technology
2	Tertiary Sector	Electric Power	Electric Power	Electric Power
3	Electric Power	Tertiary Sector	Oil & Gas	Other
4	Oil & Gas	Oil & Gas	Other	Oil & Gas
5	Other	Other	Tertiary Sector	Tertiary Sector
6	Chemical	Construction	Construction	Construction
7	Construction	Chemical	Chemical	Metal Extraction & Processing
8	Metal Extraction & Processing	Metal Extraction & Processing	Metal Extraction & Processing	Chemical
9	Food & Beverage	Industrial Machines & Goods	Industrial Machines & Goods	Pulp & Paper
10	Pulp & Paper	Food & Beverage	Pulp & Paper	Industrial Machines & Goods
11	Industrial Machines & Goods	Pulp & Paper	Food & Beverage	Agriculture & Fisheries
12	Agriculture & Fisheries	Agriculture & Fisheries	Agriculture & Fisheries	Food & Beverage

In the other perspective, Figure 14b exhibits the average loan portfolio concentration of banks towards sectors and Table 6 reports the sector rankings. We can check that banks that maintain operations with firms of the oil and gas sector usually have more concentrated loan portfolios than other banks. We see a contrast point in the food and beverage sector: while the financing portfolio concentration of firms of that sector is somewhat diversified, the loan portfolio concentration of banks that provide loans to that sector is relatively more concentrated. Opposed to the highly concentrated financing portfolio of firms in the technology sector, banks that provide loans to that sector are the most diversified in our sample. In this way, while technological firms rely on few banks to finance themselves, these banks do not specialize in only financing this kind of sector.

Figures 15a and 15b supply scatter plots of the sector riskiness against the average financing portfolio concentration of sector firm, $\text{HHI}^{(\text{finance})}$, and the average loan portfo-



(a) *Average financing portfolio concentration of sector firms*



(b) *Average loan portfolio concentration of banks towards sectors*

Figure 14: *Herfindahl-Hirschman Index (HHI) using two perspectives: 1) concentration of financing firms receive from banks and 2) concentration of loans banks provide to sectors.*

lio concentration of banks towards sectors, HHI$^{(\mathrm{loan})}$, respectively. Again, we flat out the time dimension to get a rough picture of the dependency of these two quantities. We get small correlation coefficients: -0.09 for Fig. 15a and 0.23 for Fig. 15b. Recall that by assuming flatted out time series of the feedback-based systemic risk measure and the con-

Table 6: *Sector ranking of the loan portfolio concentration of banks towards sectors. We provide sector rankings per each year.*

Rank	Year 2012	Year 2013	Year 2014	Year 2015
1	Food & Beverage	Oil & Gas	Food & Beverage	Food & Beverage
2	Oil & Gas	Food & Beverage	Oil & Gas	Chemical
3	Chemical	Agriculture & Fisheries	Chemical	Agriculture & Fisheries
4	Agriculture & Fisheries	Tertiary Sector	Tertiary Sector	Oil & Gas
5	Construction	Pulp & Paper	Construction	Technology
6	Tertiary Sector	Chemical	Agriculture & Fisheries	Tertiary Sector
7	Electric Power	Construction	Pulp & Paper	Construction
8	Metal Extraction & Processing	Metal Extraction & Processing	Metal Extraction & Processing	Pulp & Paper
9	Other	Electric Power	Other	Metal Extraction & Processing
10	Pulp & Paper	Other	Industrial Machines & Goods	Electric Power
11	Industrial Machines & Goods	Industrial Machines & Goods	Electric Power	Other
12	Technology	Technology	Technology	Industrial Machines & Goods

centration indices, we lose the time dependency of time series and assume homogeneity of sectors. Nonetheless, this simplification gives us a gist as to how these two quantities relate to each other.

3.4.3 Banks' propensity to diffuse financial stress

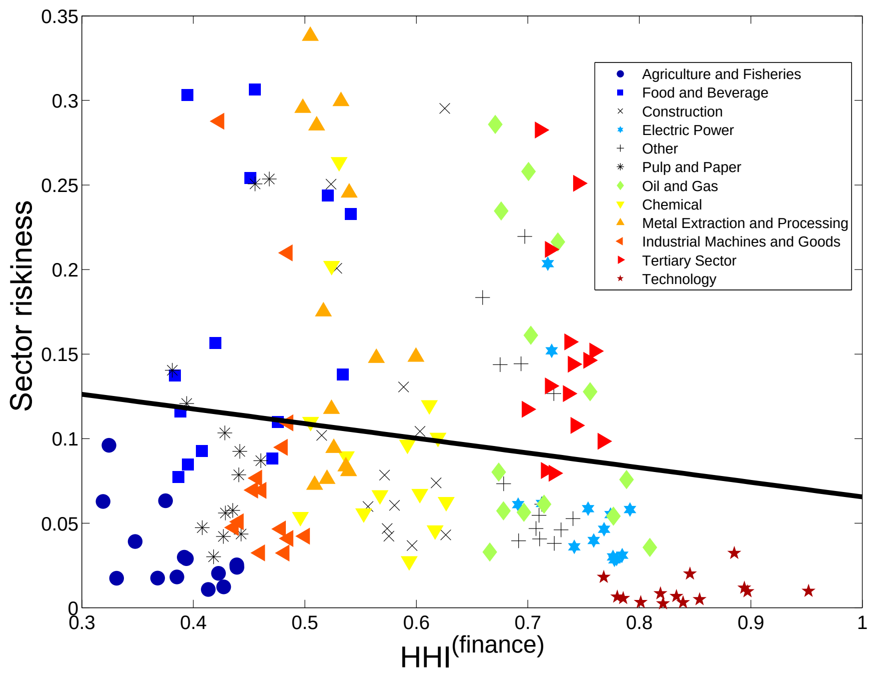
Another fundamental element that increases sector riskiness to banks relates to the type of bank firms connect to in terms of sources of stress diffusion. We argue that firms that connect to banks that are strong stress diffusers contribute more to systemic risk in the financial sector.

