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Commercial Platforms With Heterogeneous Participants*

Gabriel Garber**

Márcio Issao Nakane***

Abstract

We study two-sided markets where there are buyers and sellers, with heterogeneous participants on each side. Buyers care about the quality of the good purchased, but sellers care only about the price they get. When there is informational asymmetry about types between the sides, the role of a platform as a certifier that guarantees a minimum quality becomes central to the transactions. We analyze first-best (perfect information) and pooling equilibria without platforms and a monopolist platform that coexists with an external pooling. We also show there is no equilibrium in a simultaneous game with two platforms.

Keywords: platforms, two-sided markets, heterogeneous agents, certification

JEL Classification: D42, D43, D47, D82, D85, L12, L13, L15, L81

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1 Introduction

In two sided markets[1], the generation of value for participants requires the presence of groups from each side[2]. Commonly, the interaction between these groups is favored, or even made possible, by a firm, called a platform.

The organization of a market by one or more platforms occurs because they reduce transaction costs arising from the interaction of agents, which makes some activities central to the platform business[3]. Price setting is probably the aspect that has received the largest amount of attention from the economics literature, given its particularities when compared with one-sided markets and the main objective of obtaining adequate participation of each group. The presence of many participants in one side increases the probability that an agent from the other side is able to take part in a transaction, thus making the platform more attractive to him. It is common that one side pays a price smaller than the marginal cost, which can be zero or negative[4], even in the absence of predatory price practices. That is an important result concerning the regulation of these markets[5].

The design of the environment, both regarding rules and the form of interaction, is also essential to platforms, and may take different forms, according to the type of business in which they operate. Examples of rules range from the prohibition that a merchant turns down any card of a brand he is acquired to accept to the mandatory dress code of a nightclub. As for interaction design, direct examples are websites and shopping centers, but it also includes elements like hiding magazine indexes in order to make readers browse through more advertisements.

Finally, there is the aspect we focus on in this paper: platforms perform participant certification in varying degrees. That activity may be as simple as identifying participants

[1] Analogously, there may be markets with more sides, with the participation of the corresponding number of agent groups.
[2] We call the agents that use platforms “clients” or “participants”. When we say simply “agents”, we mean all the elements that compose the two sides of the market.
[3] See Evans and Schmalensee (2005) for a detailed explanation of these tasks and most of the examples used in this introduction.
[4] Real examples of this are widespread: nightclubs that admit women free of charge in certain days of the week, broadcast television, newspapers that charge exclusively advertisers or readers, online commerce and real estate agencies that charge only sellers, and payment cards without maintenance fees and cashback or other rewards for cardholders.
[5] See, for example, Wright (2004b).

using personal information[6] or go as far as rejecting clients who do not have a certain income level, as was the case in the traditional American Express credit card model. These certification processes are intended to avoid the access of participants who, according to the business model, do not qualify to belong in the portfolio of partners the platform intends to offer. For example, for online commerce, credit cards acquirers spend resources to verify real existence of merchants and the type of products they sell, in order to reduce the risk that cardholders become victims of fraud or make transactions with firms that also perform illegal businesses, like trafficking weapons. On the other hand, in American Express credit card model, the objective is to build a set of cardholders with high income, who are particularly interesting for stores to have as customers.

In addition, an issue related to certification is the history of participants' interactions. Generally, platform participants take part in many interactions, although they do not necessarily occur with the same partners. Thus, the platform has an advantage to collect information about participants. It is common for platforms to gather opinions from clients and build ratings. That is the case of E-bay's feedback score or Easy Taxi app rating system.

Finally, the punishment and, eventually, the exclusion of participants who do not respect the platform rules may also be interpreted as a sort of certification, given that the result is a set of possible partners whose probability of behaving in the expected manner is higher. One example of that are punishments applied by Google (exclusion or being sent to the end of the list) on sites that try to fool its search by displaying words unrelated to the featured content. Health insurance companies also may exclude doctors following insured patient's complaints. Feedback processes also reduce the potential advantage of a seller in frustrating buyers' expectations.

These issues receive platforms' attention because they affect directly how attractive they are for their clients. When an agent chooses to participate, he affects the utility that agents on the other side of the market expect to obtain by also participating. This effect, called network externality, may be positive (e.g. the increase of acquired merchant group in a payment card scheme improves the utility of cardholders) or negative (e.g. when the magazines we read bother us with an excessive amount of advertisement).

[6] For example, an online dating service may simply require participants to show they are real people. It may also offer extra certification (and thus signaling), for a certain fee.

The largest share of the literature concentrates on network externality stemming from the size of the platform, i.e. the number of participants on each side. However, if a single agent has the capacity of making the platform more attractive by participating, he is said to have a high network value[7].

Although the heterogeneity of network value among participants is a theme of great relevance for the functioning of platforms, very few authors have analyzed it from an economics perspective. This paper aims to understand its implications, regarding allocations and relevant strategies, for monopoly and simultaneous duopoly markets in which the incentives for buyers and sellers are well characterized and thus different.

In our main setting, business may occur inside a monopolist certifying platform or outside of it, without certification. We find the platform allows the participation of the sellers with quality above a threshold, but does not necessarily provide the incentives for the highest quality sellers (who would opt out of the market if signaling quality was impossible) to participate. This is because in many settings the platform appropriates most of the gain of certification, charging a high participation fee. In other settings, a group of intermediate quality sellers choose not to participate, because they are not "good enough" to enter the platform and not "bad enough" to join the outside uncertified pooling. We also analyze the cases with perfect information and asymmetric information between platforms and the case with two platforms playing simultaneously. In the latter case, we show that there is no equilibrium.

We review the relevant literature in section 2. In section 3 we lay out the model we use to analyze the issue, focusing on the asymmetry of incentives between buyers and sellers. Section 4 concludes. All the demonstrations and some variations are displayed in the Appendix.

[7] Although network externalities between groups are a distinguishing characteristic of two-sided markets, the difficulty of observing them empirically in some cases produced the understanding that in markets deemed "mature", in which there is large participation of each group, they might disappear as marginal effects. Thus, they are not a requirement to define a market as two-sided. A more comprehensive definition is offered by Rochet and Tirole (2006): "A market is two-sided if the platform can affect the volume of transactions by charging more to one side of the market and reducing the price paid by the other side by an equal amount; in other words, the price structure matters, and platforms must design it so as to bring both sides on board.(…)".

2 Literature

Several papers employ heterogeneous agents to model platforms' clients. That device, however, is more frequently used only to setup supply and demand, instead of serving the purpose of studying quality certification. Wright (2004a), for example, builds a model in which there is a continuum of consumers and industries. Consumers shop from every industry of the economy and can choose, in each of them, between two merchants, as in a Hotelling linear city setup. Merchants may choose whether to accept payment cards or not. A model along these lines is one of the setups used also by Hayashi (2006). Heterogeneity serves only to capture taste for partners, who may participate in the platform or not.

However, two pairs of authors have focused on the importance of client heterogeneity to determine platform strategies and agents' participation.

The first of these pairs, Caillaud and Jullien (2001), highlight the fact that asymmetry of network externalities between the two groups of agents generates the possibility that the platform subsidizes one side of the market in order to increase its attractiveness for the other. In their model, each side of the market is composed of a continuum of agents. Each individual can only transact with a given partner from the other side. The platform can observe the relevant information and connect participants to their pairs, if both of them participate. *Ex ante*, the chance of meeting this "ideal" partner in a platform is unrelated to the specific value of the heterogeneous characteristic. It depends only on the proportion of agents that participate in it. The authors study the cases of monopoly and of competition between platforms.

A fundamental fact towards which that paper draws attention is the possibility of coordination failures, given that the participation decision (and, according to the context, the platform choice) depends not only on prices, but also on each agent's beliefs about participation of the other side. For example, when prices are nonnegative, there is always an equilibrium called pessimistic, in which every agent believes the participation of the other side will be null, and therefore no one participates. By the same token, when there is competition between platforms, they use negative prices to create dominant strategies for one side, therefore making participation in it the only consistent belief for agents on the other side. That kind of strategy, which the authors call “divide-and-conquer” in Caillaud and Jullien (2003), will play a major role in our duopoly section.

According to Caillaud and Jullien (2003) informational intermediation consists of services such as search, certification, advertising, and price discovery. These tasks relate to the quantity of participants on each side of the market and to the available variety of potential partners as well. In a similar environment as the 2001 paper, the authors also find the platform strategy of subsidizing one side to obtain payoff from the other. One drawback of the framework pointed by the authors is the disregarding of the possibility that final clients understand prices as quality signals.

Our purpose, in this paper, is to understand the consequences of allowing platforms to select the clients they allow to participate, according to their network value. In the model we will present, there are several possible partners for each participant, but the value generated for the buyer depends on the quality of the seller. Evans and Schmalensee (2005) claim that platforms may find it optimal to limit their own size and pre-select clients from both sides in order to increase the probability of matching. We show a phenomenon similar to this.

The second pair of authors we mention, Damiano and Li (2008), build a model in which there are two platforms. Inside each of them, a pair is randomly drawn for each participant. That eliminates the size effect, i.e. agents do not care about how many participants are on the other side, just the expected quality of their partner. Each one of the symmetrical sides of the market consists of agents whose one-dimensional quality is private information.

A participant derives utility from the multiplication of her quality by the quality of her pair, less the participation price. Since the own quality affects the agent’s payoff, participation prices cause them to self-select: an agent with a very low quality will not find it attractive to pay a relatively large sum to participate, even if he expects to find a high quality partner[8]. Reservation utility is zero for all types. In case a participant is not

[8] Damiano and Li (2007) use the same mechanism to study the case of a monopolist online dating platform that uses different prices to make agents self-select into exclusive dating places.

assigned a partner, his payoff will be null, but that possibility is not analyzed in detail, since only symmetric equilibria are considered.

In the game from that paper, platforms choose price and disclose it to potential participants, so they can choose in which platform to participate, if in any. In the solution, by backward induction, two "strategies" are outlined for platforms: overtaking, which consists in excluding the rival from the market by charging an adequately higher price, attracting the higher quality participants, and undercutting, that excludes the rival by charging a sufficiently lower price. There is no equilibrium when platforms choose prices simultaneously. The paper offers, alternatively, an equilibrium for the sequential version of the game, in which a firm sets prices before the other.

In the model we study in this paper, buyers payoff have a similar structure to the one for participants in Damiano and Li (2008), but sellers do not care about the partner they get. They are only affected by the price they receive. That breaks the self-selection mechanism, bringing the model closer to a commercial platform.

3 The model

In this section, we develop a commercial platform model with heterogeneous agents, whose roles as buyers or sellers are well defined. Their payoffs are laid out in subsection 3.1. In subsection 3.2, we study how the market would function in the case of perfect information. From subsection 3.3 on, we analyze cases with asymmetric information about the quality of goods or services sold, starting with a pooling equilibrium, in which there is no platform present in the market. In subsection 3.4, platforms enter the model, first as a monopolist platform that coexists with the possibility of a pooling of nonparticipants, and afterwards as two competing platforms.

3.1 Basic structure

We model a market as two different sets of participants: buyers and sellers. Each seller has a unit of a good and each buyer may purchase one unit.

Sellers are identified by a one-dimensional characteristic, s[9], of the good that they have for sale. According to the context, this characteristic, which represents the quality of the good, may reflect a merchant's ability, his network value, etc. The distribution of s is uniform over the support $[0,1]$. The opportunity cost of selling a good of quality s is αs, indicating that the utility of a seller who chooses not to take part in this market is increasing in the quality of his good[10].

Therefore, the net utility of a seller who owns good s is given by:

$$U_s = \begin{cases} P_s - \alpha s, & \text{if the good is sold} \\ 0, & \text{otherwise} \end{cases}$$

On the buyers' side, there is also a continuum of agents, identified by the marginal utility of the quality of the good they consume, represented by a, also uniformly distributed on interval $[0,1]$. The marginal disutility of paying for the good is increasing in its price.

Specifically, we represent the buyer's utility by:

$$U_a = \begin{cases} as - P_s^2, & \text{if the good is consumed} \\ 0, & \text{otherwise} \end{cases}$$

[9] We use indistinctively s to refer to the quality of the good as well as to the seller to whom that quality corresponds.
[10] For example, in a payment card scheme, a seller with high network value is one with whom many buyers would like to trade. In case such a seller decides to stop accepting cards from a certain scheme, she will probably end up losing much less clients than a low network value seller would lose.

3.2 First best equilibrium

In this subsection, we study the case in which the information about the sellers' quality is public and we characterize the allocation and prices that would prevail in a competitive context. By competitive, we mean that buyers and sellers, aware of the qualities of transacted products, could freely make proposals to their potential partners. One consequence of that is that an equilibrium allocation must be such that, given prices and partners of each agent, no one of them could make an offer to someone on the other side that would result in mutual benefit

Although this is a perfect information problem, the solution strategy resembles a screening problem, since individual rationality and incentive compatibility constraints must be respected.

Using this sort of non-arbitrage condition, in first place, note that on the supply side, equilibrium prices must be increasing in quality. On the demand side, take two individuals a_0 and a_1 such that $a_1 > a_0$, who are buying at some equilibrium allocation, respectively, s_0 at price P_0 and s_1 at price P_1. Then, the buyers' incentive compatibility constraints are:

$$a_0 s_0 - P_0^2 \geq a_0 s_1 - P_1^2 \qquad \text{(IC0)}$$

$$a_1 s_1 - P_1^2 \geq a_1 s_0 - P_0^2 \qquad \text{(IC1)}$$

The single crossing condition (i.e. the fact that values of a are different) implies that one of the ICs must have a strict inequality. Thus, switching sides in IC0 and subtracting it from IC1, we have that:

$$(a_1 - a_0)(s_1 - s_0) > 0 \qquad \text{(FB1)}$$

Therefore, $a_1 > a_0$ implies $s_1 > s_0$ and a function $a(s)$, defined to connect each seller to its correspondent buyer, must be increasing. Assuming (temporarily) all individual rationality constraints are satisfied, given the symmetry of members of both sides, the result would be that $a = s$ in an equilibrium resembling a competitive one[11].

We would like to define a function $P(s)$, which tells the price of every good sold. Such a function should be continuous within a continuous interval of transacting sellers[12]. Take again IC1. Terms may be reorganized to obtain:

$$a_1 \geq \frac{P_1^2 - P_0^2}{s_1 - s_0}$$

However, since the case under analysis has an infinite number of types in a continuous interval, this expression must hold with equality in the limit as $s_0 \to s_1$, with P_0 correspondingly converging to P_1. Using function $P(s)$, we may write:

$$a_1 = \lim_{s_0 \to s_1} \frac{P^2(s_1) - P^2(s_0)}{s_1 - s_0}$$

That defines the derivative for $P(s)$. Using the fact that this equation would also be valid had we chosen $a_0 > a_1$, we have:

$$a_1 = 2P(s_1)P'(s_1)$$

Substituting $a = s$ and integrating both sides, we obtain:

$$\frac{s^2}{2} = P^2 + C$$

Where C is a constant with a value yet to be defined. Given the participation of type $s = 0$, the result is $C = 0$. Thus:

$$P(s) = s/\sqrt{2}$$

[11] In terms of Damiano and Li (2008) and Damiano and Li (2007), this matching maximizes the values generated by interactions. In those papers, this feature results from the complementarity between qualities in agents' payoff functions. Here, this characteristic lays entirely on the buyers' side.