To exemplify, consider an arbitrary bank $i \in \mathscr{B}$. If its neighbors are vulnerable to i, the default or an increase in the distress position of bank i is largely absorbed by these neighboring banks due to their high stress sensitivity, possibly leading them into bankruptcy. From this perspective, our systemic risk measure increases as it relates positively to the bank stress levels. Thus, sectors become riskier the more their corresponding firms connect to banks that are sources of stress diffusion. In view of that, we formulate the following hypothesis:

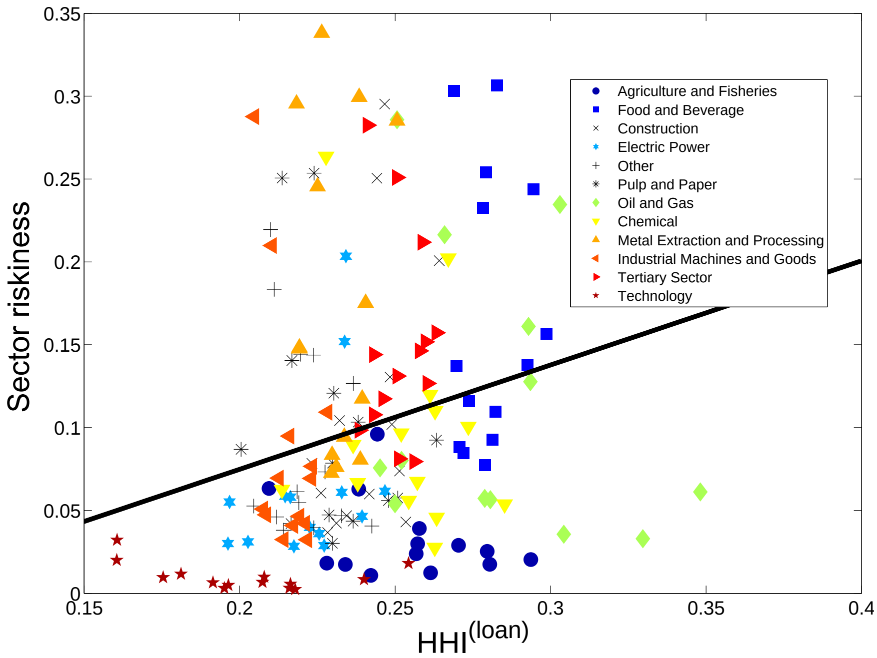
Hypothesis 4. *Sector riskiness increases as the firms of that sector connect to banks that are sources of stress diffusion.*

A suitable proxy to evaluate how strong a bank is as a source of stress diffusion is the network measurement called impact diffusion influence. In essence, the impact diffusion influence is a vertex-level measure that estimates the potential influence exercised by a bank on diffusing impacts throughout the financial network. We formally define and explore this network measurement in detail in Appendix A.

We now discuss how to compute the average impact diffusion influence of banks that connect to a specific sector $s \in \mathscr{S}$. For that end, we take a weighted convex linear combination of the impact diffusion influence indices of banks that maintain connections with firms of sector s:



***(a)** Sector riskiness vs. average financing portfolio concentration of sector firms*



***(b)** Sector riskiness vs. average loan portfolio concentration of banks towards sectors*

Figure 15: *Scatter plot of* $\text{HHI}^{(\text{finance})}$ *and* $\text{HHI}^{(\text{loan})}$ *vs. sector riskiness. We also plot a linear fit for the data in the two black continuous lines appearing in each figure, whose equations are* $\text{SR} = 0.152 - 0.087\text{HHI}^{(\text{finance})}$ *and* $\text{SR} = -0.051 + 0.630\text{HHI}^{(\text{loan})}$*, respectively. The dependent variable* SR *stands for sector riskiness.*

$$\bar{I}(s) = \frac{1}{TL(s)} \sum_{j \in \mathscr{B}} \mathbf{A}_{js}^{(\text{bank}-\text{sector})} I(j), \tag{43}$$

in which $\bar{I}(s)$ symbolizes the average impact diffusion influence of banks that connect to firms of sector s, $I(j)$ represents the impact diffusion influence of bank $j \in \mathscr{B}$, and $TL(s)$ characterizes the total loan amounts that sector s receives from banks.

Silva et al. (2015) investigate the Brazilian bank-bank network with respect to the banks' impact diffusion influence. They show that large banking institutions have large impact diffusion influence. In contrast, non-large banking institutions have strong heterogeneity in terms of the impact diffusion influence. Silva et al. (2015) reveal that there is a small subset of non-large banks whose impact diffusion influence surpasses that of large banks. Nonetheless, the majority of these banking institutions has smaller impact diffusion influence in relation to large institutions.

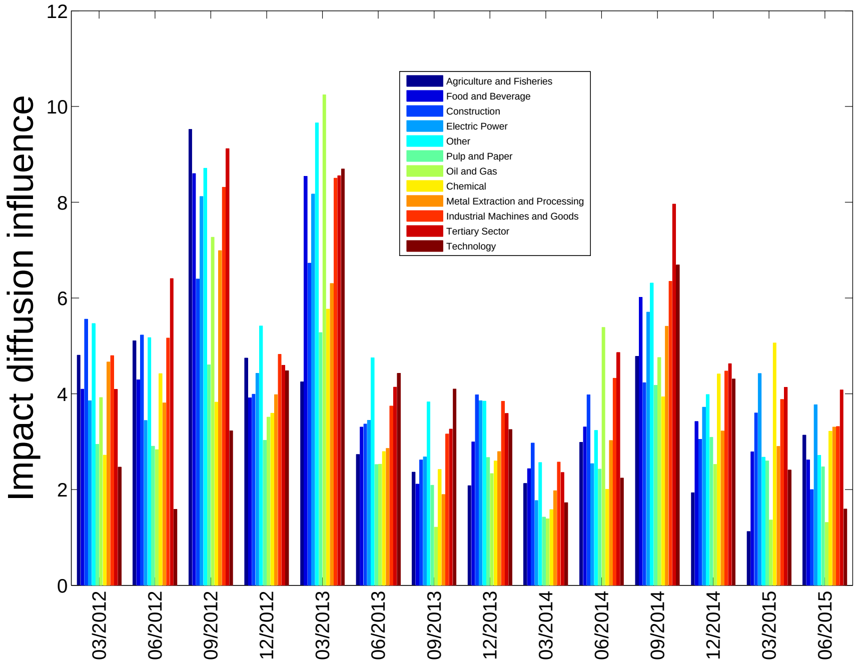


Figure 16: *Average weighted impact diffusion influence of banks that connect to firms of a same sector.*

Figure 16 depicts the weighted average impact diffusion influence of banks connected to the economic sectors. We compute that measure using (43). We also report in Table 7 the average sector ranking with respect to the weighted average impact diffusion influence indices.

We see that firms of the oil and gas sector maintain financing operations mostly with banks that have small impact diffusion influence. That is, banks that are exposed to the oil and sector often are not the main sources of stress diffusion in the interbank market. In our sample, this is one of the possible factors that the oil and sector is not the riskiest sector of the economy: even though the oil and gas sector takes massive amounts of loans from banks, these banks are not the main players in diffusing stress in the network. Thus, sectors that take the largest amounts of loans are not necessarily the riskiest sectors in the

Table 7: *Sector ranking of the average impact diffusion influence of banks that connect to firms of a same sector. We provide sector rankings per each year.*

Rank	Year 2012	Year 2013	Year 2014	Year 2015
1	Other	Other	Tertiary Sector	Electric Power
2	Agriculture & Fisheries	Technology	Industrial Machines & Goods	Tertiary Sector
3	Tertiary Sector	Tertiary Sector	Other	Chemical
4	Industrial Machines & Goods	Industrial Machines & Goods	Food & Beverage	Industrial Machines & Goods
5	Construction	Electric Power	Construction	Metal Extraction & Processing
6	Food & Beverage	Construction	Technology	Food & Beverage
7	Metal Extraction & Processing	Food & Beverage	Electric Power	Construction
8	Electric Power	Oil & Gas	Metal Extraction & Processing	Other
9	Oil & Gas	Chemical	Oil & Gas	Agriculture & Fisheries
10	Chemical	Metal Extraction & Processing	Agriculture & Fisheries	Pulp & Paper
11	Technology	Agriculture & Fisheries	Chemical	Technology
12	Pulp & Paper	Pulp & Paper	Pulp & Paper	Oil & Gas

economy. In fact, though the total loans perspective is important, it is not a sufficient nor necessary condition to increased sector riskiness. We also must look at other dimensions, such as the risk dimension: if exposed banks to that sector are strong sources of stress propagation, then the corresponding firms of that sector can trigger large potential losses in the interbank market through these stress diffusers should an external shock hit them.

In contrast, firms of the tertiary, construction, food and beverage, and metal extraction and processing sectors connect to banks, on average, with large impact diffusion influence. According to our data, recall from Table 3 that the metal extraction and processing and tertiary sectors are placed among the top 2 riskiest sectors to banks. This observation suggests that large impact diffusion influence values are associated to increased sector riskiness.

Figure 17 exhibits the scatter plot of the sector riskiness to banks against the average weighted impact diffusion influence. Likewise before, we flat out the time dimension to get a rough picture of the dependency of these two quantities. We get correlation coefficients of 0.43 for Fig. 17. The positive angular coefficient in the linear fit corroborates our hypothesis about the positive relation between sector riskiness and the impact diffusion influence.

3.4.4 Econometric model

In this section, we define the empirical specification that we employ to identify and assess the determinant factors that cause sector riskiness to banks. Our analysis takes into account results of both the bank-bank and bank-firm contagion channels through the financial accelerator engine. We use the following static panel:[26]

[26]We first try to fit in a linear dynamic panel-data estimation put forward by Arellano and Bover (1995) and Blundell and Bond (1998), using instruments to account for possible endogeneity among the regressors. However, we consistently get estimates for the autoregressive coefficients that are statistically insignificant. Thus, we opt to use a static panel.

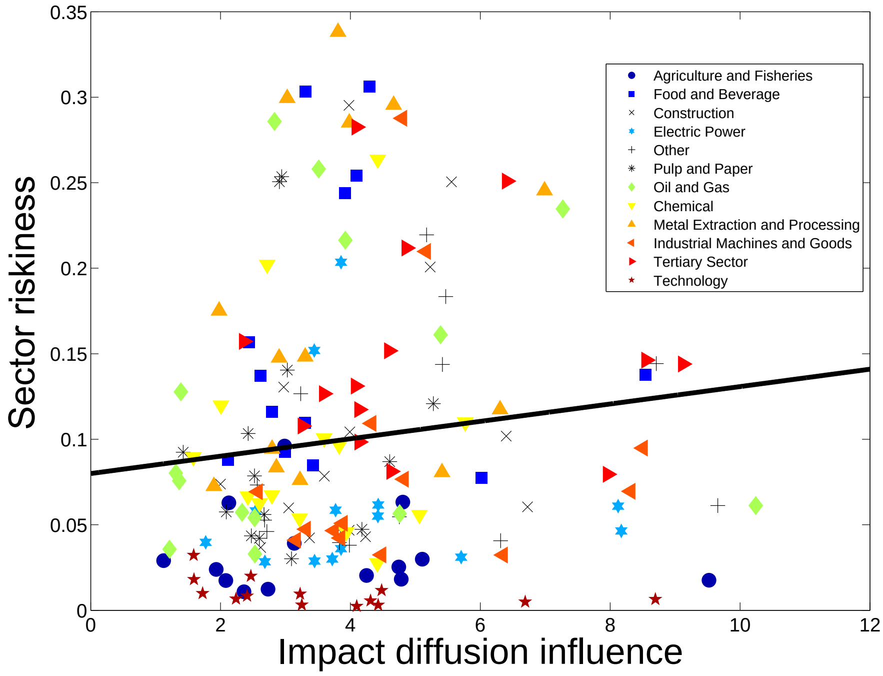


Figure 17: *Scatter plot of the impact diffusion influence* (IDI) *vs. sector riskiness. We also plot a linear fit for the data in the black continuous line, whose equation is* $\mathrm{SR} = 0.080 + 0.005\mathrm{IDI}$. *The dependent variable* SR *stands for sector riskiness.*

$$Y_{it} = \beta_0' X_{it} + \beta_1' C_{it} + \beta_2' D_t + \upsilon_i + \varepsilon_{it} \tag{44}$$

in which $'$ is the transpose operator. We define the terms in (44) as follows:

- Y_{it} is the sector riskiness of sector i at time t. Recall that we proxy sector i's riskiness as the feedback-based systemic risk measure of banks when we assume as initial shock the default of all firms of sector i.
- X_{it} is the feature vector of sector i at time t. We load up this vector with the determinants that we are interested in studying. They are:
  - Total loans of banks to sectors.
  - Average financing portfolio concentration of firms of a sector.
  - Average weighted loan portfolio concentration of banks towards firms of the sector.
  - Average weighted impact diffusion influence of banks.
- β_0 is the vector of estimates with respect to the feature vector X_{it}. We are interested in the sign, magnitude, and the statistical significance of these estimates.

- C_{it} is a set of controls that we use in our econometric analysis. We control for:
  - The average leverage of banks that connect to firms of sector i at time t.
  - The average bank dependency (or debt to total assets) of firms of sector i at time t.
- β_1 is the estimates for the controls in C_{it}.
- D_t is a vector of time dummies that we use to absorb time effects in the estimations.
- β_2 is the estimates for D_t.
- υ_i represents the non-observable individual factors of sector i.
- $\varepsilon_{i,t}$ is the error term that, by hypothesis, is identically and independently distributed with zero mean and constant variance σ_ε^2, i.e., $\varepsilon_{i,t} \sim \mathrm{IID}(0, \sigma_\varepsilon^2)$.

The terms υ_i and D_t account for the sector and time fixed effects in the model. Our strategy is design model specifications in which we vary between the presence or absence of these fixed effects. In addition, we first start off with the simple linear panel-data estimation through multiple OLS regression, in which we do not account for possible endogeneity in the regressors. Afterwards, we supply robustness tests using a linear panel-data estimation through the Generalized Method of Moments (GMM) with instrumental variables (IV), in which we deal with potential endogeneity problems. In summary, we find that the results of the OLS and GMM/IV are very similar.