[12] To see that $P(s)$ should be continuous within a continuous interval of transacting sellers, suppose a seller $\hat{s}$ interior to a continuous interval of transacting sellers. Suppose further that $\lim_{s \to \hat{s}-} P(s) = \hat{P}$ and $\lim_{s \to \hat{s}+} P(s) = \hat{P} + \Delta$, with $\Delta > 0$. Then, a seller arbitrarily close to the left of $\hat{s}$ could benefit from offering his product to the partner of a seller arbitrarily close to $\hat{s}$ from the right side, since the variation of quality would be infinitesimal and the variation of price discrete. That situation violates the equilibrium necessary condition of no arbitrage.

Regarding individual rationality restrictions, we must check the participation of buyers and sellers.

Returning to the analysis with indexes, buyers participation requires that:

$$a_0 s_0 - P_0^2 \geq 0 \quad \text{(IR0)}$$

$$a_1 s_1 - P_1^2 \geq 0 \quad \text{(IR1)}$$

As usual, IR0 and IC1 imply IR1. Hence, from all types, the only individual rationality constraint that may be binding is the one forthe lowest quality participating type, which we have already imposed.

On the supply side, participation will be guaranteed if $\alpha \leq 1/\sqrt{2}$. Cases with $\alpha > 1/\sqrt{2}$ are consistent with equilibrium situations in which the "worst" sellers and the "best" buyers participate.

We show the complete analysis in the Appendix. There, we also analyze equilibrium with a constant marginal disutility for the price paid. In that case, there is a threshold for α up from which no transactions occur.

3.3 Pooling Equilibrium

Let's consider now a case where a seller's quality is private information and transactions occur at a unique price P_p.

All sellers for whom $P_P - \alpha s \geq 0$ will participate in that market. Define s_p as the s for which $P_P - \alpha s_P = 0$. The value of s_p is also equivalent to the amount of participating sellers. Therefore, we have a supply curve:

$$s_P = P_P/\alpha$$

On the buyers' side, there will be participation of types for which $aE\left[s \;/\; P_p\right] - P_P^2 \geq 0$, where $E\left[s \;/\; P_p\right]$ is the expected quality for a participating seller, given price P_P.

Given the uniform distribution of s and the supply curve, $\left[s \;/\; P_p\right] = P_P/2\alpha$, the buyers' participation constraint therefore is:

$$\frac{aP_P}{2\alpha} - P_P^2 \geq 0$$

Or, considering $P_P > 0$, $\frac{a}{2\alpha} - P_P \geq 0$. Defining a_P as the buyer for whom this condition is observed with equality, and considering the uniform distribution of buyers, we may write a demand curve (modified in order to incorporate expected seller quality) as:

$$1 - a_P = 1 - 2\alpha P_P$$

where the left-hand side is the quantity of buyers.

Equilibrium requires that $s_P = 1 - a_P$, therefore:

$$P_P = \alpha/(1 + 2\alpha^2)$$

$$s_P = 1/(1 + 2\alpha^2)$$

$$a_P = 2\alpha^2/(1+2\alpha^2)$$

As it is easy to verify, the increase of opportunity cost α reduces the number of transactions. Additionally, $\alpha = 1/\sqrt{2}$, which is the maximum cost to allow full participation in the case of perfect information, determines $s_P = a_P = 1/2$ in the pooling equilibrium. That means that the lowest quality half of the buyers and the highest quality half of the sellers are excluded from the market when α has that value.

3.4 Market Structures with Platforms

3.4.1 Monopoly

Now we introduce an agent capable of observing the quality of the sellers and selecting those for whom it allows participation in a separate environment, a two-sided platform. Inside it, buyers and sellers form pairs randomly. We call the mean quality of a seller inside the platform m_m.

The platform will choose a participation price for sellers, p_m. Buyers do not pay (directly) for participation, disbursing only price P_m , for the purchase of the good. The platform admits an equal number of buyers and sellers[13,14] and its cost is given by o per seller, coming from the certification process, i.e., quality verification.

Buyers and sellers who do not participate in the platform can either make transactions without certification or simply opt out of the market as a whole. Exchange

[13] We postpone the discussion of how this happens to the next subsection, since it would introduce unnecessary complexity at this point. We will show that this requirement together with the platform decision variables determine P_m.

[14] With this, if buyers also paid for participation there would be neutrality of the platform's price structure, i.e. the situation in which only the sellers' pay for participation is equivalent to another one where both sides pay, (considering reductions in the purchase price of the good and participation price of the sellers equal to the participation price of the buyers). That neutrality, which would cause the market not to match the two-sided market definition from Rochet e Tirole (2006) is actually not present given infinitesimal costs that we explain in the next section. The case of duopoly is clearly two sided from the outset.

outside the platform occurs in what we now call external pooling, at price P_{-m}. We call the corresponding mean quality m_{-m}.

Therefore, the seller's payoff is:

$$U_s = \begin{cases} P_m - p_m - \alpha s, \text{ if he participates in the platform} \\ P_{-m} - \alpha s, \text{ if he makes a transaction out of the platform} \\ 0 \text{ , if he does not make any transaction} \end{cases}$$

The expected payoff of the buyer is:

$$U_a = \begin{cases} am_m - P_m^2, \text{ if he participates in the platform} \\ am_{-m} - P_{-m}^2, \text{ if he makes a transaction out of the platform} \\ 0 \text{ , if he does not make any transaction} \end{cases}$$

We use the term "buyers and sellers allocation" to refer to the sets of buyers and sellers that participate in the platform, in the external pooling or opt out of the market. We are interested in characterizing this allocation and the prices associated with equilibrium situations, since a complete description of the strategies involved is quite complex and out of the scope of our analysis.

The approach we employ to find an equilibrium allocation is to evidence a number of necessary no-regret conditions in order to restrict the set of situations under analysis before we proceed to define the monopolist's optimal choice, since the platform plays in a stage prior to buyers' and sellers' move.

Necessary Condition 1 – Rationality comparing platform and external pooling: $m_m \geq m_{-m}$, and $P_m \geq P_{-m}$ (with strict inequalities if $p_m > 0$)

Proof.: In equilibrium, a seller must not envy the situation of another if that situation is obtainable by him under the rules of the game. Thus, given that external pooling is available to all sellers, one who transacts inside the platform will not be able to attain a higher payoff by selling out of it. That requires $P_m - p_m \geq P_{-m}$, which implies $P_m \geq P_{-m}$. On the buyers' side, this means that $m_m \geq m_{-m}$, establishing that the platform must operate with a set of sellers with mean quality at least as high as the one of sellers working out of it. With a strictly positive p_m, these inequalities will be strict. □

Necessary Condition 2 – Continuity of participants' sets: The buyers in $]a_m, 1]$ and the sellers in $]s_L, s_H[$ will participate in the platform, where a_m and s_H are thresholds to be defined by equilibrium conditions and s_L is platform choice. Furthermore, sellers in $]s_H, 1[$ will not participate in the trade.

Proof: First, suppose a seller s_1 who participates in the platform. Then, we have that
$P_m - p_m - \alpha s_1 \geq P_{-m} - \alpha s_1$ and $P_m - p_m - \alpha s_1 \geq 0$. That means that for some $s_2 > s_1$, $P_m - p_m - \alpha s_2 \geq P_{-m} - \alpha s_2$. On the other hand, $P_m - p_m - \alpha s_2$ does not have a defined sign. Thus, if a seller participates in the platform, those with higher quality than him, may either prefer to participate too or not transact, but they would never choose the external pooling, given the choice. On its turn, take a seller $s_0 < s_1$. We know that $P_m - p_m - \alpha s_0 \geq P_{-m} - \alpha s_0$ and $P_m - p_m - \alpha s_0 > 0$. Therefore, a seller with lower quality than s_1 prefers participating too, weakly over the external pooling and strictly over not transacting.

On the buyers' side, if a certain a_1 participates in the platform, we have $a_1 m_m - P_m^2 \geq a_1 m_{-m} - P_{-m}^2$ and $a_1 m_m - P_m^2 \geq 0$. Considering some $a_2 > a_1$ and $m_m > m_{-m}$ (i.e. eliminating just the possibility of $p_m = 0$), both inequalities become strict. Since the platform does not select buyers according to their heterogeneous characteristic, that implies that all buyers with a quality superior to a threshold a_m will participate in the platform.

From the platform's viewpoint, starting from some set of participating sellers, the substitution of a subset of them for one with equal size but superior mean quality, brings the possibility of increases in P_m and p_m, with increase in profits while keeping buyers at least indifferent to the initial situation. Therefore, it will never be optimal for the platform to simultaneously allow the participation of some s_1 and forbid it for a seller $s_2 > s_1$. As a result, the platform must only define an inferior quality threshold, s_L, taking into account that a superior threshold s_H will result from prices P_me p_m.

The sellers belonging to $]s_L, s_H[$ participate in the platform. Under the uniform distribution of s, that implies $m_m = \frac{s_H+s_L}{2}$. The sellers belonging to $]s_H, 1[$ choose not to transact[15]. □

Necessary Condition 3 – Platform equilibrium: $a_m = 1 - (s_H - s_L)$ (NC3)

Proof: It follows directly by imposing that supply equals demand, NC2 and uniform distributions of a and s. □

Since s_H may be smaller than one, the external pooling is solved with more "potential" buyers than sellers.

Necessary Condition 4 – Frontiers of the external pooling: Sellers in $[0, s_{-m}[$ and buyers in $[a_{-m}, a_m[$ will trade in the external pooling, where $s_{-m} = Min\left\{\frac{P_{-m}}{\alpha}; s_L\right\}$ and $a_{-m} = \frac{P^2_{-m}}{m_{-m}}$.

[15] There is the possibility that $s_H = 1$ and that he participates in the platform.

Proof: It follows from the individual rationality constraints of buyer and sellers of qualities below those of the platform participants. In particular, for a_{-m} we have that:

$$a_{-m}m_{-m} - P_{-m}^2 = 0 \qquad \text{(NC4)}$$

Necessary Condition 5 – Equilibrium in the external pooling: $a_m - a_{-m} = s_{-m}$ (NC5)

Proof: It follows directly by imposing supply equals demand and uniform distributions of a and s. □

General scheme of buyers and sellers allocations

In Figure 1, we represent the necessary conditions 1 through 5. It helps visualizing spaces occupied by the platform and by the external pooling, and noticing the necessary conditions that follow below.

It also highlights that, while low quality buyers and high quality sellers tend to be excluded from trade, which might be expected, there is the possibility that some intermediate quality sellers, with $s_{-m} < s < s_L$, are excluded too.

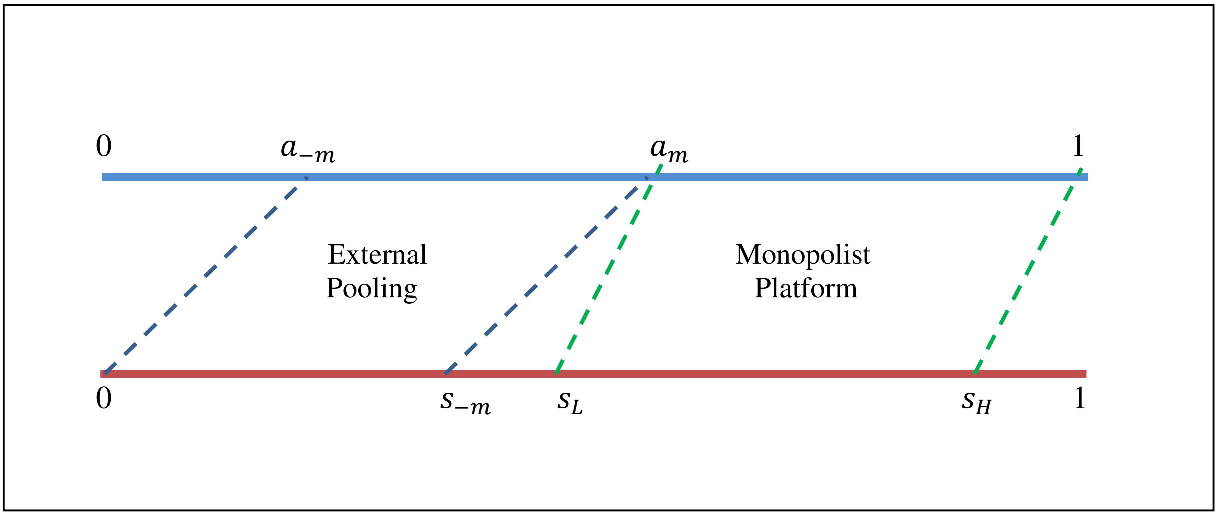


Figure 1 – Monopolist Platform: Equilibrium

Necessary Condition 6 – Indifference of a_m: $a_m m_m - P_m^2 = a_m m_m - P_{-m}^2$ (NC6)

Proof: For a_m, the incentive compatibility constraint implies that $a_m m_m - P_m^2 \geq a_m m_m - P_{-m}^2$. However, if the inequality was strict, by continuity there would be a positive mass of buyers with $a < a_m$, outside the platform, for whom it would also be strictly preferable to transact in it. Thus, the restriction must hold with equality. Since $m_m > m_{-m}$, it is warranted that any buyer with $a < a_m$ will strictly prefer the external pooling to the platform, with the reverse holding for $a > a_m$. □

Given the necessary conditions for equilibrium, it is useful to separate the solution in cases, according to whether the inequality constraints are binding or not. In order to do that, we analyze constraints $s_{-m} \leq s_L$ and $s_H \leq 1$.

Constraint $s_{-m} \leq s_L$:

As Figure 1 shows, s_{-m} is limited from above by s_L. This constraint being binding or not alters the solution of the external pooling, resulting in different strategies for the monopolist. That is to say, ultimately, the monopolist is choosing between the best option with $s_{-m} < s_L$ and the best one with $s_{-m} = s_L$.

For s_L, we know that $P_m - p_m - \alpha s_L \geq P_{-m} - \alpha s_L$ and $P_m - p_m - \alpha s_L \geq 0$. Once defined the ordering between $P_{-m} - \alpha s_L$ and 0, one of the inequalities will imply the other. If $P_{-m} - \alpha s_L > 0$, only the first inequality needs to be taken into account. In that case, all sellers with $s < s_L$ will trade in the external pooling, i.e. $s_{-m} = s_L$. On the other hand, if $P_{-m} - \alpha s_L < 0$, we must take into account the second inequality and we will find that $s_{-m} < s_L$. The case where $P_{-m} - \alpha s_L = 0$ is borderline.