We also apply a log transformation on all of the independent and dependent variables in the econometric model. In this way, we can interpret the estimates in terms of elasticity. In addition, we report robust standard errors to account for possible heteroskedasticity problems. To verify to what extent the regressors are correlated, which can lead to increased standard errors in the econometric model, we report in Table 8 their pairwise cross-correlation. Overall, the pairwise cross-correlation is small, reaching a maximum of 0.37 between total loans and bank financing portfolio.

3.4.5 Discussion of the results

Table 9 reports the estimates for our panel regressions using plain OLS. Note that the coefficients seem to maintain their sign and statistical significance among different specifications when we gradually add sector and/or time fixed effects.

We control for the average bank leverage and the average firms' bank dependency (or firms' debt to assets) in our regressions. We employ them as proxies for bank and firm individual riskiness. By absorbing the effects of individual riskiness of banks and

Table 8: Cross-correlation between the regressors we employ in our analysis.

	Total loans	Firm HHI	Bank HHI	Impact Diffusion	Bank leverage	Bank dependency
Total loans	1.00					
Firm HHI	0.06	1.00				
Bank HHI	0.37	-0.22	1.00			
Impact diffusion	-0.01	0.07	-0.04	1.00		
Bank leverage	-0.15	-0.17	-0.09	0.21	1.00	
Bank dependency	0.09	-0.16	0.23	-0.15	0.02	1.00

firms in these controls, we are able to better estimate the role of the network topology in explaining sector riskiness to banks.

Our results reveal that the average bank leverage yields positive and statistically significant coefficients in all four specifications. In this way, sectors that connect to banks that are highly leveraged tend to show higher riskiness levels to the banking sector. We also see that the firms' bank dependency relates positively to sector riskiness, but the significance is not as robust as the coefficients of the average bank leverage. Thus, sectors with firms that have high levels of debt tend to be more riskier to the banking sector.

We see that the coefficient representing the average total loans sectors receive from banks is positive and statistically significant throughout the four specifications. In this way, sectors that hold larger loan volumes tend to be more risky to exposed banks. This empirical evidence supports our Hypothesis 1.

The portfolio concentration of banks and firms is negatively related to sector riskiness to banks. Our estimates show that the financing portfolio concentration of firms is more robust than the average loan concentration of banks when explaining sector riskiness. The results indicate that the more diversified banks or firms are (or less concentrated), the higher the sector riskiness is to exposed banks. This finding may be related to the propagation speed of contagion. If banks or firms engage in several financial operations, then an initial shock has greater chances of disseminating through the network through the large direct neighborhood of these banks or firms. Once the direct neighborhood is hit, another round of contagion may occur in case these banks or firms are not able to absorb the incoming losses. Therefore, shocks can survive more easily in the network by effectively "visiting" several potential banks or firms. This observation gives us evidences in favor of Hypothesis 2 and 3.

We also see a positive and statistically significant coefficient for the average diffusion power of banks. In this way, sectors that connect to banks that have more propensity

Table 9: *Panel regressions with OLS on the relative importance of network topology and bank and firm characteristics in determining sector riskiness.*

Independent variable	**Model 1**	**Model 2**	**Model 3**	**Model 4**
Dependent variable: sector riskiness				
Feature vector variables				
Total loans$_{it}$	0.670***	0.638***	0.638***	0.638***
	(0.085)	(0.084)	(0.087)	(0.084)
Firm HHI$_{it}$	-0.141***	-0.143***	-0.143***	-0.143***
	(0.049)	(0.046)	(0.047)	(0.045)
Bank HHI$_{it}$	-0.171	-0.235**	-0.235**	-0.235**
	(0.132)	(0.108)	(0.113)	(0.108)
Bank impact diffusion$_{it}$	0.881***	0.887***	0.890***	0.890***
	(0.015)	(0.009)	(0.010)	(0.009)
Control variables				
Bank leverage$_{it}$	0.117***	0.085***	0.085***	0.085***
	(0.010)	(0.007)	(0.008)	(0.007)
Bank dependency$_{it}$	0.213*	0.255	0.241	0.247***
	(0.118)	(0.183)	(0.210)	(0.096)
Sector fixed effects	NO	YES	NO	YES
Time fixed effects	NO	NO	YES	YES
Observations	168	168	168	168
R^2	0.480	0.618	0.482	0.627
Wald	0.000	0.000	0.000	0.000

We report the p-value for the Wald test (F-test). Standard errors are in parentheses. *** , ** , * stand for 1, 5 and 10 percent significance levels respectively. We omit the constant term in the table.

to diffuse stress inside the network are more risky to the banking sector. In this way, network topology plays a crucial role in deciding for the riskiest sectors in the economy. Therefore, we find empirical evidence in favor of Hypothesis 4.

3.4.6 Robustness tests

For robustness, we now employ the GMM for linear static panel-data estimation with fixed effects. Now we account for the presence of potential endogeneity problems, which can arise when there is correlation between the explanatory variables and the error term. Endogeneity can occur as a result of measurement error, regression with autocorrelated errors, simultaneity and omitted variables. A common cause of endogeneity is a loop of causality between the independent and dependent variables of a model, often termed mutual causality. We address the endogeneity problem by instrumenting the potential endogenous regressors using lagged independent variables. We consider as potentially endogenous all of the regressors in the feature vector X_{it}.

To check the validity of our estimates due to the introduction of instrumental variables, we perform two statistical tests: 1) test of overidentifying restrictions of all instruments and 2) test of underidentification of the GMM system.

To test for the validity of the overidentifying restrictions of the GMM system, we use the Hansen J test of overidentification. The joint null hypothesis is that the instruments are valid instruments, i.e., uncorrelated with the error term, and that the excluded instruments are correctly excluded from the estimated equation. A rejection casts doubt on the validity of the instruments. The Hansen J statistic is consistent in the presence of heteroskedasticity and autocorrelation. In our analysis, we must not be able to reject this test so as to have valid instruments.

To test for the underidentification of the GMM system, we use the Kleibergen-Paap rank LM statistic. This statistical test is essentially the test of the rank of a matrix: under the null hypothesis that the equation is underidentified, the matrix is not full column rank. Note that we must reject the null hypothesis in order to have identification of our system.

Table 10 reports the estimates for our panel estimation using GMM/IV. Observe that the number of observations is smaller than that in Table 9 because we lose some observations due to instrumentalization through lagged variables.

3.5 How relevant is the stress feedback effect between the real and financial sectors?

One of the main contributions of this paper is to provide a systematic way of modeling the feedback from firms to banks and vice versa using an instance of the financial accelerator. Note that the bidirectional nature of the feedback is the characteristic that permits second- and high-order rounds of stress propagation. If the shock transmission were established unidirectionally, then shocks from one network to another would never bounce back to the first due to the incommunicability. For instance, de Castro Miranda and Tabak (2013) design stress scenarios by considering how shocks in firms propagate to the interbank market in an unidirectional way ("firm to bank" channel). That is, once the initial firm shock passes to the interbank market, it never comes back to firms. In this section, we show that the "bank to firm" channel, which to the best of our knowledge has never been considered in the literature, plays an important role in the contagion process, as it can greatly increase the sector riskiness to banks. In this way, it is of utter importance to consider feedback mechanisms between different contagion channels when designing

__Table 10:__ Robustness test. Panel regressions using GMM with instrumental variables on the relative importance of network topology and bank and firm characteristics in determining sector riskiness.

Independent variable	**Model 1**	**Model 2**	**Model 3**	**Model 4**
		Dependent variable: sector riskiness		
Feature vector variables				
Total loans$_{it}$	0.690***	0.524***	0.670***	0.613***
	(0.076)	(0.061)	(0.070)	(0.071)
Firm HHI$_{it}$	-0.223***	-0.116**	-0.238***	-0.270***
	(0.084)	(0.059)	(0.077)	(0.083)
Bank HHI$_{it}$	-0.202	-0.347*	-0.061	-0.233**
	(0.223)	(0.183)	(0.206)	(0.104)
Bank impact diffusion$_{it}$	0.790***	0.864***	0.891***	0.891***
	(0.031)	(0.040)	(0.033)	(0.031)
Control variables				
Bank leverage$_{it}$	0.123***	0.074***	0.099***	0.076***
	(0.017)	(0.013)	(0.016)	(0.014)
Bank dependency$_{it}$	0.303	0.001	0.367**	0.365
	(0.185)	(0.167)	(0.170)	(0.252)
Sector fixed effects	NO	YES	NO	YES
Time fixed effects	NO	NO	YES	YES
Observations	140	140	140	140
R^2	0.474	0.485	0.477	0.501
Wald	0.000	0.000	0.000	0.000
Hansen J	0.130	0.164	0.228	0.198
Kleibergen-Paap rank LM	0.004	0.002	0.001	0.000

We report the p-value for the Wald, Hansen J, and Kleibergen-Paap tests. Standard errors are in parentheses. *** , ** , * stand for 1, 5 and 10 percent significance levels respectively. We omit the constant term in the table.

robust risk analysis tools.

To analyze how large that multiplicative factor can be when we incorporate the feedback mechanism, we choose a time point in which the sector riskiness in the Brazilian network is large. Inspecting Fig. 7, we opt to use June 2014 as a reference date to conduct our investigation.

Our methodology to understand the importance of the two-way feedback is to first evaluate the sector riskiness to banks when we consider only the “firm to bank” contagion channel, thus disabling the “bank to firm” contagion channel that we model through increasing credit constraints of banks. Afterwards, we recompute the sector riskiness by now considering the "firm to bank" and "bank to firm" contagion channels (financial accelerator). The observed increment in the sector riskiness is then attributed to the second- and high-order rounds of stress propagation. Note that we let enabled the classical "bank-bank" contagion channel in both simulations.

Unlike the previous sections in which we simply default all firms of a sector, we are now concerned with how sector riskiness to banks relates to different initial stress levels of sectors. As we increase the initial stress level of the sector, banks become more distressed and hence the sector riskiness rises. However, we verify here that similar increases of stress levels in different sectors result in different increments in the corresponding sector riskiness to banks. We show that the reason of the heterogeneity in the stress amplification is due to the *network effect*.

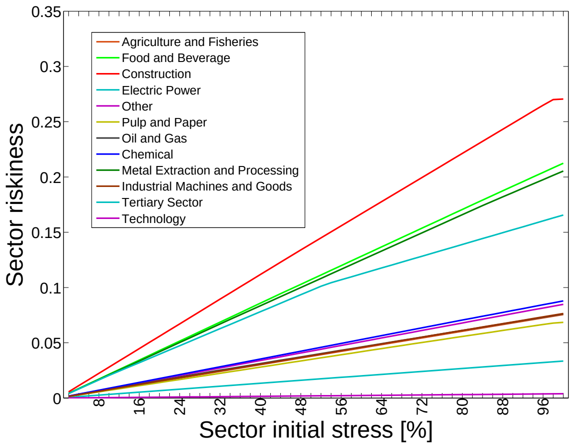
Figures 18a and 18b portray the sector riskiness to banks in June 2014 when we consider the one-way and two-way feedback mechanisms between firms and banks, respectively. The exercise consists in partially stressing firms of a same sector and then in verifying the resulting sector riskiness. As expected, we can see that the sector riskiness using bidirectional feedback is higher than when considering only the "firm to bank" contagion channel. Figure 18c displays the percentage difference between these two approaches. Applying a right-sided Wilcoxon signed rank test, we conclude that the increases in systemic risk estimates in light of the feedback effects are significant at the 1% significance level.[27]

Though not the riskiest sector, the oil and gas sector is the one that is mostly underestimated when we consider only the "firm to bank" contagion channel. This fact occurs because:

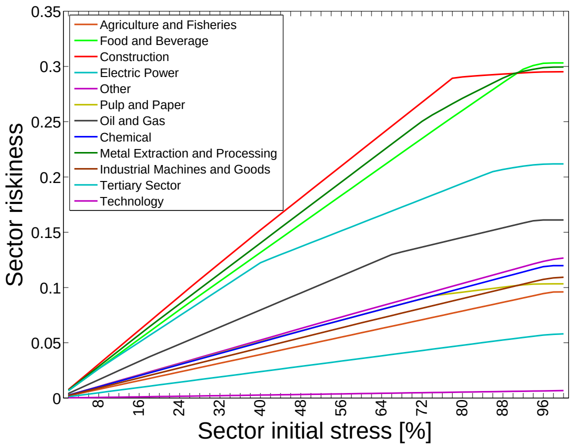
- firms of the oil and gas sector have considerable bank dependency. As such, according to the dynamic of our feedback-based model, these firms can reach high stress levels. Consequently, they are able to stress financing banks to a large extent.
- financing banks provide massive amounts of loans to firms of the oil and gas sector. In this way, the vulnerability of banks to this sector is expected to be non-negligible. As such, they are more susceptible to shocks coming from these firms.

Putting these two observations together, we can check that the stress propagation cycles from "firms and banks," due to elevated firm stress levels, and from "banks to firms," due to credit constraint, are large. Therefore, since the unidirectional feedback approach does not account for these cycles, it severely underestimates the true risk of the oil and gas sector to banks. This is an example where the financial accelerator plays an important role in the stress propagation.