Solution of the external pooling if $s_{-m} < s_L$:

$$P_{-m} = \frac{\alpha a_m}{1+2\alpha^2} \quad \text{(EP1a)}$$

$$s_{-m} = \frac{a_m}{1+2\alpha} \quad \text{(EP2a)}$$

$$a_{-m} = \frac{2\alpha^2 a_m}{1+2\alpha^2} \quad \text{(EP3a)}$$

Proof: In this configuration, s_{-m} must be indifferent between participating or not, i.e., $P_{-m} = \alpha s_{-m}$. That means that $m_{-m} = \frac{s_{-m}}{2} = \frac{P_{-m}}{2\alpha}$. Substituting this condition into (NC4) and using $P_{-m} > 0$, if there is a nonzero mass of sellers in the external pooling:

$$\frac{a_{-m}}{2\alpha} - P_{-m} = 0 \quad \text{(NC6)}$$

Using this equation jointly with (NC5 old M3), after some manipulation, we obtain the stated equations. □

These results replicate the ones in the previous section if $a_m = 1$. The value of a_m will be indirectly set by the monopolist when he chooses s_L and p_m.

Solution of the external pooling if $s_{-m} = s_L$:

$$P_{-m} = \sqrt{\frac{(a_m - s_L)s_L}{2}} \quad \text{(EP1b)}$$

$$s_{-m} = s_L \quad \text{(EP2b)}$$

$$a_{-m} = a_m - s_L \quad \text{(EP3b)}$$

Proof: In this situation, the individual rationality constraint of s_{-m} in the external pooling may not be binding, which is warranted by $P_{-m} - \alpha s_L > 0$. Using $m_{-m} = \frac{s_L}{2}$, (M2) and (M3) we obtain the stated equations. □

Constraint $s_H \leq 1$: $(P_m - p_m - \alpha s_H = 0)(1 - s_H) = 0$

Proof: We know that $P_m - p_m - \alpha s_H \geq P_{-m} - \alpha s_H$ and $P_m - p_m - \alpha s_H \geq 0$. The first inequality is guaranteed, given $P_m - p_m \geq P_{-m}$. Since it is not convenient to the platform to forbid the participation of a seller with quality superior to all its participants, we have that s_H will either be indifferent between participating in the platform and not trading ($P_m - p_m - \alpha s_H = 0$) or $s_H = 1$ (in this case s_H might strictly prefer the platform, for all we have stated so far). □

This constraint, unlike the imposition of $s_{-m} = s_L$, is simply an exogenous restriction to the problem, and not a matter of monopolist strategy.

Discussion of monopolist's strategies:

It is not possible, in equilibrium, that both s_L and s_H strictly prefer trading in the platform to the other two options, because that would imply a suboptimal monopolist profit, which could be increased by raising p_m without any transaction loss. In other terms, if in the second stage of the game there is a subgame equilibrium with all sellers who participate in the platform getting strictly higher payoffs than they could get outside of it, there would also be a subgame equilibrium for the second stage with the same number of transactions if p_m was infinitesimally higher.

Therefore, it is necessary that one of the following situations happen for an allocation to be an equilibrium:

i. s_H strictly prefers participation in the platform to not trading. So does s_L, but he is indifferent between the platform and the external pooling;
ii. s_H is indifferent between participating in the platform and not trading and
    a. s_L strictly prefers participating in the platform to not participating and this option to the external pooling,($s_L > s_{-m}$) or;
    b. s_L strictly prefers participating in the platform to not participating, weakly prefers the platform to the external pooling and that option to not trading. ($s_L = s_{-m}$).

Statement 1 – No allocation in Case I may support an equilibrium

Proof. Since $P_m - p_m - \alpha s_H > 0$, we have that $s_H = 1$. From s_L's preferences, we know that $P_m - p_m - \alpha s_L = P_{-m} - \alpha s_L$, thus $P_m - p_m = P_{-m}$, so all sellers wish to trade and are indifferent between the two possible ways of doing it. The result is that $P_{-m} > 0$ and market equilibrium requires participation of all buyers, constrains which cannot be simultaneously met. □

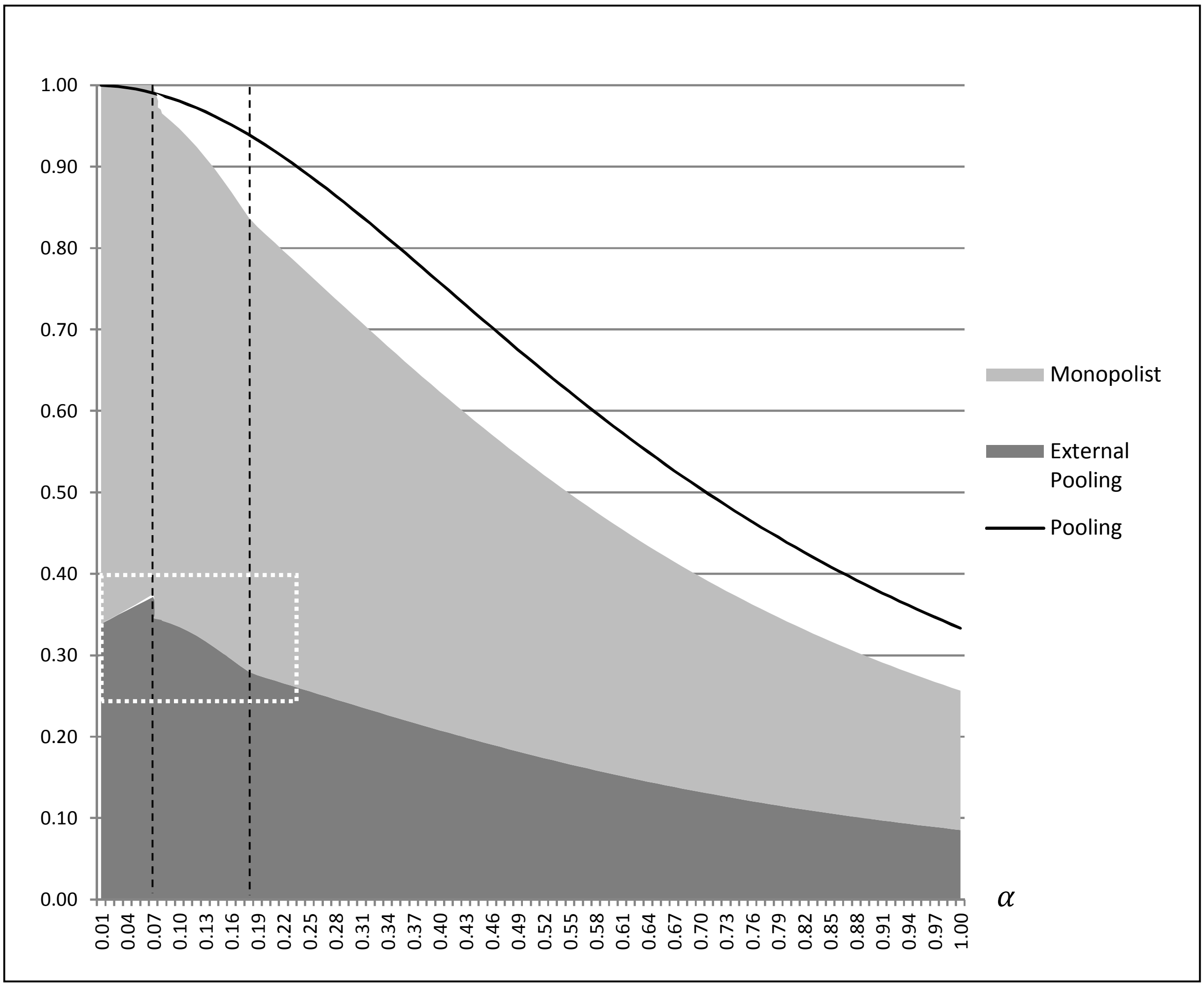
Case ii:

Cases ii-a and ii-b are solved using for the external pooling, respectively, equations (EP1a) through (EP3a) and (EP1b) through (EP3b). There are parameter value combinations for which case ii-a is not possible, given that constraint $s_L \geq s_{-m}$ would be violated, indicating that an equilibrium must be sought in case ii-b. In all other situations, the profits in both cases must be compared, for each combination of (α, o), considering in each case the optimal monopolist choices for p_m and s_L.

We show the calculations in the Appendix. Note that it is necessary to separate, within case ii-a, the situations in which $s_H = 1$, and within case ii-b, the ones that require $P_m - p_m = P_{-m}$.

Results – example with $o = 0$

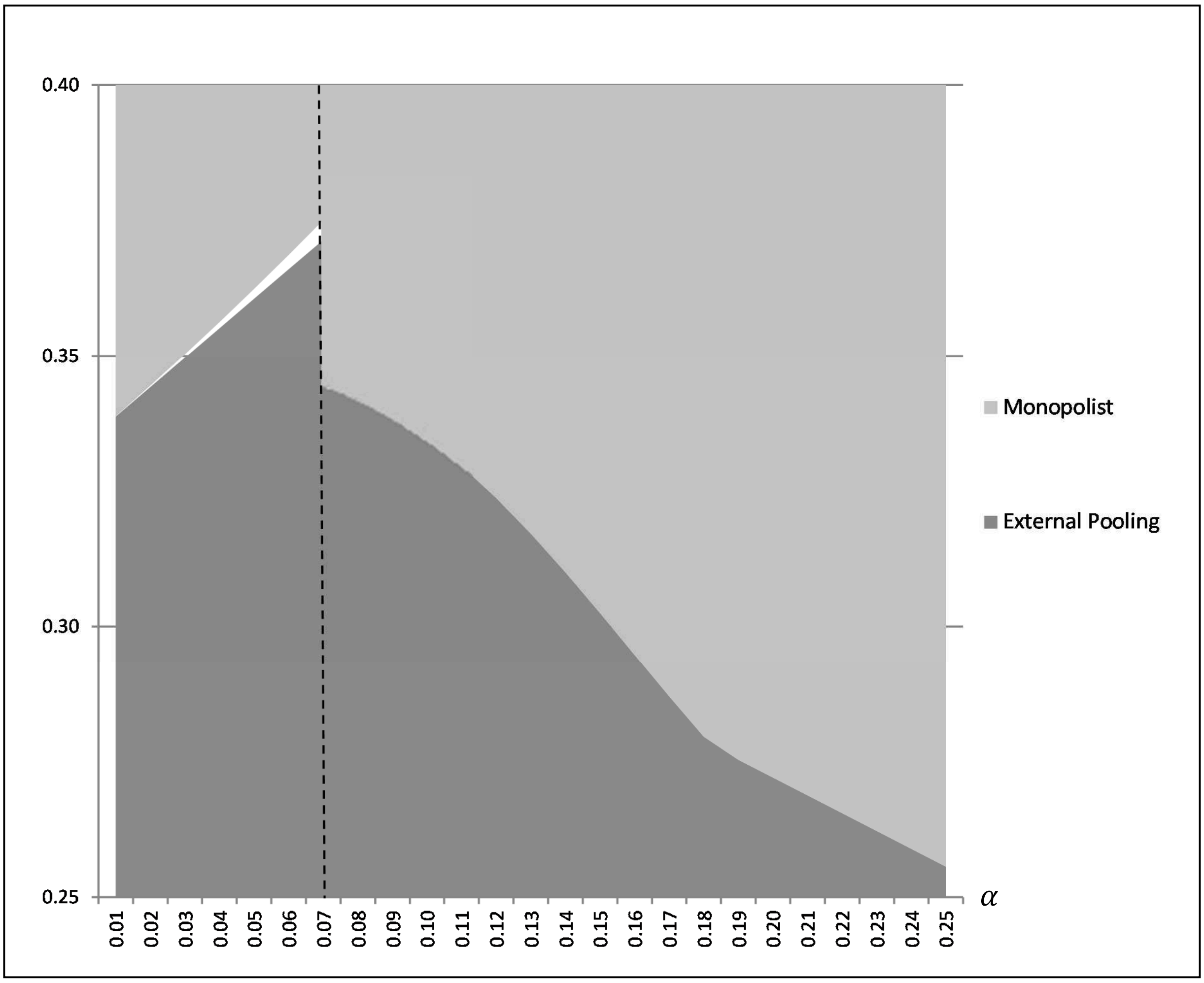
It is interesting to compare the market with and without a platform. The simplest case for that comparison is when there is no certification cost, since it enables us to see the changes that result merely from market structure.



Graph 1 – Spaces occupied by the platform and the external pooling

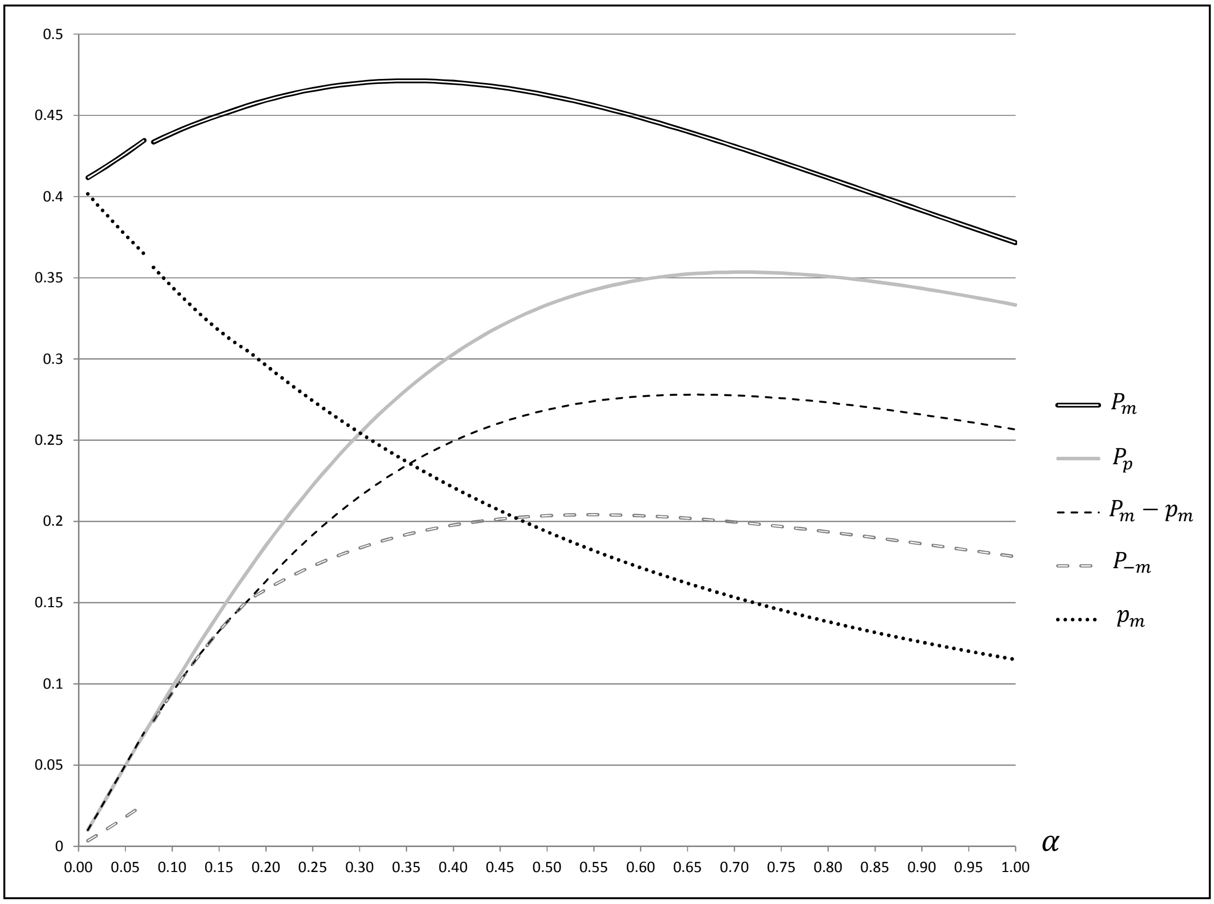
Graph 1 depicts, for various values of α, which sellers trade and where, according to their qualities. The maximum value of $\alpha = 1$ serves only as an example, given that α is not limited from above. As we may see, in this example, the entry of a monopolist platform in a market that previously operated without certification harms the high quality sellers in most of the graph, since they quit trade. These newly excluded sellers are those with qualities laying between the solid black line and the top of the light gray area. The vertical dashed lines mark the changes in the constraints we mentioned between binding and not binding. At the left of the graph we find case ii-a, imposing $s_H = 1$. Only in that configuration, the best quality sellers benefit as compared with the no-platform market situation, in which they do not trade. However, in that situation there is a number of sellers excluded from trade by the entry of the platform, corresponding to interval $[s_{-m}, s_L]$. The area within the white dotted rectangle is zoomed in Graph 2, which allows easier verification of that detail. In the area between the two dashed lines, case ii-b with

$P_m - p_m = P_{-m}$ prevails, while at the right of the graph, we find case ii-b with $P_m - p_m > P_{-m}$.



Graph 2 – Spaces occupied by the platform and the external pooling (detail)

Graph 3 allows us to compare prices. Only for values of $\alpha < 0.07$ there are situations in which trading in the platform ($P_m - p_m$) is superior for sellers than doing it in the market with no platform (P_P), although the difference is too small to be visible in the graph. Most of the time, sellers are better off without certification than when it is monopolized. Additionally, we note the reduction in the participation price with the increase in sellers' opportunity cost, along the different cases that prevail.



Graph 3 - Prices[16]

[16] P_m: Price of the good in the monopolist platform;
P_{-m}: Price of the good in the external pooling the (outside the monopolist platform);
p_m: Platform participation price;
$P_m - p_m$: Price of the good in the monopolist platform, net of the participation price;
P_p: Price of the good under pooling (without the presence of a platform).

3.4.2 The process of supply and demand equalization

In the previous section, we simply assume that the platform admits equal numbers of agents in both sides of the market. It is, however, necessary to model that process in a more detailed fashion, since in our stylized game, the platform only plays before buyers and sellers.

The possibility of admitting different quantities of participants from each side would make the model more complex, since it requires the definition of which participants will trade. A possibility would be random draws from the side that is in excess. The main disadvantage of this is that it brings into the model the traditional network externality effects related to the number of participants, when we are actually interested in purging them out and analyzing a setting where only the quality of one side affects incentives of the participants of the other side. That is why we choose to keep equal participation from each side in the present model.

One way of making equilibrium candidates presenting that property is using a deterministic rule, pointing to the exclusion of a participant as a function of actions chosen by all players, while maintaining a payment for participation. Then, whenever there is a participant excluded from trade within the platform, he will regret at least not having chosen to abstain from trading, given the platform's first stage decision.

A particular case of that approach is to impose that the participants with lowest quality from the side in excess will be excluded, until participation is equalized between sides. Thus, although in our model the distribution of participation price between buyers and sellers is irrelevant (thus, we chose to make only sellers pay a discrete amount) it is important to keep at least some infinitesimal participation cost paid by each side. We assume that cost is equal to the cost of trying to trade in the external pooling, but that it may be avoided at all by choosing not to trade.

The conditions stated in the previous section, to compare utilities of agents trading in and out of the platform, are still valid when we include these costs, given that starting from a situation in which both sides are present in equal amounts, each agent could individually change his position without altering the mass of agents present in his side.

To sum up, with these assumptions we can guarantee equal participations of buyers and sellers in the platform and in the external pooling.[17]

3.4.3 Duopoly

In this section we analyze a problem of two platforms with simultaneous choice about their prices and the qualities admitted. We indicate these agents by sub-index $k = i, j$. We maintain the considerations that make equal participation in both sides necessary to equilibrium. Since the focus is in the competition between platforms, for simplicity, we exclude the possibility of trading outside both of them (which we previously called external pooling) and the possibility of participating in both of them (multihoming).

Just like in the monopoly case, the pairing of buyers and sellers within a platform is random, with the probability of each type proportional to its participation. We define as m_k the expected quality of a seller trading in platform k.

It is important to define prices paid by buyers and sellers in each platform. Although in equilibrium there must still be the neutrality of participation price distribution between sides, outside of it there is the possibility of some participating agent not transacting. The question of separate prices gains importance when we intend to study the competition between two-sided platforms. In particular, it is common that one side is "subsidized" and that incentives stemming from that fact are key for competition. That has been observed in real cases, like the regulation of interchange fees in Australia, and registered in the literature, as in Caillaud and Jullien (2001) and Caillaud and Jullien (2003). In particular, we allow participation prices to be positive or negative. We keep purchase price nonnegative, in order not to mischaracterize the roles of buyers and sellers in the model.

[17] When we consider, ahead, the possibility of negative prices, we may think that, if there are more sellers than buyers, the platform would have benefited from excluding some of them.

Therefore, P_k, p_{ak}, p_{sk} and o_k stand for, respectively, good purchase price, participation prices of buyers and sellers and unit cost for platform k. We consider a case with $o_k = 0$.

Sellers' payoff is:

$$U_s = \begin{cases} P_k - p_{sk} - \alpha s, \text{ if he trades in platform } k \\ -p_{sk}, \text{ if he participates in platform } k \text{ and does not trade} \\ 0, \text{ if he does not participate in any platform} \end{cases}$$

Buyers payoff is:

$$U_a = \begin{cases} am_k - g(P_k + p_{ak}), \text{ if he trades in platform } k \\ -g(p_{ak}), \text{ if he participates in platform } k \text{ and does not trade} \\ 0, \text{ if he does not participate in any platform} \end{cases}$$

$$g(x) = [\mathbb{I}_{x>0} - \mathbb{I}_{x\leq 0}]x^2$$

where $\mathbb{I}_{x>0}$ assumes value 1 if $x > 0$ and 0 otherwise and $\mathbb{I}_{x\leq 0}$ assumes value 1 if $x \leq 0$ and 0 otherwise. It is necessary to define function $g(x)$, since we want to preserve the sign of the argument, in spite of the quadratic form.

The result we find is that there is no Nash equilibrium in this game. That relates to a platform always being able to take the place of the other in an almost perfect way. The analysis of this game is examined in detail in the Appendix. Here we just quickly review the path used in the demonstration.

In first place, the argument from the monopoly section that a platform is only interested in forbidding the participation of sellers with quality under some threshold is still valid. We name such thresholds s_{Lk}.

Next, we show that there could not be an equilibrium in which platforms had superposition, i.e., that both of them included as participants more than one common type of buyer or seller. The proof is divided in two parts: $m_i = m_j$ and $m_i \neq m_j$[18].

[18] Specifically, we analyze $m_i < m_j$, but the platform index is interchangeable.

We verify, additionally, that there could not be an equilibrium in which some platform occupied discontinuous quality spaces. On the other hand, the operation of each platform in a continuous segment in $[0,1]$ implies different expected seller qualities between platforms.

Since each platform might seize for itself a place arbitrarily close to the one occupied by the other, for example by making the sellers' participation price negative (or marginally more negative) while compensating revenue with increases in buyers' participation price, we conclude that the platforms might not have different profits in equilibrium.

Finally, we show that equal profits also do not support any Nash equilibrium. First, we consider equal positive profits, and find that the condition that each platform is maximizing profits given the strategy of the other cannot be met. In the sequence, we show that a situation with null profits for both platforms would always bring the possibility of profitable deviations, both when both platforms operate and when none of them operates.

4 Conclusions

In this article we analyzed two-sided markets in which two characteristics play the main role: on the one hand, there is heterogeneity of the quality of the goods sold and the buyers' propensity to pay, on the other, sellers and buyers have different (even contrary) interests.

In most of the paper, we analyze asymmetric information situations, in which the buyers can only see the quality of the good she purchased after the transaction. However, we start studying the perfect information situation, which turns out to have a similar solution structure to a screening problem, given the individual rationality and incentive compatibility restrictions that arise from the setting with continuous intervals of different buyers and sellers. We find, in that context, that high quality sellers trade with high propensity to pay buyers. It is possible that the buyers with the lowest propensities to pay and the sellers with the highest qualities opt out of the market, if sellers' opportunity cost is high enough.

The exclusion of the best sellers and worst buyers is intensified in the case of pooling, in which the quality of the good sold is unknown to the buyers and all transactions occur at a single price. That setting is used as a benchmark to analyze the case in which a monopolist platform enters the market, and business may happen inside or outside of it. The platform observes sellers' qualities and limits participation. We find that the platform forbids participation of those who are not good enough, in order to maintain a superior expected quality for buyers than the one found in the external pooling. That kind of behavior is in line with the description in Evans and Schmalensee (2005). In addition, the platform faces a tradeoff, since it can only increase the number of transactions by allowing the participation of lower quality sellers. This resembles a congestion effect.

Although the intuition is that the presence of a separate market for selected goods, in which high quality is certified, would benefit the high quality sellers, that turns out not to be guaranteed, since the platform tries to appropriate most of the gains from signaling. In the examples we analyze, the entry of a monopolist platform in a market that previously operated in pooling equilibrium resulted in the participation of some formerly excluded sellers only in cases with very low seller opportunity cost. Also, there are situations in with the quality space that the platform chooses to operate in results in some intermediate quality sellers preferring not to trade. That is because they are not good enough to be admitted in the platform and not bad enough (and, correspondingly, low cost enough) to accept trading in the outside pooling.

As an illustration, we may think of a market of online merchants, who sell goods through their own websites, without the possibility of signaling quality, or even the authenticity of the offerings. If, then, a certifying platform enters the market, it will attract the merchants whose products are of good quality and whose deliveries occur as arranged. The separation of sellers in two groups, one inside the platform and one outside, has an ambiguous effect on the income of the high quality merchants. On the one hand, enabling them to signal their quality (although not perfectly) enables them to increase the price they ask for their products, as compared with the previous situation. On the other hand, the high quality merchants are in an inconvenient position to resist the extraction of this extra income through the participation fee. That is because, should they want to trade outside the platform, now the pool of uncertified merchants is constituted only by lower quality ones, not admitted by the platform. In combination with the situation of high market power, that may result in some high quality merchants, who formerly did not trade, starting to sell their goods. However, in other situations, it may also result in the exclusion of the highest quality merchants that were active before the entry. Additionally, there may also be some intermediate quality merchants who stop trading.

Finally, we analyze a game in which two platforms choose simultaneously the prices and admitted qualities, in a first stage, while buyers and sellers choose whether to trade and in which platform, in a second stage. In that setting, the possibility of charging negative prices from one of the sides gains great importance. That is what Caillaud and Jullien (2001) call “divide and conquer”. We show that there is no pure strategy Nash equilibrium in that game, which arises from the possibility of one platform taking the place of the other by paying the sellers to participate in it and making it, thus, a dominant strategy for them. The inexistence of this kind of equilibrium in a simultaneous game is also found by Damiano and Li (2008). We do not pursue the investigation of sequential or mixed strategies equilibria, because we do not think that they would be representative of the structures we focus on, thus escaping the scope of the paper.

As a research agenda, it would be interesting to seek empirical applications, using actual data. The entry of platforms in a large set of diversified markets (taxi cabs, delivery food, hotel reservations, real estate trading, etc.) makes convenient environments to study the case of former pooling markets under transformation.

Furthermore, in our model the heterogeneity of quality is the fundamental characteristic. It would be interesting to analyze how results could change if there was some non-null mass agent with high quality. In particular, we might ask if, in the case of duopoly, platforms would dispute that agent or, in the case of monopoly, if he would have some bargain power against the platform.

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Appendixes

Appendix 1 – First Best

In this appendix, we analyze in detail the First Best equilibrium, without the assumption of full participation used in the main text. The approach is to examine buyers' incentive compatibility constraints together with individual rationality constraints from both sides of the market, in order to sort out possible values of C and α into relevant cases.

As we have argued, FB1 implies an increasing $a(s)$ function and buyers' incentive compatibility constraints, given the participation of some given buyer, guarantee the participation of all buyers with a larger a than him. In particular, participation is ensured for $a = 1$ if there is any transaction. That is because if some seller $s > 0$ is trading, the buyer with $a = 1$ could obtain utility higher than the partner of that seller is getting by offering him an infinitesimally larger price.

We name a_L the participant buyer with lowest quality. It is useful to notice that if $a_L > 0$, his payoff will be null. With a strictly positive payoff, $a_L > 0$ implies the existence of some buyer $\hat{a}$, not participant and arbitrarily close to a_L, who would have a null payoff but would be willing to offer infinitesimally more than a_L to trade with his partner, s_L. This shows that such a situation could not be an equilibrium. Thus, the set of participants on the buyers' side is continuous between a_L and 1[19].

For any continuous set of participant sellers, there must be, in equilibrium, a set of buyers who are their partners. Furthermore, there must be the same mass of participants between two buyers in such a set as the mass of buyers between their partners: $a(s_1) - a(s_0) = s_1 - s_0$. This implies that in any of such continuous sets, $a(s)$ is an affine function, with a unit coefficient. Given this fact, participation of $a = 1$ implies $a(s) \geq s$, therefore:

[19] Assume some $\hat{a} > a_L$ participating and some $\tilde{a} > \hat{a}$ not participating. Such a situation cannot be observed in equilibrium, since $\hat{s}$, such that $a(\hat{s}) = \hat{a}$, is larger than zero and $\tilde{a}$ would be willing to pay infinitesimally more than $\hat{a}$ to have $\hat{s}$ for a partner.

$$a(s) = s + d, \quad \text{where} \quad d \geq 0 \tag{A1.1}$$

Taking again IC1, we may calculate the limit, within a continuous set of participant sellers, of $s_0 \to s_1$, in the same fashion as we do in the main text, obtaining: $a = 2P(s)P'(s)$. Therefore, within each continuous subset of $[0,1]$ of participant sellers, $P(s)^2 = s^2/2 + ds - C$. In case the set of participant sellers does not form a unique continuous subset of $[0,1]$, each continuous disjoint subset will have a different value for d, which should be higher the more to the left the subset is on $[0,1]$.

Now we show that the set of participant sellers will be a unique continuous subset of $[0,1]$. Assume two subsets of $[0,1]$, $[s_{0L}, s_{0H}]$ and $[s_{1L}, s_{1H}]$ of participant sellers, such that $s_{0L} < s_{0H} < s_{1L} < s_{1H}$. Assume further that there are no participant sellers with s such that $s_{0H} < s < s_{1L}$. In the lowest quality subset, we must have that $P(s)^2 = s^2/2 + d_L s - C_L$, while in the one with superior quality $P(s)^2 = s^2/2 + d_H s - C_H$, with $d_H < d_L$.