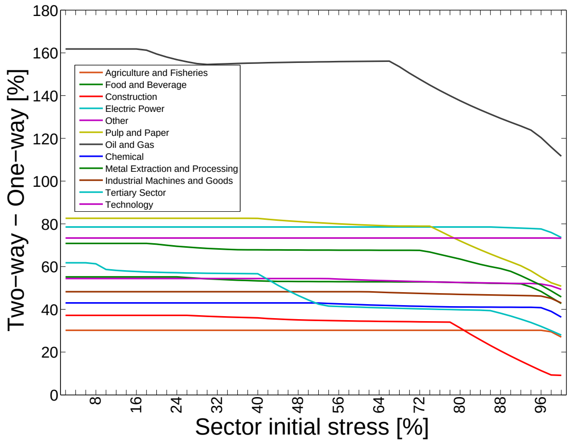
[27]We conduct the same exercise using the right-sided paired-sample t-test and also conclude for the statistical significance of the feedback effects in estimating systemic risk in the real and financial sectors.



***(a)** One-way (firm to bank) feedback*



***(b)** Two-way feedback*

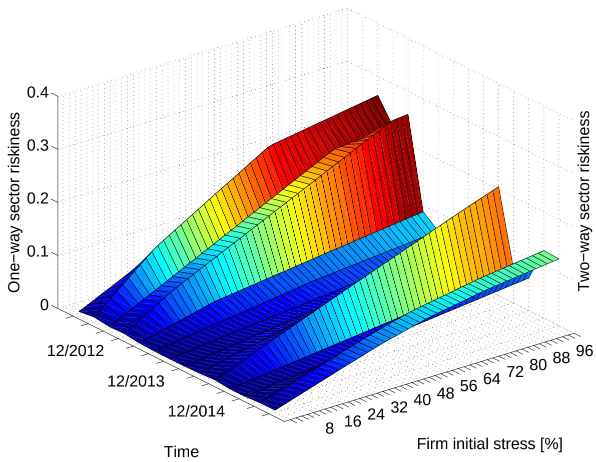


***(c)** Difference between the two- and one-way approaches (in %)*

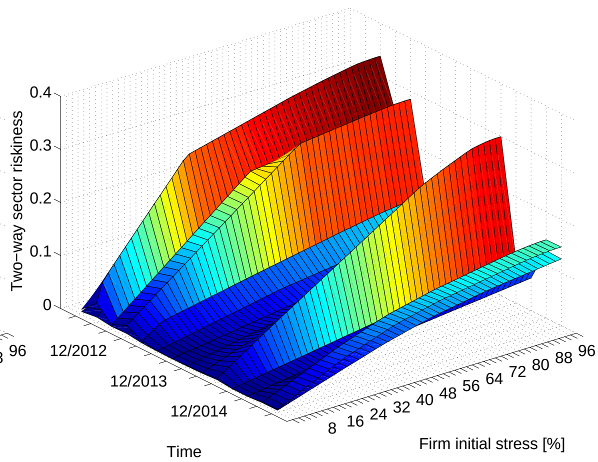
***Figure 18:** Curves showing sector riskiness as a function of both time and the sector initial stress level in June 2014. In (a), we only account for the firm to bank feedback, thus disabling the "bank to firm" contagion channel. In (b), we enable both the "firm to bank" and "bank to firm" contagion channels. In (c), we provide the percentage difference between the two approaches. Observe that the bidirectional feedback is important because it allows for second- and high-order rounds of stress propagation.*

Still looking at the Figs. 18a and 18b, when we solely consider the "firm to bank" contagion channel, the construction sector is the riskiest sector to banks in June 2014 for any combination of partial stress levels. When we add the reversal "bank to firm" contagion channel, thus establishing the financial accelerator engine, the construction sector only remains as the riskiest sector when partial stress levels of sectors are below the 90% mark. After that point, the metal extraction and processing and the food and beverage sectors surpass the construction sector in terms of sector riskiness to banks.

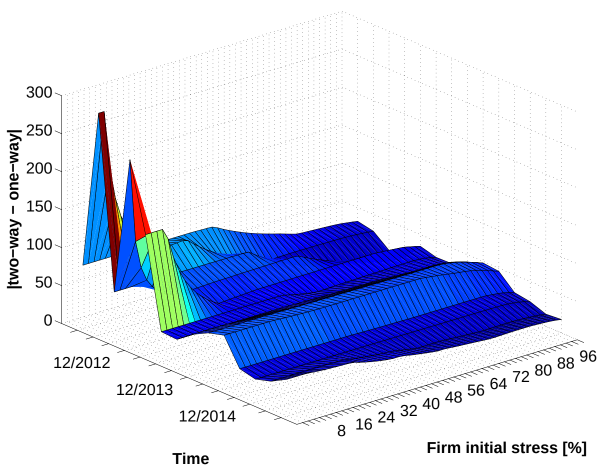
In special, when all of the firms of a same sector default, then the metal extraction and processing sector is the riskiest one. To get a picture of how the riskiness of this sector evolves with respect to different partial stress initial scenarios as a function of time, we depict in Figs. 19a and 19b the surfaces of the riskiness of the metal extraction and processing sector to banks as a function of time and initial stress levels when we consider unidirectional and bidirectional feedback between banks and firms, respectively. Figure 19c portrays the surface representing the percentage difference between the two approaches. We see that the underestimation of sector riskiness to banks can be underestimated up to 300% in some parts of the surface. On average, the underestimation is of 47.76% in the analyzed period for the metal extraction and processing sector. In this way, we can see that, by avoiding the two-way feedback between firms and banks, we may be incurring in large underestimation errors when computing risk-related network measures. Thus, we argue that it becomes essential to model two-way feedback mechanisms between different contagion channels. This task becomes even more crucial in financial networks, in that economic agents are heavily interconnected in a complex way.



(a) One-way (firm to bank) feedback



(b) Two-way feedback



(c) Difference between the two- and one-way approaches (in %)

Figure 19: *Surface showing the riskiness of the metal extraction and processing sector to banks as a function of both time and the sector initial stress level. In (a), we only account for the firm to bank feedback, thus disabling the "bank to firm" contagion channel. In (b), we enable both the "firm to bank" and "bank to firm" contagion channels. In (c), we provide the percentage difference between the two approaches.*

Going back to Figs. 18a and 18b, we attribute the increment in the sector riskiness due to an infinitesimal increase in the sector stress level to the network amplification effect. Define $\mathrm{NAE}(f_s)$, the network amplification effect of sector $s \in \mathscr{S}$ when its stress level is $f_s \in [0,1]$, as:

$$\mathrm{NAE}(f_s) = \frac{\partial}{\partial f_s} r(f_s), \tag{45}$$

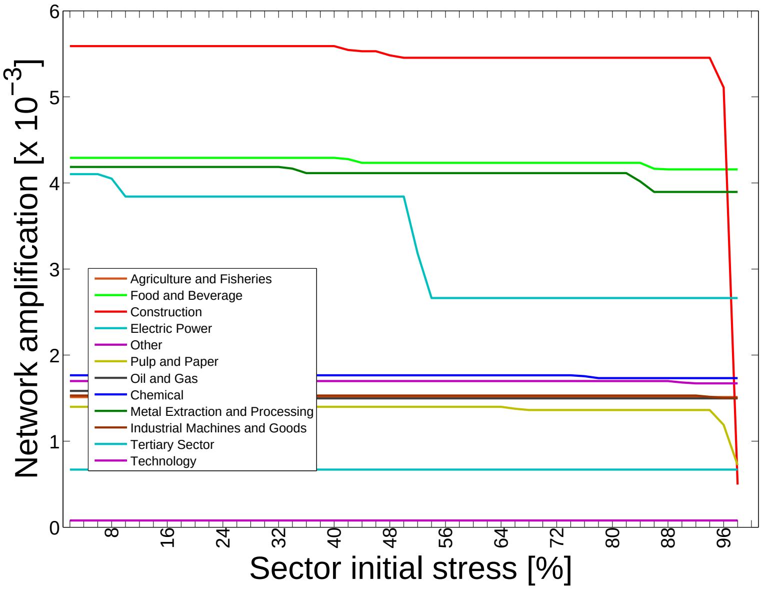
in which $r(f_s)$ denotes the riskiness of sector s to banks when it has a stress level exactly equal to f_s. Note that increments in the stress level of a specific sector s that largely increase riskiness to banks yield a large network amplification effects in view of the large derivative values in (45).

Figures 20a and 20b display the network amplification effect when we consider the one-way and two-way feedback mechanisms, respectively. We can see that the network amplification effects of sectors are roughly non-increasing piecewise linear. One linear region connects to the subsequent one through an adjacent *critical point*. Critical points dampen the network amplification effect as we move far from the graph origin. Consequently, the further from the origin critical points are, the more harmful a small increase in the stress levels of firms of that sector can be.

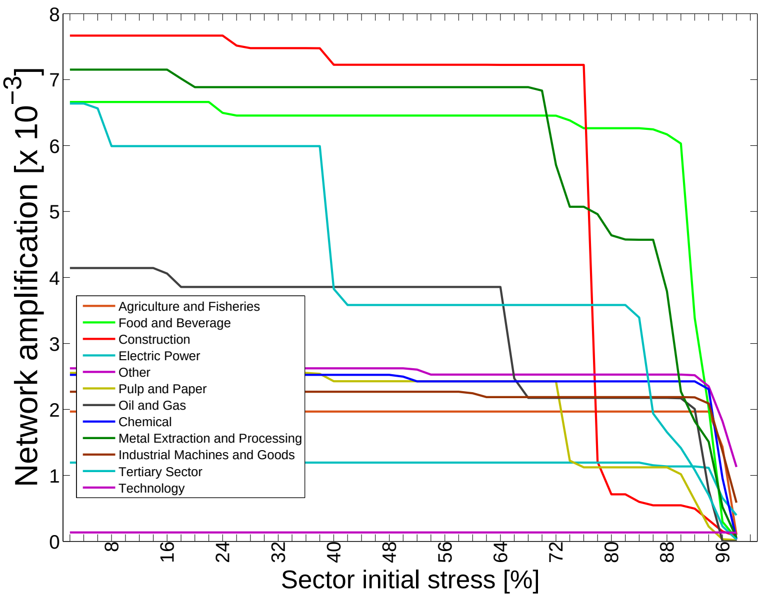
We can check that there are much fewer critical points when we only use the "firm to bank" contagion channel in detriment to the bidirectional "firm to bank" and "bank to firm" contagion channels. In addition, correspondent critical points of latter are shifted to the right in relation to the former.

In the case we enable the bidirectional feedback mechanism, the construction sector has a critical point that accounts for a large drop-off of its network amplification effect at the 75% mark. Similarly, the food and beverage has a large drop-off critical point at the 90%. The metal extraction and processing sector has several critical points that gradually dampen the network amplification effect. In contrast, we see that the technology sector does not have critical points, hence its network amplification effect is constant for different initial sector stress levels.

We can also relate the existence of the critical points in Fig. 20 to phase transition. We can say that there is phase transition for those critical points in which an infinitesimal increase in the initial stress level of firms causes large changes in the network amplification effect. Looking at Fig. 20b, the construction sector shows a perceptible phase transition when its firms have initial stress levels at the surroundings of the 75% mark. In this respect, we see that the network goes from a more stress amplifying behavior to a less stress amplifying behavior as we cross the 75% mark. Our result corroborates Acemoglu et al. (2015b)'s findings that networks exhibit some sort of phase transition that is



***(a)** One-way (firm to bank) feedback*



***(b)** Two-way feedback*

***Figure 20:** Network amplification effect of the sector riskiness to banks when we increment the sector initial stress levels. The network effect is given by the derivative of the sector riskiness to banks with respect to the sector initial stress levels. The curves in (a) and (b) correspond to the derivatives of the curves in Figs. 18a and 18b.*

dependent on the initial shock magnitude they receive. While we uncover the existence of phase transition in the initial shock magnitude to the potential stress that the financial sector suffers, Acemoglu et al. (2015b) study how the magnitude of initial shocks starting from the financial sector relate to the integration and diversification of banks.

3.6 Policy implications

There is an ongoing discussion carried out by the Basel Committee regarding bank's risk-weighting of assets. Studies report a non-negligible variability on bank's regulatory capital ratios. Such variance is not fully explained by differences in the riskiness of banks' portfolios, which weakens the confidence in capital ratios. Efforts to address this issue comprises the revision of standardized approaches for the calculation of risk-weighted assets (RWA), rendering them more risk-sensitive, the revision of the role of internal models, the promotion of greater consistency of disclosure requirements related to RWA, the enhancement of the comparability of RWA calculated by different internal rating-based models and the ongoing monitoring of risk-weighted assets variation (BCBS (2014a, 2015)).

Our study can shed some light on this debate, providing a tool for setting the weights to loans to specific economic sectors. Our econometric model highlights the determinants of sector riskiness (total loans, firm HHI, bank HHI, bank impact diffusion, bank leverage and bank dependency) and enables to assess which sectors are more/less risky to the financial system. Financial regulators can use this information to weigh loans by a higher/lower factor in the composition of the RWA depending on the current riskiness of the network topology.