Assume $\hat{a} \in [a_L, 1]$, defined such that $\lim_{s \to s_{0H}-} a(s) = \hat{a}$. Given the continuity of the participant buyers set, it must also be that $\lim_{s \to s_{1L}+} a(s) = \hat{a}$. The payoffs of participating buyers must form a continuous schedule; otherwise, a discontinuity at some a would enable some buyers arbitrarily close to him to increase his utility. Therefore, we obtain:

$$\hat{a}s_{0H} - P(s_{0H})^2 = \hat{a}s_{1L} - P(s_{1L})^2$$

On the other hand, consider the payoffs of s_{0H} and s_{1L}. We assumed that $s_{0H} < s_{1L}$, so there are sellers with null payoff between them. Since there are non-participants arbitrarily close to s_{0H} and s_{1L}, it would be necessary that they should also have null payoff:

$$P(s_{0H}) - \alpha s_{0H} = P(s_{1L}) - \alpha s_{1L} = 0$$

Thus, $P(s_{0H}) = \alpha s_{0H}$ and $P(s_{1L}) = \alpha s_{1L}$. Nonparticipant sellers with $s \in]s_{0H}, s_{1L}[$ also have null payoff; however, they would be willing to participate for any price higher than αs. Therefore, the fact that $\hat{a}s - \alpha^2 s^2$ is strictly concave, implies that the situation described cannot be compatible with an equilibrium, since $\hat{a}$ would rather pay more than αs to trade with any of these non-participant sellers than trading with s_{0H} or s_{1L}[20].

Consequently, $a(s)$ is an affine function with unit coefficient defined over the set of participant sellers. Considering the possibility of $d > 0$, participant buyers' incentive compatibility constraints imply:

$$P(s)^2 = s^2/2 + ds - C \qquad \text{(A1.2)}$$

For nonparticipants, the incentive compatibility constraint is that the null payoff is higher than what they could get paying the necessary price to convince a seller to trade with them. That may be the opportunity cost of participating or of abandoning another partner, depending on the initial situation.

Buyers' individual rationality constraint is $a(s)s - P(s)^2 \geq 0$. Combining it with (A1.2) and substituting (A1.1), we get:

$$(s + d)s - \left(\frac{s^2}{2} + ds - C\right) \geq 0$$

$$\therefore \qquad \frac{s^2}{2} + C \geq 0$$

To analyze this expression, it is useful to define three cases:

Case I – $C > 0$: the obedience of the participation restriction is guaranteed for all buyers, with a strictly positive value, whoever his partner is. That implies full participation on the buyers´ side ($a_L = 0$), which, in equilibrium, may only take place with full participation on the sellers' side, meaning $d = 0$.

Case II – $C = 0$: $a_L \geq 0$, and his payoff is null. This implies $s_L = 0$, because if there is full participation, then $s_L = a_L = 0$ and, if there is not, that would be the only s yielding null payoff to some $a_L > 0$.

[20] Name $\tilde{s}$ one of these sellers. $\tilde{s} = \rho s_{0H} + (1 - \rho)s_{1L}$, with $\rho \in]0,1[$. Then: $\hat{a}\tilde{s} - \alpha^2\tilde{s}^2 > \rho(\hat{a}s_{0H} - \alpha^2 s_{0H}^2) + (1-\rho)(\hat{a}s_{1L} - \alpha^2 s_{1L}^2) = \hat{a}s_{0H} - \alpha^2 s_{0H}^2 = \hat{a}s_{1L} - \alpha^2 s_{1L}^2$.
$\hat{a}\tilde{s} - \alpha^2\tilde{s}^2 = \hat{a}[\rho s_{0H} + (1-\rho)s_{1L}] - \alpha^2[\rho s_{0H} + (1-\rho)s_{1L}]^2 = \rho(\hat{a}s_{0H} - \alpha^2 s_{0H}^2) + (1-\rho)(\hat{a}s_{1L} - \alpha^2 s_{1L}^2) + \alpha^2\rho(1-\rho)(s_{0H} - s_{1L})^2$.

Case III – $C < 0$: $a_L > 0$, since the payoff of $a = 0$ would be necessarily negative were him to participate. Given that the payoff of a_L is null, we may find s_L from:

$$\frac{s_L^2}{2} + C = 0$$

$$\therefore \qquad s_L = \sqrt{-2C}$$

$$\text{and} \qquad a_L = d + \sqrt{-2C} > 0$$

On the other hand, sellers' participation restriction may be written as $P(s) \geq \alpha s$, or $P(s)^2 \geq \alpha^2 s^2$. Using (A1.2):

$$\frac{s^2}{2} + ds - C \geq \alpha^2 s^2$$

$$\therefore \qquad s^2\left[\frac{1}{2} - \alpha^2\right] + ds - C \geq 0 \qquad \text{(A1.3)}$$

We know that the participation of sellers is continuous between s_L and s_H. Thus, $s_L > 0$ or $s_H < 1$ imply that the payoffs of these sellers should be null, since if they were strictly postive, there would be non-participating sellers arbitrarily close to them who would be willing to steal their partners by charging an inferior price.

It is useful to define the roots of $s^2\left[\frac{1}{2} - \alpha^2\right] + ds - C = 0$, as:

$$r_1 = \frac{-d + \sqrt{d^2 + 4\left[\frac{1}{2} - \alpha^2\right]C}}{2\left[\frac{1}{2} - \alpha^2\right]}$$

$$r_2 = \frac{-d - \sqrt{d^2 + 4\left[\frac{1}{2} - \alpha^2\right]C}}{2\left[\frac{1}{2} - \alpha^2\right]}$$

Again, we divide the analysis in Cases A, B and C and check if they are compatible with Cases I trough III, defined before:

Case A - $\left[\frac{1}{2} - \alpha^2\right] < 0$ (or $\alpha > \frac{1}{\sqrt{2}}$): Expression $s^2\left[\frac{1}{2} - \alpha^2\right] + ds - C$ describes a concave quadratic function. In order to verify if there is sellers' participation implies that this function should assume nonnegative values, which guarantees that the roots are defined, although they may not belong to $[0,1]$. Furthermore, we know that $r_2 > r_1$.

Root r_1 can be written as:

$$r_1 = \frac{d - \sqrt{d^2 - 4\left[\alpha^2 - \frac{1}{2}\right]C}}{2\left[\alpha^2 - \frac{1}{2}\right]}$$

Therefore, $C > 0$, would imply $\sqrt{d^2 - 4\left[\alpha^2 - \frac{1}{2}\right]C} < d$, and thus $s_L = r_1 > 0$. However, this is not compatible with Case I, in which $s_L = 0$.

On its turn, $C = 0$ implies $s_L = r_1 = 0$ and $r_2 = d/\left[\alpha^2 - \frac{1}{2}\right]$, or, $s_H = Min\left\{d/\left[\alpha^2 - \frac{1}{2}\right]; 1\right\}$.

This situation is compatible with what we found in Case II. It is necessary to solve for the value of d that supports equilibrium. Assume d is such that $d/\left[\alpha^2 - \frac{1}{2}\right] \leq 1$. Then, as equilibrium conditions we would have that $a_L = a(0) = d$ and $a(s_H) = s_H + d = 1$. Therefore:

$$d/\left[\alpha^2 - \frac{1}{2}\right] + d = 1$$

∴

$$d\left[1 + \alpha^2 - \frac{1}{2}\right] = \left[\alpha^2 - \frac{1}{2}\right]$$

∴

$$d = \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]} < 1$$

Given the assumption of Case A, we also know that $d > 0$. Thus, we find an equilibrium where the highest quality sellers and lower quality buyers are excluded.

On the other hand, $d/\left[\alpha^2 - \frac{1}{2}\right] > 1$ is not compatible with equilibrium, since it implies $d > \left[\alpha^2 - \frac{1}{2}\right] > 0$ which cannot happen simultaneously with $d = 0$, which results from $a(s_H) = a(1) = 1 + d = 1$.

Finally, with $C < 0$, we have that $r_1 < r_2$ e $r_1 < 0$, implying $s_L = 0$, which is incompatible with Case III, in which $s_L > 0$.

Case B - $\left[\frac{1}{2} - \alpha^2\right] = 0$ (or $\alpha = \frac{1}{\sqrt{2}}$): Here, (A1.3) reduces to $ds \geq C$. Thus, $C > 0$ implies nonparticipation of the lowest quality sellers, which is not compatible with Case I. On the other hand, if $C \leq 0$, the participation constraint will be observed for all sellers. Therefore, for an equilibrium it must be that $d = 0$, because $d > 0$ would imply a strictly positive payoff for s_L. The situation with $C = 0$ is compatible with Case II, with full participation. Finally, with $C < 0$ the payoff is strictly positive for all sellers, which is not compatible with the requirement of $s_L > 0$ of Case III.

Case C - $\left[\frac{1}{2} - \alpha^2\right] > 0$ (or $\alpha < \frac{1}{\sqrt{2}}$): (A1.3) is a convex quadratic function, always increasing for $s > 0$. Hence, if there is any participation of sellers, the highest quality seller participates, i.e. $s_H = 1$. In case there is any seller for whom (A1.3) is not observed, the roots we defined before will be valid, although in contraposition to Case A, here $r_2 < r_1$.

With $C > 0$, the payoff of $s = 0$ is strictly negative, so that $s_L = r_1 > 0$. However, this is not compatible with full participation implied by Case I. Alternatively, with $C = 0$, $r_1 = 0$, so that (A1.3) is valid for all sellers. This is compatible with Case II, with full participation. Finally, with $C < 0$, the payoff is strictly positive for all sellers, which is not compatible with Case III.

As a conclusion, we found that the only possibilities of equilibrium are with $C = 0$. Furthermore, if $\alpha \leq \frac{1}{\sqrt{2}}$, there will be full participation and, if $\alpha > \frac{1}{\sqrt{2}}$, the highest quality sellers and lowest quality buyers will be excluded. This means that when the sellers opportunity cost is relatively low, it is possible for all of them to trade, contrarily to what happens when such a cost is high.

Finally, to prove the stability of these equilibria, we show that given an initial situation fitting one of them, no seller could make to any buyer an offer that would be a mutually beneficial deviation. For that, assume a random seller, $\hat{s}$, in one of the equilibria shown, trying to devise such an offer.

Assume condition $\alpha \leq \frac{1}{\sqrt{2}}$. Then we have that $\hat{a} = a(\hat{s}) = \hat{s}$ and $P(s)^2 = s^2/2$. Thus, the utility of a buyer in such an equilibrium is $a^2/2$. The price $\hat{P}$ which $\hat{s}$ might ask from a is, in the limit, one that makes him indifferent: $a\hat{s} - \hat{P}^2 = a^2/2$. Since that $\alpha\hat{s}$ is constant in the payoff of $\hat{s}$ given participation, the best partner $\hat{s}$ starting from that situation is the one that maximizes $\hat{P}$, or, in the same way $\hat{P}^2$. That buyer would be $a = \hat{s}$, validating the equilibrium. On the other hand, if some buyer $\hat{a}$ would think of switching sellers, he would need to pay his new partner at least the same price as this new seller is getting, and the switching would be unattractive to $\hat{a}$, which is guaranteed by the buyers' incentive compatibility constraint.

Alternatively, under condition $> \frac{1}{\sqrt{2}}$, participants are buyers for whom $a \geq d = \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}$ and sellers for whom $s \leq 1 - \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}$. Price satisfies $P(s)^2 = \frac{s^2}{2} + \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]} s$ and the pairing is given by $\hat{a} = a(\hat{s}) = \hat{s} + \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}$. Thus, the utility of a participant buyer in such an equilibrium is:

$$as - P(s)^2 = a\left(a - \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}\right) - \frac{\left(a - \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}\right)^2}{2} - \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}\left(a - \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}\right)$$

$$= \left(a - \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}\right)^2 - \frac{\left(a - \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}\right)^2}{2} = \frac{1}{2}\left(a - \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}\right)^2$$

Therefore, the highest price that some seller $\hat{s}$ could ask from a buyer $a \geq d$ should satisfy $a\hat{s} - \hat{P}^2 = \frac{1}{2}\left(a - \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}\right)^2$. Note that $\alpha\hat{s}$ is a constant term in the payoff of $\hat{s}$. Thus, the best partner $\hat{s}$ could get starting from that situation is the one that maximizes $\hat{P}$, or, equivalently $\hat{P}^2$ requiring:

$$\hat{s} - \left(a - \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}\right) = 0$$

or

$$a = \hat{s} + \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}$$

This validates the proposed equilibrium. Then again, if $\hat{s}$ is not a participant, this condition could not be satisfied, since it implies $a > 1$. Considering $a = 1$, we have $\hat{s} - \hat{P}^2 = \frac{1}{2}\left(1 - \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}\right)^2$ or $\hat{P}^2 = \hat{s} - \frac{1}{2}\left(1 - \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}\right)^2$. The payoff $\hat{s}$ would obtain making an offer to $a = 1$ would be given by: $\sqrt{\hat{s} - \frac{1}{2}\left(1 - \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}\right)^2} - \alpha\hat{s}$.

That payoff is positive if, and only if:

$$\hat{s} - \frac{1}{2}\left(1 - \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}\right)^2 > \alpha^2 \hat{s}^2$$

or

$$\hat{s} - \frac{1}{2}\left(1 - \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}\right)^2 - \alpha^2 \hat{s}^2 > 0 \qquad \text{(A1.4)}$$

However, the value of the expression on the left-hand side of (A1.4) is zero for $\hat{s} = 1 - \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}$ and its derivative, $1 - 2\alpha^2\hat{s}$, is negative for $\hat{s} \geq 1 - \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}$.

For $\hat{s} = 1 - \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}$:

$$\hat{s} - \frac{1}{2}\left(1 - \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}\right)^2 - \alpha^2 \hat{s}^2 = 1 - \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]} - \alpha^2\left(1 - \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}\right)^2 - \frac{1}{2}\left(1 - \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}\right)^2 =$$

$$1 - \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]} - \left(\alpha^2 + \frac{1}{2}\right)\left(\frac{\left[\alpha^2 + \frac{1}{2}\right] - \left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]}\right)^2 = 1 - \frac{\left[\alpha^2 - \frac{1}{2}\right]}{\left[\alpha^2 + \frac{1}{2}\right]} - \frac{1}{\left[\alpha^2 + \frac{1}{2}\right]} =$$

$$\frac{\left[\alpha^2+\frac{1}{2}\right]-\left[\alpha^2-\frac{1}{2}\right]-1}{\left[\alpha^2+\frac{1}{2}\right]}=\frac{\frac{1}{2}+\frac{1}{2}-1}{\left[\alpha^2+\frac{1}{2}\right]}=0$$

In turn, the derivative of the left-hand side of (A1.4), for $\hat{s}\geq 1-\frac{\left[\alpha^2-\frac{1}{2}\right]}{\left[\alpha^2+\frac{1}{2}\right]}$ is :

$$1-2\alpha^2\hat{s}\leq 1-2\alpha^2\left[1-\frac{\left[\alpha^2-\frac{1}{2}\right]}{\left[\alpha^2+\frac{1}{2}\right]}\right]=1-2\alpha^2\left[\frac{1}{\left[\alpha^2+\frac{1}{2}\right]}\right]=\frac{\alpha^2+\frac{1}{2}-2\alpha^2}{\left[\alpha^2+\frac{1}{2}\right]}=\frac{\frac{1}{2}-\alpha^2}{\left[\alpha^2+\frac{1}{2}\right]}<0$$

Thus, for all $\hat{s}>1-\frac{\left[\alpha^2-\frac{1}{2}\right]}{\left[\alpha^2+\frac{1}{2}\right]}$ it is more convenient not to participate.