One important aspect of financial regulation that is often left aside by the literature is of its endogenous nature with respect to the observed network topology. Economic agents make decisions based on what the current regulation permits or forbids. Should the financial regulation change incentives, broaden or reduce the list of in-law possibilities of economic agents, the network topology is expected to change as well to reflect these innovations. Moreover, decisions that would be optimal to regulators in pursuing financial stability, such as to force one economic agent to engage in connections with less risky counterparties may not always induce the desired global properties in the financial system. For instance, even if we force these connections, the collectiveness of all of the economic agents' decisions may still provide network topologies with higher systemic risk levels. Studies that bring together the consequences on network topology and systemic risk as a response to financial regulation are important questions that still lack proper study in the literature.

4 Conclusion

We develop a general framework to estimate systemic risk that accounts for feedback effects between different contagion transmission channels. To the best of our knowledge, this is the first work to recognize and quantify the importance of feedback effects in contagion models. We show that the model has strong theoretical properties, such as the existence of a unique fixed point.

We elaborate on an innovative financial accelerator engine to model the feedback effects between the real and financial sectors by using contagion transmission channels such as loan defaults, bank credit crunches, deposit withdrawals, and deposit defaults. The financial accelerator models the fact that, if a shock hits the real sector, it also generates distress in the financial sector, which then feedbacks into the real sector, and once again bounces back to the financial sector, and so forth.

We illustrate the model using unique data sets from Brazil that have all loans made between banks and all loans made by banks to firms. We find that the feedback effects between the real and financial sectors are economically significant, which imply that models that do not incorporate feedback effects may be significantly underestimating systemic risk.

Our approach is relevant for the design of proper economic policies and financial regulation. Also, it can be used for the design of stress testing and to evaluate the relative importance of a variety of shocks in the economy and the interactions between the real and the financial sector.

Our model is flexible and its extension anchor consists in designing suitable vulnerability matrices to account for new behaviors. In this spirit, there are several ways in which we can extend the model. If an external shock originates in the banking system, firms can trigger bank runs that would exacerbate such shocks. In addition, we may incorporate fire sales and liquidity constraints as new contagion transmission channels.

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Appendix A Impact diffusion influence

This network measure is introduced by Silva et al. (2015). The gross exposure matrix $\mathbf{A}$ is not a representative candidate for evaluating possible contagion routes among pairs of banks, as it does not convey the notion of banks' capability of absorbing impacts coming from their direct exposures. Rather, the exposure matrix $\mathbf{A}$ only numerically quantifies the pairwise exposures in an absolute manner. The vulnerability matrix is a more suitable candidate for risk-related network measurements, because it considers the loss absorbing capabilities of banks.

The impact diffusion influence relies on the truncated vulnerability matrix $\bar{\mathbf{V}}$ that we compute as follows:

$$\bar{\mathbf{V}}_{ij} = \begin{cases} 1, & \text{if } \mathbf{V}_{ij} = 1. \\ 0, & \text{otherwise.} \end{cases} \tag{46}$$

in which $\mathbf{V}_{ij}$ is the usual vulnerability matrix as defined in (3).

The truncated vulnerability network not only provides information of direct contagion, but indirect contagion once we take higher powers of $\bar{\mathbf{V}}$. For instance, the entry $(\bar{\mathbf{V}}^k)_{ij} \in \mathbb{N}$ indicates the quantity of contagion paths of length k that starts from i and are transmitted to j due to high vulnerabilities of other banks in the path.

The impact diffusion influence measures the potential influence exercised by a bank on the diffusion or propagation of impacts in the network. Thus, it gives us a proxy of how harmful is one member of the network to the others. The impact diffusion influence can be seen as a centrality measure of the bank in the network or the overall dependence of other members in the network to a specific bank.

The concept of the impact diffusion influence is built upon the notion of communicability in networks. In this respect, the communicability from i to j is computed as:

$$\mathbf{G}_{ij}(\bar{\mathbf{V}}) \triangleq \frac{1}{s!}\mathbf{P}_{ij} + \sum_{k>s} \frac{1}{k!}(\bar{\mathbf{V}}^k)_{ij} = (e^{\bar{\mathbf{V}}})_{ij}, \tag{47}$$

in which $\mathbf{P}_{ij}$ denotes the number of paths with the shortest length from i to j; s is the length of such paths. The term $\bar{\mathbf{V}}_{ij}^{(k)}$ is the (i, j)-th element of the k^{th} power of matrix $\bar{\mathbf{V}}$, which gives the number of walks of length k from i to j along the truncated vulnerability matrix $\bar{\mathbf{V}}$, where $k > s$. Note here that we are quantifying not only shortest paths between pairwise banks, but also longer paths that contagion routes can materialize from due to lower capital buffers of banks in these peculiar paths. In any case, we are always attenuating the influence of these walks in accordance with their lengths, so as to prefer direct to indirect contagion.

The diffusion influence of a bank i can be understood in terms of the variation it provokes on the communicabilities between all of the participants when i's power of diffusing impacts is removed from the network that is built up from the truncated vulnerability matrix. This can be effectively performed by deleting all of the out-edges emanating from i. This type of filtering transforms i in a sink vertex in the network, for every path that reaches i must end in there.

The reasoning behind that procedure is as follows. If i is responsible for diffusing a significant portion of impact throughout the network, then its removal will reduce the communicability indices of all of the banks. In contrast, if i does not potentially diffuse impact to the network, then the communicability indices will remain unaltered or slightly altered.

In light of that, we define the potential influence that i exerts on diffusing impact to the network as:

$$I_i(\bar{\mathbf{V}}, P^{(\text{value})}) \triangleq \sum_{j \in \mathscr{B}} \sum_{\substack{r \in \mathscr{B} \\ r \neq j}} \left[\mathbf{G}_{jr}(\bar{\mathbf{V}}) - \mathbf{G}_{jr}(\bar{\mathbf{V}}^{(i-)})\right] \cdot P_r^{(\text{value})}, \tag{48}$$

in which $\bar{\mathbf{V}}^{(i-)}$ denotes the modified truncated vulnerability matrix, in which all of the out-edges that emanate from i are removed. $P^{(\text{value})}$ is a proxy for the value or importance of all of the banks in the market. In this work, we proxy the banks' importance as the total liabilities they hold inside the interbank network. The factor $\left[\mathbf{G}_{jr}(\bar{\mathbf{V}}) - \mathbf{G}_{jr}(\bar{\mathbf{V}}^{(i-)})\right]$ indicates the communicability index of walks from j to r that visit i. This term is evaluated by first computing the communicability index of j to r in the original truncated vulnerability network. From that, we subtract the fraction of that communicability that is not due to a path that has i along the way. Consequently, Equation (48) effectively quantifies the shortfall, which is weighted by the bank's importance, occurred in the network when i's power of diffusing impacts is disabled.