On the buyers' side, the incentive compatibility constraint was imposed directly on participants. The maximum payoff that could be obtained by buyer $\hat{a}$ by making an offer to some participating seller s would be $\hat{a}s-\frac{s^2}{2}-\frac{\left[\alpha^2-\frac{1}{2}\right]}{\left[\alpha^2+\frac{1}{2}\right]}s$, which is maximized with $s=\hat{a}-\frac{\left[\alpha^2-\frac{1}{2}\right]}{\left[\alpha^2+\frac{1}{2}\right]}$. For nonparticipants, i.e. those for whom $a<\frac{\left[\alpha^2-\frac{1}{2}\right]}{\left[\alpha^2+\frac{1}{2}\right]}$, the condition would imply $s<0$. Regarding the possibility of making an offer to $s=0$, the payoff of these buyers would remain null, generating no incentives to deviation, i.e. switching of partners.

Appendix 2 – First Best with an alternative formulation of buyers payoff

In this appendix, we consider a case in which the marginal disutility stemming from paying a purchase is constant. The payoff of buyers' is given by:

$$U_a = \begin{cases} as - P_s, & \text{if good } s \text{ is consumed} \\ 0, & \text{otherwise} \end{cases}$$

As we argued in Appendix 1, it is still true that $a(s)$ is increasing and that incentive compatibility constraints, given the participation of some buyer, guarantee the participation of all buyers with a larger than his and, in particular, of $a = 1$. It is also true that the payoff of a_L must be null whenever $a_L > 0$ and that, within any continuous subset of sellers $a(s) = s + d$, for some $d \geq 0$.

Like in the main text formulation, the comparison between two incentive compatibility constraints, taking the limit as the difference of sellers' qualities converges to zero, produces:

$$a_1 = \lim_{s_0 \to s_1} \frac{P(s_1) - P(s_0)}{s_1 - s_0}$$

$$a_1 = P'(s_1)$$

Particularly, if there is trade, the buyer with $a = 1$ participates, leading to (A1.1). Hence:

$$s + d = P'(s)$$

$$\therefore \frac{s^2}{2} + ds = P + C$$

$$\therefore P(s) = \frac{s^2}{2} + ds - C \qquad \text{(A2.1)}$$

Continuity considerations about the set of participating sellers are still valid, although here the reasoning is somewhat different.

Affirmative 1: The set of participant sellers in the first best under the alternative formulation is continuous.

Proof: Assume two subsets of $[0,1]$, $[s_{0L}, s_{0H}]$ and $[s_{1L}, s_{1H}]$, of participating sellers, such that $s_{0L} < s_{0H} < s_{1L} < s_{1H}$. Assume further, that there is no mass of participating sellers with s such that $s_{0H} < s < s_{1L}$. In the lower quality subset, price is determined by $P(s) = s^2/2 + d_L s - C_L$, while in the higher quality one $P(s) = s^2/2 + d_H s - C_H$, with $d_H < d_L$.

Assume $\hat{a} \in [a_L, 1]$, defined such that $\lim_{s \to s_{0H}-} a(s) = \hat{a}$. Given the continuity of the participant buyers, it also happens that $\lim_{s \to s_{1L}+} a(s) = \hat{a}$. Therefore, it must be that the payoff of participating buyers is continuous, since discontinuity in some given a would enable some buyers arbitrarily close to a to increase his payoff by deviating. Thus, we obtain:

$$\hat{a}s_{0H} - P(s_{0H}) = \hat{a}s_{1L} - P(s_{1L})$$

On the other hand, consider the payoffs of s_{0H} and s_{1L}. If, as we assume, $s_{0H} < s_{1L}$, there are sellers with zero payoff between them. Since there are nonparticipants arbitrarily close to each of them, they must have zero payoff too:

$$P(s_{0H}) - \alpha s_{0H} = P(s_{1L}) - \alpha s_{1L} = 0$$

Thus, $P(s_{0H}) = \alpha s_{0H}$ and $P(s_{1L}) = \alpha s_{1L}$ (the nonparticipant sellers with $s \in]s_{0H}, s_{1L}[$ also have zero payoff). Substituting this into the indifference of $\hat{a}$, we obtain:

$$\hat{a}s_{0H} - \alpha s_{0H} = \hat{a}s_{1L} - \alpha s_{1L}$$

∴

$$(\hat{a} - \alpha)s_{0H} = (\hat{a} - \alpha)s_{1L}$$

This means that, either $\hat{a} = \alpha$ or $s_{0H} = s_{1L}$. In the second case, there is no discontinuity. In the first case, this implies that the payoff of $\hat{a}$ is null, which would imply that no buyer with $a < \hat{a}$ would participate, since if any buyer in that situation had his individual rationality constraint observed, $\hat{a}$ could get a positive payoff by stealing his partner. However, the participation of buyers in that condition would be necessary for an equilibrium with discontinuity in the set of participating sellers. □

Substituting (A1.1) and (A2.1) into buyers' payoff and taking into account the participation constraint, we obtain:

$$(s + d)s - \left(\frac{s^2}{2} + ds - C\right) = \frac{s^2}{2} + C \geq 0$$

This condition is identical to the one found in Appendix I, from which it follows that Cases I through III for the possible values of C are still valid.

On the other hand, sellers' individual rationality constraints are:

$$P(s) - \alpha s = \frac{s^2}{2} + (d - \alpha)s - C \geq 0 \quad \text{(A2.2)}$$

Considering (A2.2) with equality, the equation describes a convex quadratic function, with roots (if defined) given by:

$$r_1 = -(d - \alpha) + \sqrt{(d - \alpha)^2 + 2C}$$

and

$$r_2 = -(d - \alpha) - \sqrt{(d - \alpha)^2 + 2C}$$

We know that $r_2 \leq r_1$. For the roots to be well defined, which is equivalent to expression $\frac{s^2}{2} + (d - \alpha)s - C$ presenting some non-positive value in $\mathbb{R}$, we need $(d - \alpha)^2 + 2C \geq 0$. In case $(d - \alpha)^2 + 2C < 0$, there would be no real roots, implying participation of all sellers. However, in that situation we would have $2C < -(d - \alpha)^2 \leq 0$, i.e., $C < 0$, which would imply that we would be in Case III, which is not compatible with total participation of sellers.

Thus, in a potential equilibrium, it must be that r_1 and r_2 are well defined. The sellers with positive payoff are those for whom either $s < r_2$ or $s > r_1$. However, it cannot be that there are sellers in both these conditions, since it would imply a discontinuous participation of sellers.

In order to restrict the situations under analysis, assume that $r_2 > 0$, with participation of sellers in $[0, r_2]$. Then, it must be that $r_1 > 1$. We assume that is the case. Then:

$$r_2 = -(d - \alpha) - \sqrt{(d - \alpha)^2 + 2C} > 0$$

$\therefore$

$$\sqrt{(d - \alpha)^2 + 2C} < -(d - \alpha)$$

This inequality may not be observed if $d \geq \alpha$. If $d < \alpha$, both sides are positive and we may analyze the expression:

$$(d-\alpha)^2 + 2C < (d-\alpha)^2$$

This implies $C < 0$. This is compatible with Case III, which brings us to a contradiction, since in that case $s_L > 0$. Thus, we assume that $r_2 \leq 0$, implying that the sellers' participation in equilibrium must be $[r_1, 1]$.

It would also not be possible that $r_1 \leq 0$, which would require full participation of sellers, and therefore $d = 0$, since that would imply $r_1 = \alpha + \sqrt{\alpha^2 + 2C} \leq 0$.

Finally, assuming $0 < r_1 \leq 1$, we would have the participation of $s = 1$, which would imply also that $d = 0$. The equilibrium condition would be $s_L = r_1 > 0$ which puts us in Case III:

$$\sqrt{-2C} = \alpha + \sqrt{\alpha^2 + 2C}$$

∴

$$\alpha = \sqrt{-2C} - \sqrt{\alpha^2 + 2C}$$

∴

$$\alpha^2 = -2C - 2\sqrt{-2C}\sqrt{\alpha^2 + 2C} + \alpha^2 + 2C$$

∴

$$-2\sqrt{-2C}\sqrt{\alpha^2 + 2C} = 0$$

Since Case III is defined with $C < 0$, we have that $\alpha^2 + 2C = 0$, therefore $C = -\frac{\alpha^2}{2}$. The equilibrium is described as:

$$a_L = s_L = \alpha$$

$$a_H = s_H = 1$$

$$P(s) = \frac{s^2 + \alpha^2}{2}$$

The participating buyers' utility is given by $\frac{a^2-\alpha^2}{2}$, while sellers' utility is $\frac{s^2+\alpha^2}{2} - \alpha s = \frac{(s-\alpha)^2}{2}$.In order to check the equilibrium we found, analyze the decision of some buyer $\hat{a} \geq \alpha$. Given the price schedule in equilibrium, he would maximize $\hat{a}s - \frac{s^2+\alpha^2}{2}$, obtaining $s = \hat{a}$ as the optimal solution, thus getting utility $\frac{\hat{a}^2-\alpha^2}{2}$. We also see that $\hat{a}$ would not be interested in making an offer to some nonparticipating buyer, since his utility would be limited to $\hat{a}s - \alpha s = (\hat{a} - \alpha)s < (\hat{a} - \alpha)\alpha$. By computing the difference between this alternative payoff and the utility obtained in equilibrium, we show that such difference is negative:

$$(\hat{a} - \alpha)\alpha - \frac{\hat{a}^2 - \alpha^2}{2} = \frac{1}{2}(2\alpha\hat{a} - 2\alpha^2 - \hat{a}^2 + \alpha^2) = -\frac{1}{2}(\hat{a}^2 - 2\alpha\hat{a} + \alpha^2)$$
$$= -\frac{(\hat{a} - \alpha)^2}{2}$$

On the other hand, take some buyer with $\hat{a} < \alpha$ (nonparticipant), and assume that he makes an offer to some participating seller. Then, $\hat{a}$ would have a payoff $\hat{a}s - \frac{s^2+\alpha^2}{2}$, which in this case, would be maximized at $s = \alpha$, with a negative result: $\hat{a}\alpha - \frac{\alpha^2+\alpha^2}{2} = \alpha(\hat{a} - \alpha)$. If a nonparticipating seller were considered, then the payoff would be limited to $\hat{a}s - \alpha s = (\hat{a} - \alpha)s < 0$.

As may be noted, in this formulation, there will be no transactions in the First Best if $\alpha > 1$.

Appendix 3 – Monopoly Platform Equilibrium

In this appendix, we study the possible cases of equilibrium in monopoly. We start the analysis by subcase ii-a without imposing some of the restrictions that would imply that the result is actually contained in it. Next, we evaluate when these restrictions would be violated, meaning that the equilibrium should be sought in the other subcases. We then follow the analysis with case ii-b. In the end, we show the areas occupied by each strategy, which combined lead to Graph 1 in the main text.

A3.1 – Subcase ii-a – unrestricted analysis

In this section we assume that $s_H \leq 1$ and $s_L \geq s_{-m}$. In the end of it, we analyze parameters values combinations that would make any of these conditions fail. We start from the indifference of s_H ($P_m - p_m - \alpha s_H = 0$):

$$s_H = \frac{P_m - p_m}{\alpha} \qquad \text{(A3.1)}$$

Note that if (A3.1) implies $s_H > 1$, it will also mean that $s = 1$ prefers strictly the participation in the platform, so that we would not be in Case ii. We come back to the situation in which $s_H = 1$ in the next session. We use equation (A3.1) to switch the monopolist´s choice variable, which simplifies the calculations that follow.

Substituting (A3.1) into platform equilibrium, represented by (NC3) we obtain:

$$\frac{P_m - p_m}{\alpha} - s_L = 1 - a_m$$

$$\therefore \qquad p_m = \alpha a_m - \alpha - \alpha s_L + P_m \qquad \text{(A3.2)}$$

On the other hand, we have the indifference of a_m between the platform and the external pooling, expressed by (NC6). Substituting (EP1a):

$$a_m m_m - P_m^2 = a_m m_{-m} - \left(\frac{\alpha a_m}{(1 + 2\alpha^2)}\right)^2$$

Using $m_{-m} = \frac{s_{-m}}{2} = \frac{a_m}{2(1+2\alpha^2)}$ and $m_m = \frac{s_L + s_H}{2}$ we obtain:

$$a_m\left(\frac{s_L+s_H}{2}\right)-P_m^2=a_m\frac{a_m}{2(1+2\alpha^2)}-\left(\frac{\alpha a_m}{(1+2\alpha^2)}\right)^2$$

$\therefore$
$$P_m^2=a_m\left(\frac{s_L+s_H}{2}\right)-a_m\frac{a_m}{2(1+2\alpha^2)}+\left(\frac{\alpha a_m}{(1+2\alpha^2)}\right)^2$$

Substituting s_H with (NC3):

$$P_m^2=a_m\left(\frac{s_L+1-a_m+s_L}{2}\right)-a_m\frac{a_m}{2(1+2\alpha^2)}+\left(\frac{\alpha a_m}{(1+2\alpha^2)}\right)^2$$

$\therefore$
$$P_m^2=\left(\frac{1}{2}+s_L\right)a_m-\frac{1+\left(1+2\alpha^2\right)^2}{2(1+2\alpha^2)^2}a_m^2$$

Given that $P_m \geq 0$, for each value of a_m, there is only one possible value of P_m given s_L[21], expressed as:

$$P_m(a_m,s_L)=\sqrt{a_m\left(\frac{1}{2}+s_L\right)-\frac{1+(1+2\alpha^2)^2}{2(1+2\alpha^2)^2}a_m^2}$$

(A3.3)

Note that, by (A3.2), (A3.3) implies that p_m can also be written as a function of a_m and s_L.

The monopolist platform maximizes profit:

$$\Pi_m=(1-a_m)(p_m-o)$$

Substituting p_m with (A3.2):

$$\Pi_m=(1-a_m)(\alpha a_m-\alpha-\alpha s_L+P_m(a_m,s_L)-o)$$

We use (a_m,s_L) as monopolist's choice variables to simplify calculations, instead of (p_m,s_L). The correspondence between a_m and p_m, given s_L follows from (A3.2) and (A3.3).

[21] For $P_m > 0$, which is necessary for some sellers with $s > 0$ to participate in the platform, we have that $a_m > 0$ and must check that $\left(\frac{1}{2}+s_L\right)-\frac{1+\left(1+2\alpha^2\right)^2}{2(1+2\alpha^2)^2}a_m > 0$.

The first order condition for s_L is:

$$\frac{\partial \Pi_m}{\partial s_L} = (1 - a_m)\left(-\alpha + \frac{\partial P_m(a_m, s_L)}{\partial s_L}\right) = 0$$

Assuming that the platform operates, we know that $a_m < 1$, so that $\frac{\partial \mathrm{P_m(a_m,s_L)}}{\partial \mathrm{s_L}} = \alpha$. Using (A3.3):

$$\frac{\partial P_m(a_m, s_L)}{\partial s_L} = \frac{1}{2} P_m^{-1} a_m = \alpha$$

$$\therefore \quad P_m = \frac{a_m}{2\alpha} \qquad \text{(FOCs}_\text{L}\text{)}$$

This expression may be conveniently used to simplify (A3.3):

$$P_m^2 = a_m\left(\frac{1}{2} + s_L\right) - \frac{1 + (1 + 2\alpha^2)^2}{2(1 + 2\alpha^2)^2} a_m^2$$

$$\therefore \quad \left(\frac{a_m}{2\alpha}\right)^2 = a_m\left(\frac{1}{2} + s_L\right) - \frac{1 + \left(1 + 2\alpha^2\right)^2}{2(1 + 2\alpha^2)^2} a_m^2$$

Assuming $a_m > 0$, which is a necessary condition for $P_m > 0$ and, as a consequence, for participation on the seller side, we obtain:

$$\frac{a_m}{4\alpha^2} = \left(\frac{1}{2} + s_L\right) - \frac{1 + (1 + 2\alpha^2)^2}{2(1 + 2\alpha^2)^2} a_m$$

$$\therefore \quad s_L = \frac{1}{2}\left[\left(\frac{\left(1 + 2\alpha^2\right)^3 + 2\alpha^2}{2\alpha^2(1 + 2\alpha^2)^2}\right) a_m - 1\right] \qquad \text{(A3.4)}$$

The first order condition for a_m is:

$$\frac{\partial \Pi_\mathrm{m}}{\partial \mathrm{a_m}} = -(\alpha \mathrm{a_m} - \alpha - \alpha \mathrm{s_L} + \mathrm{P_m}(\mathrm{a_m}, \mathrm{s_L}) - \mathrm{o}) + (1 - \mathrm{a_m})\left(\alpha + \frac{\partial \mathrm{P_m}(\mathrm{a_m}, \mathrm{s_L})}{\partial \mathrm{a_m}}\right) = 0$$

$$\therefore \quad \frac{\partial \Pi_m}{\partial a_m} = -(\alpha a_m - \alpha - \alpha s_L + P_m(a_m, s_L) - o) +$$

$$(1 - a_m)\left(\alpha + \frac{1}{2} P_m^{-1}\left[\left(\frac{1}{2} + s_L\right) - \frac{1 + \left(1 + 2\alpha^2\right)^2}{(1 + 2\alpha^2)^2} a_m\right]\right) = 0 \qquad \text{(FOCa}_\text{m}\text{)}$$

Substituting FOCs$_\mathrm{L}$ into FOCa$_\mathrm{m}$:

$$-\left(\alpha a_m - \alpha - \alpha s_L + \frac{a_m}{2\alpha} - o\right)$$
$$+ (1 - a_m)\left(\alpha + \frac{\alpha}{a_m}\left[\left(\frac{1}{2} + s_L\right) - \frac{1 + (1 + 2\alpha^2)^2}{(1 + 2\alpha^2)^2} a_m\right]\right) = 0$$

$$\therefore \qquad -\left(\frac{(1+2\alpha^2)}{2\alpha}a_m - \alpha - \alpha s_L - o\right) + (1-a_m)\alpha\left(1 + \frac{\left(\frac{1}{2}+s_L\right) - \frac{1+(1+2\alpha^2)^2}{(1+2\alpha^2)^2}a_m}{a_m}\right) = 0$$

Finally, substituting s_L with (A3.4):

$$-\left(\frac{(1+2\alpha^2)}{2\alpha}a_m - \alpha - \alpha\frac{1}{2}\left[\left(\frac{(1+2\alpha^2)^3+2\alpha^2}{2\alpha^2(1+2\alpha^2)^2}\right)a_m - 1\right] - o\right) +$$

$$(1-a_m)\alpha\left(1 + \frac{\left(\frac{1}{2}+\frac{1}{2}\left[\left(\frac{(1+2\alpha^2)^3+2\alpha^2}{2\alpha^2(1+2\alpha^2)^2}\right)a_m - 1\right]\right) - \frac{1+(1+2\alpha^2)^2}{(1+2\alpha^2)^2}a_m}{a_m}\right) = 0$$

$$a_m(o) = \left[\frac{1}{2} + \frac{\alpha^2(1+2\alpha^2)^2}{(1+2\alpha^2)^3-2\alpha^2}\right] + \frac{2\alpha(1+2\alpha^2)^2}{(1+2\alpha^2)^3-2\alpha^2}o \qquad \text{(A3.5)}$$

Hence, we found a_m in the monopolist's optimal choice. It is an affine function of the certifying cost o, and both the intercept and the coefficient of o are positive. Substituting (A3.5) respectively into (A3.4) and (FOCs$_L$):

$$s_L(o) = \left[\frac{1}{4(1+2\alpha^2)} + \frac{1}{8\alpha^2} + \frac{\alpha^2}{(1+2\alpha^2)^3-2\alpha^2}\right] + \frac{1}{2}\frac{(1+2\alpha^2)^3+2\alpha^2}{\alpha((1+2\alpha^2)^3-2\alpha^2)}o \qquad \text{(A3.6)}$$

$$P_m(o) = \left[\frac{1}{4\alpha} + \frac{\alpha(1+2\alpha^2)^2}{2((1+2\alpha^2)^3-2\alpha^2)}\right] + \frac{(1+2\alpha^2)^2}{(1+2\alpha^2)^3-2\alpha^2}o \qquad \text{(A3.7)}$$

For $0 < a_m < 1$ and $P_m > 0$ the second order sufficient conditions for a maximum are observed.[22]

To obtain the variables referring to the external pooling, we substitute (A3.5) respectively into (EP1a), (EP2a) and (EP3a):

[22] After substitutions and simplifications, we obtain:

$$\frac{\partial^2 \Pi_m}{\partial s_L^2} = -\alpha^2\frac{(1-a_m)}{P_m} < 0$$

$$\frac{\partial^2 \Pi_m}{\partial a_m^2} = -\frac{(1-a_m)}{P_m}\frac{(8\alpha^2+12\alpha^4+8\alpha^6+1)^2}{16\alpha^2(1+2\alpha^2)^4} - \frac{4\alpha^2+12\alpha^4+8\alpha^6+1}{2\alpha(1+2\alpha^2)^2} < 0$$

$$\frac{\partial^2 \Pi_m}{\partial s_L \partial a_m} = \frac{(1+2\alpha^2)^2+2\alpha^2\left((1+2\alpha^2)^2+1\right)}{4(1+2\alpha^2)^2}\frac{(1-a_m)}{P_m}$$

$$\frac{\partial^2 \Pi_m}{\partial s_L^2}\frac{\partial^2 \Pi_m}{\partial a_m^2} - \left(\frac{\partial^2 \Pi_m}{\partial s_L \partial a_m}\right)^2 = \alpha^2\left(\frac{4\alpha^2+12\alpha^4+8\alpha^6+1}{2\alpha(1+2\alpha^2)^2}\right)\frac{(1-a_m)}{P_m} > 0$$

$$P_{-m}(o) = \left[\frac{\alpha}{2(1+2\alpha^2)} + \frac{\alpha^3(1+2\alpha^2)}{(1+2\alpha^2)^3-2\alpha^2}\right] + \frac{2\alpha^2(1+2\alpha^2)}{(1+2\alpha^2)^3-2\alpha^2} o$$

(A3.8)

$$s_{-m}(o) = \left[\frac{1}{2(1+2\alpha^2)} + \frac{\alpha^2(1+2\alpha^2)}{(1+2\alpha^2)^3-2\alpha^2}\right] + \frac{2\alpha(1+2\alpha^2)}{(1+2\alpha^2)^3-2\alpha^2} o \qquad \text{(A3.9)}$$

$$a_{-m}(o) = \left[\frac{\alpha^2}{(1+2\alpha^2)} + \frac{2\alpha^4(1+2\alpha^2)}{(1+2\alpha^2)^3-2\alpha^2}\right] + \frac{4\alpha^3(1+2\alpha^2)}{(1+2\alpha^2)^3-2\alpha^2} o \qquad \text{(A3.10)}$$

Therefore, once again we find, in (A3.6) through (A3.10), affine functions with positive intercepts and coefficients of o. The unrestricted optimization will not produce negative values for any of these variables.

From (NC3), we know that $s_H = 1 - a_m + s_L$. Substituting (A3.4):

$$s_H = \frac{1}{2} + \left(\frac{1}{4\alpha^2} + \frac{1}{2(1+2\alpha^2)^2} - \frac{1}{2}\right) a_m \qquad \text{(A3.11)}$$

$s_H > 0$ is guaranteed, since $a_m \leq 1$. Substituting (A3.5):

$$s_H = \left(\frac{1}{8\alpha^2} + \frac{1}{4(1+2\alpha^2)^2} + \frac{1}{2}\frac{\left(1+2\alpha^2\right)^2}{(1+2\alpha^2)^3-2\alpha^2}\right) + \left(\frac{1}{2\alpha^2} + \frac{1}{(1+2\alpha^2)^2} - 1\right)\left(\frac{\alpha\left(1+2\alpha^2\right)^2}{(1+2\alpha^2)^3-2\alpha^2}\right) o$$

(A3.12)[23]

Substituting FOCs$_L$ and (A3.4) into (A3.2), the participation price in the platform will be given by:

$$p_m = \frac{\left(1+2\alpha^2\right)^3-2\alpha^2}{4\alpha(1+2\alpha^2)^2} a_m - \frac{\alpha}{2} \qquad \text{(A3.13)}$$

Substituting (A3.5):

$$p_m = \frac{\left(1+2\alpha^2+4\alpha^4\right)}{8\alpha(1+2\alpha^2)^2} + \frac{1}{2} o \qquad \text{(A3.14)}$$

A3.2 – Subcase ii-a – boundaries

Given the solution found in A.1, we need to know for which parameter values we will in fact obtain a solution in ii-a. In particular, we need to check that $s_H \leq 1$ and $s_L \geq s_{-m}$.

---

[23] $s_H = \left(\frac{1}{8\alpha^2} + \frac{1}{4(1+2\alpha^2)^2} + \frac{1}{2}\frac{\left(1+2\alpha^2\right)^2}{(1+2\alpha^2)^3-2\alpha^2}\right) + \left(\frac{4\alpha^2-4\alpha^4-8\alpha^6+1}{2\alpha^2(1+2\alpha^2)^2}\right)\left(\frac{\alpha\left(1+2\alpha^2\right)^2}{(1+2\alpha^2)^3-2\alpha^2}\right) o$

Thus, s_H is an increasing function of o for $\alpha > 0.789614$.

Using (NC5) and (A3.4), we obtain:

$$s_L - s_{-m} = \frac{(1+2\alpha^2)^2 + 8\alpha^6}{4\alpha^2(1+2\alpha^2)^2} a_m - \frac{1}{2}$$

Thus:

$$s_L - s_{-m} \geq 0 \leftrightarrow \frac{(1+2\alpha^2)^2 + 8\alpha^6}{2\alpha^2(1+2\alpha^2)^2} a_m \geq 1$$

This condition guarantees $p_m > 0$[24]. Isolating a_m:

$$s_L - s_{-m} \geq 0 \leftrightarrow a_m \geq \frac{2\alpha^2\left(1+2\alpha^2\right)^2}{(1+2\alpha^2)^2+8\alpha^6} \tag{A3.15}$$

Using (A3.5) to express this condition in terms of o:

$$s_L - s_{-m} \geq 0 \leftrightarrow o \geq \frac{96\alpha^8 - 16\alpha^4 - 6\alpha^2 + 96\alpha^{10} - 1}{4\alpha(1+2\alpha^2)^2((1+2\alpha^2)^2+8\alpha^6)} \tag{A3.16}$$

The numerator of the right hand side may be positive or negative. In $\mathbb{R}^+$, it will be negative only for $\alpha < 0.693194$, and, if that happens, the condition will always be satisfied.

On the other hand, using (A3.11):

$$s_H \leq 1 \leftrightarrow \left(\frac{1}{4\alpha^2} + \frac{1}{2(1+2\alpha^2)^2} - \frac{1}{2}\right) a_m \leq \frac{1}{2}$$

$$\therefore \quad s_H \leq 1 \leftrightarrow \frac{4\alpha^2 - 4\alpha^4 - 8\alpha^6 + 1}{2\alpha^2(1+2\alpha^2)^2} a_m \leq 1 \tag{A3.17}$$

The numerator of the coefficient of a_m does not have a definite sign. If it is negative or null, the condition will be guaranteed. That happens in $\mathbb{R}^+$ for $\alpha \geq 0.789614$. For higher values of α, it will be easier to have the expression in terms of o, so we do not need to alter the direction of the inequality. Substituting (A3.5) we obtain:

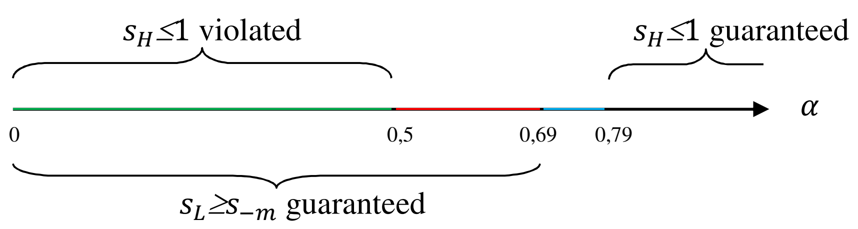
$$s_H \leq 1 \leftrightarrow o \leq \frac{64\alpha^6 - 8\alpha^4 - 6\alpha^2 + 352\alpha^8 + 544\alpha^{10} + 256\alpha^{12} - 1}{4\alpha(1+2\alpha^2)^2(4\alpha^2 - 4\alpha^4 - 8\alpha^6 + 1)} \tag{A3.18}$$

[24] From (A3.13), we know that $p_m > 0 \leftrightarrow \frac{\left(1+2\alpha^2\right)^3 - 2\alpha^2}{4\alpha^2(1+2\alpha^2)^2} a_m = \frac{\left[\left(1+2\alpha^2\right)^2 + 8\alpha^6\right] + 8\alpha^4}{4\alpha^2(1+2\alpha^2)^2} a_m > \frac{1}{2}$. But this is guaranteed if $\frac{\left[\left(1+2\alpha^2\right)^2 + 8\alpha^6\right]}{4\alpha^2(1+2\alpha^2)^2} a_m \geq \frac{1}{2}$.

In $\mathbb{R}^+$ with $\alpha < 0.789614$, the sign of the numerator will be positive for $\alpha > 0.500859$. Therefore, for α between 0.500859 and 0.789614, the observation of $s_H \leq 1$ in the unrestricted solution depends on the magnitude of o. For $\alpha < 0.500859$, the numerator will be negative (and the denominator positive), always violating (A3.18).

Figure A1 condenses what we know about the solutions of the unrestricted problem, regarding the possibility of violating any of the restrictions we need to observe.

Figure A1



Thus, as far as we have shown, the only values of α that could support simultaneous violation of both restrictions would be the ones between 0.693194 and 0.789614.

Finally, it is interesting to notice that both restriction will never be simultaneously violated. To verify that, first, note that if $4\alpha^2 - 4\alpha^4 - 8\alpha^6 + 1 \leq 0$, it is guaranteed that $s_H < 1$. If this condition is not observed, the violation of $s_L \geq s_{-m}$ guarantees $s_H < 1$, by the following:

$$s_L - s_{-m} < 0 \leftrightarrow a_m < \frac{2\alpha^2(1+2\alpha^2)^2}{(1+2\alpha^2)^2+8\alpha^6} \qquad \text{(A3.19)}$$

But $\frac{2\alpha^2(1+2\alpha^2)^2}{(1+2\alpha^2)^2+8\alpha^6} < \frac{2\alpha^2(1+2\alpha^2)^2}{4\alpha^2-4\alpha^4-8\alpha^6+1}$, hence, if $s_L - s_{-m} < 0$, $a_m < \frac{2\alpha^2(1+2\alpha^2)^2}{4\alpha^2-4\alpha^4-8\alpha^6+1}$, which, given $4\alpha^2 - 4\alpha^4 - 8\alpha^6 + 1 > 0$, guarantees $s_H < 1$ by (A3.17).

Finally, we can verify the limit for $a_m \leq 1$ directly from (A3.5):

$$o \leq \frac{1+2\alpha^2+4\alpha^4}{4\alpha(1+2\alpha^2)^2} \tag{A3.20}$$

Taking the difference between the limits expressed in (A3.18) and (A3.20):

$$\frac{64\alpha^6 - 8\alpha^4 - 6\alpha^2 + 352\alpha^8 + 544\alpha^{10} + 256\alpha^{12} - 1}{4\alpha(1+2\alpha^2)^2(4\alpha^2 - 4\alpha^4 - 8\alpha^6 + 1)} - \frac{1 + 2\alpha^2 + 4\alpha^4}{4\alpha(1+2\alpha^2)^2}$$
$$= \frac{(1+4\alpha^2)(2\alpha^2 + 4\alpha^4 - 1)(4\alpha^2 + 12\alpha^4 + 8\alpha^6 + 1)}{2\alpha(1+2\alpha^2)^2(4\alpha^2 - 4\alpha^4 - 8\alpha^6 + 1)}$$

The expression. $2\alpha^2 + 4\alpha^4 - 1$ is increasing in $\mathbb{R}^+$ and has a root equal to 0.555893. Additionally, for sellers with $\alpha < 0.789614$, where condition (A3.17) may be violated, the denominator is also positive. Therefore, we know that for $0.555893 < \alpha < 0.789614$ it is also sufficient to consider (A3.20). As for $\alpha < 0.555893$, it is enough to consider (A3.17).

A3.3 – Subcase ii-a with $s_H = 1$

We know that s_H is indifferent between participating in the platform and not trading. Thus:

$$P_m - p_m - \alpha = 0$$
$$\therefore \quad p_m = P_m - \alpha \tag{A3.21}$$

Equation (NC3) implies $a_m = s_L$, so that the monopolist's choice reduces only to one variable. Using this equality and substituting $s_H = 1$, $m_{-m} = \frac{s_{-m}}{2}$ and (EP1a) through (EP3a) into (M4), we obtain:

$$P_m(a_m) = \sqrt{\frac{(1+2\alpha^2)^2 a_m + 4\alpha^2(1+\alpha^2)a_m^2}{2(1+2\alpha^2)^2}} \tag{A3.22}$$

Substituting (A3.21) and (A3.22) into monopolist's profit:

$$\Pi_m = (1 - a_m)(p_m - o)$$

$$\Pi_m = (1 - a_m)(P_m(a_m) - \alpha - o) \quad \text{(A3.23)}$$

Then, the first order conditions become:

$$\frac{\partial \Pi_m}{\partial a_m} = -(P_m(a_m) - \alpha - o) + (1 - a_m)\frac{dP(a_m)}{da_m} = 0$$

and

$$\frac{\partial \mathrm{P_m}}{\partial a_m} = \frac{1}{2} P_m^{-1} \frac{\left(\alpha + 2\alpha^2\right)^2 + 8\alpha^2(1+\alpha^2)a_m}{2(1+2\alpha^2)^2}$$

Doing the algebra and substituting (A3.22), we obtain:

$$\left(16\alpha^2(1+\alpha^2)\right)^2 a_m^4 + 32\alpha^2(1+\alpha^2)(4\alpha^2 + 4\alpha^4 + 3)a_m^3 + \left((9 - 152\alpha^4 - 384\alpha^6 - 368\alpha^8 - 128\alpha^{10} - 8\alpha^2) - 32\alpha^2(2o\alpha + o^2)(1+2\alpha^2)^2(1+\alpha^2)\right)a_m^2 - (2(4\alpha^2 + 4\alpha^4 + 3)(1+2\alpha^2)^2 + 8(\alpha + o)^2(2+2\alpha^2)^4)a_m + (1+2\alpha^2)^4 = 0 \quad (\mathrm{FOCa}_{m}, s_H{=}1)$$

Equation (FOCa_{m},s_H=1) requires a numeric solution and squaring both sides of the equation while manipulating terms may produce a real and otherwise acceptable solution for the problem that needs to be discarded because of a step prior to that operation[25].

A3.4 – Subcase ii-b – unrestricted analysis

In this subsection, we study the possibility that the monopolist chooses a position adjacent to the external pooling, so that $s_L = s_{-m}$. Substituting (NC3) and (EP1b) into (NC6), we obtain:

$$P_m = \sqrt{\frac{a_m - (a_m - s_L)^2}{2}} \quad \text{(A3.24)}$$

Substituting p_m with the indifference of s_H and (A3.24) into the monopolist's profit, we obtain the following objective function:

$$\Pi_m = (1 - a_m)(p_m - o)$$

∴

[25] On the other hand, the second other condition is easier to obtain and has a definite sign:

$$\frac{\partial^2 \Pi}{\partial a_m^2} = -P_m^{-1}\frac{1}{2(1+2\alpha^2)^2} - a_m \frac{3}{2} P_m^{-1} \frac{8\alpha^2(1+\alpha^2)}{2(1+2\alpha^2)^2} - (1 - a_m)\frac{1}{4} P_m^{-3}\left(\frac{\left(1+2\alpha^2\right)^2 + 8\alpha^2\left(1+\alpha^2\right)^2 a_m}{2(1+2\alpha^2)^2}\right)^2 < 0$$

$$\Pi_m = (1 - a_m)(P_m(a_m, s_L) - \alpha + \alpha a_m - \alpha s_L - o)$$

Hence, the first order condition for s_L is given by:

$$\frac{\partial \Pi_m}{\partial s_L} = (1 - a_m)\left(\frac{\partial P_m}{\partial s_L} - \alpha\right) = 0$$

where $$\frac{\partial P_m}{\partial s_L} = \frac{a_m - s_L}{2P_m}$$

Since the operation of the platform implies $a_m < 1$, it follows that:

$$P_m = \frac{a_m - s_L}{2\alpha} \qquad (\text{FOC}s_L, s_L = s_{-m})$$

Substituting ($\text{FOC}s_L, s_L = s_{-m}$) into (A3.24), we obtain:

$$P_m = \sqrt{\frac{a_m}{2(1+2\alpha^2)}} \qquad (\text{A3.25})$$

The first order condition for a_m is given by:

$$\frac{\partial \Pi_m}{\partial a_m} = -(P_m - \alpha + \alpha a_m - \alpha s_L - o) + (1 - a_m)\left(\frac{\partial P_m}{\partial a_m} + \alpha\right) = 0$$

where $$\frac{\partial P_m}{\partial a_m} = \frac{s_L - a_m + \frac{1}{2}}{2P_m}$$

Therefore:

$$\frac{\partial \Pi_m}{\partial a_m} = -(P_m - \alpha + \alpha a_m - \alpha s_L - o) + (1 - a_m)\left(\frac{s_L - a_m + \frac{1}{2}}{2P_m} + \alpha\right) = 0 \qquad (\text{FOC}a_m, s_L = s_{-m})$$

Substituting (A3.24) and (A3.25) into ($\text{FOC}a_m, s_L = s_{-m}$)[26], we obtain:

$$-6(1 + 2\alpha^2)P_m^2 + 4(o + \alpha)P_m + 1 = 0$$

The unique nonnegative root of this equation is:

$$P_m = \frac{2(o+\alpha) + \sqrt{4(o+\alpha)^2 + 6(1+2\alpha^2)}}{6(1+2\alpha^2)} \qquad (\text{A3.26})$$

[26] After substitutions, the second order condition is given by (signs are evaluated for $P_m > 0$ and $a_m < 1$):

$$\frac{\partial^2 \Pi_m}{\partial s_L^2} = -\frac{1 + 2\alpha^2}{2}(1 - a_m)P_m^{-1} < 0$$

$$\frac{\partial^2 \Pi_m}{\partial a_m^2} = -\frac{1}{2}P_m^{-1} - (1 - a_m)\left(\frac{1+2\alpha^2}{2}P_m^{-1} - \frac{1}{2}\alpha P_m^{-2} + \frac{1}{16}P_m^{-3}\right) < 0$$

$$\frac{\partial^2 \Pi_m}{\partial s_L \partial a_m}(1 - a_m)\frac{P_m^{-1}}{2}\left(1 + (a_m - s_L)\left(\alpha P_m^{-1} - \frac{1}{4}P_m^{-2}\right)\right) > 0$$

$$\frac{\partial^2 \Pi_m}{\partial s_L^2}\frac{\partial^2 \Pi_m}{\partial a_m^2} - \left(\frac{\partial^2 \Pi_m}{\partial s_L \partial a_m}\right)^2 = \frac{1}{4}(1 - a_m)P_m^{-2}\left(2\alpha^2 + \frac{1}{8}P_m^{-2}(1 - a_m) + 1\right) > 0$$

Substituting (A3.26) into (A3.25), we obtain:

$$a_m = \frac{4(o+\alpha)^2+2(o+\alpha)\sqrt{4(o+\alpha)^2+6(1+2\alpha^2)}+3(1+2\alpha^2)}{9(1+2\alpha^2)} \tag{A3.27}$$

Using (A3.26) and (A3.27) with (A3.24), we obtain that:

$$s_L = \frac{4(o+\alpha)^2-6\alpha(o+\alpha)+(2o-\alpha)\sqrt{4(o+\alpha)^2+6(1+2\alpha^2)}+3(1+2\alpha^2)}{9(1+2\alpha^2)} \tag{A3.28}$$

Substituting (A3.27) and (A3.28) into (M1):

$$s_H = 1-\alpha\frac{2(o+\alpha)+\sqrt{4(o+\alpha)^2+6(1+2\alpha^2)}}{3(1+2\alpha^2)} \tag{A3.29}$$

Note that it is guaranteed that $s_H < 1$.

With the indifference of s_H, (A3.29) and (A3.26) we recover p_m:

$$p_m = \frac{2(o-2\alpha)+\sqrt{4(o+\alpha)^2+6(1+2\alpha^2)}}{6} \tag{A3.30}$$

Substituting (A3.27) respectively into (M6b) and (M8b):

$$P_{-m} = \sqrt{\frac{\alpha\left(8(o+\alpha)^3-8(2\alpha-o)(o+\alpha)^2+18o(1+2\alpha^2)+\sqrt{4(o+\alpha)^2+6(1+2\alpha^2)}\left(4(2o-\alpha)(o+\alpha)+3(1+2\alpha^2)\right)\right)}{54(1+2\alpha^2)^2}} \tag{A3.31}$$

$$a_{-m} = \alpha\frac{2(o+\alpha)+\sqrt{4(o+\alpha)^2+6(1+2\alpha^2)}}{3(1+2\alpha^2)} \tag{A3.32}$$

Within this case, it is necessary to check that the result we obtain implies $P_m - p_m \geq P_{-m}$ and $P_{-m} - \alpha s_L \geq 0$. It is worth noticing that in case it is necessary to impose $P_m - p_m = P_{-m}$, the indifference of s_H, $P_m - p_m - \alpha s_H = 0$ guarantees $P_{-m} - \alpha s_L > 0$. On the other hand, if we have to impose $P_{-m} - \alpha s_L = 0$, the indifference of s_H implies, $P_m - p_m - \alpha s_L > 0$, and, hence, $P_m - p_m > P_{-m}$. For the range of parameters under analysis, it is necessary to solve the case in which $P_m - p_m = P_{-m}$.

A3.5 – Subcase ii-b with $s_L = s_{-m}$ and $P_m - p_m = P_{-m}$:

In this configuration, equation (A3.24) is still valid. Stating from the equality between $P_m - p_m$ and P_{-m}, we use the indifference of s_H and (EP1b):

$$P_{-m} = P_m - p_m$$

$$\therefore \qquad \sqrt{\frac{(a_m - s_L)s_L}{2}} = \alpha s_H$$

Substituting (NC3):

$$\sqrt{\frac{(a_m - s_L)s_L}{2}} = \alpha(1 - a_m + s_L)$$

Both sides must be positive. Taking the squares and reorganizing terms, we obtain:

$$\alpha^2(a_m - s_L)^2 - \left(2\alpha^2 + \frac{s_L}{2}\right)(a_m - s_L) + \alpha^2 = 0$$

Hence:

$$a_m = 1 + s_L + \frac{\frac{s_L}{2} - \sqrt{2\alpha^2 s_L + \frac{s_L^2}{4}}}{2\alpha^2} \tag{A3.33}$$

This equation shows that, in this subcase, the monopolist's choice is reduced to only one variable.

Substituting (A3.33) into (A3.24), we obtain:

$$P_m(s_L) = \sqrt{\frac{\alpha^2(4\alpha^2 - 3)s_L - \frac{s_L^2}{2} + (s_L + 2\alpha^2)\sqrt{2\alpha^2 s_L + \frac{s_L^4}{4}}}{8\alpha^4}} \tag{A3.34}$$

Substituting, the indifference of s_H, (NC3) and (A3.33) into the monopolist's profit, we obtain:

$$\Pi_m = \left(\frac{\sqrt{2\alpha^2 s_L + \frac{s_L^4}{4}} - \frac{s_L}{2}}{2\alpha^2} - s_L\right)\left(P_m(s_L) + \frac{\frac{s_L}{2} - \sqrt{2\alpha^2 s_L + \frac{s_L^2}{4}}}{2\alpha} - o\right) \tag{A3.35}$$

The first order condition is given by:

$$\frac{\partial \Pi_m}{\partial s_L} = \left(\frac{\sqrt{2\alpha^2 s_L + \frac{s_L^2}{4}}^{-1}\left(2\alpha^2 + \frac{s_L}{2}\right) - 1}{4\alpha^2} - 1 \right)\left(P_m + \frac{\frac{s_L}{2} - \sqrt{2\alpha^2 s_L + \frac{s_L^2}{4}}}{2\alpha} - o \right)$$

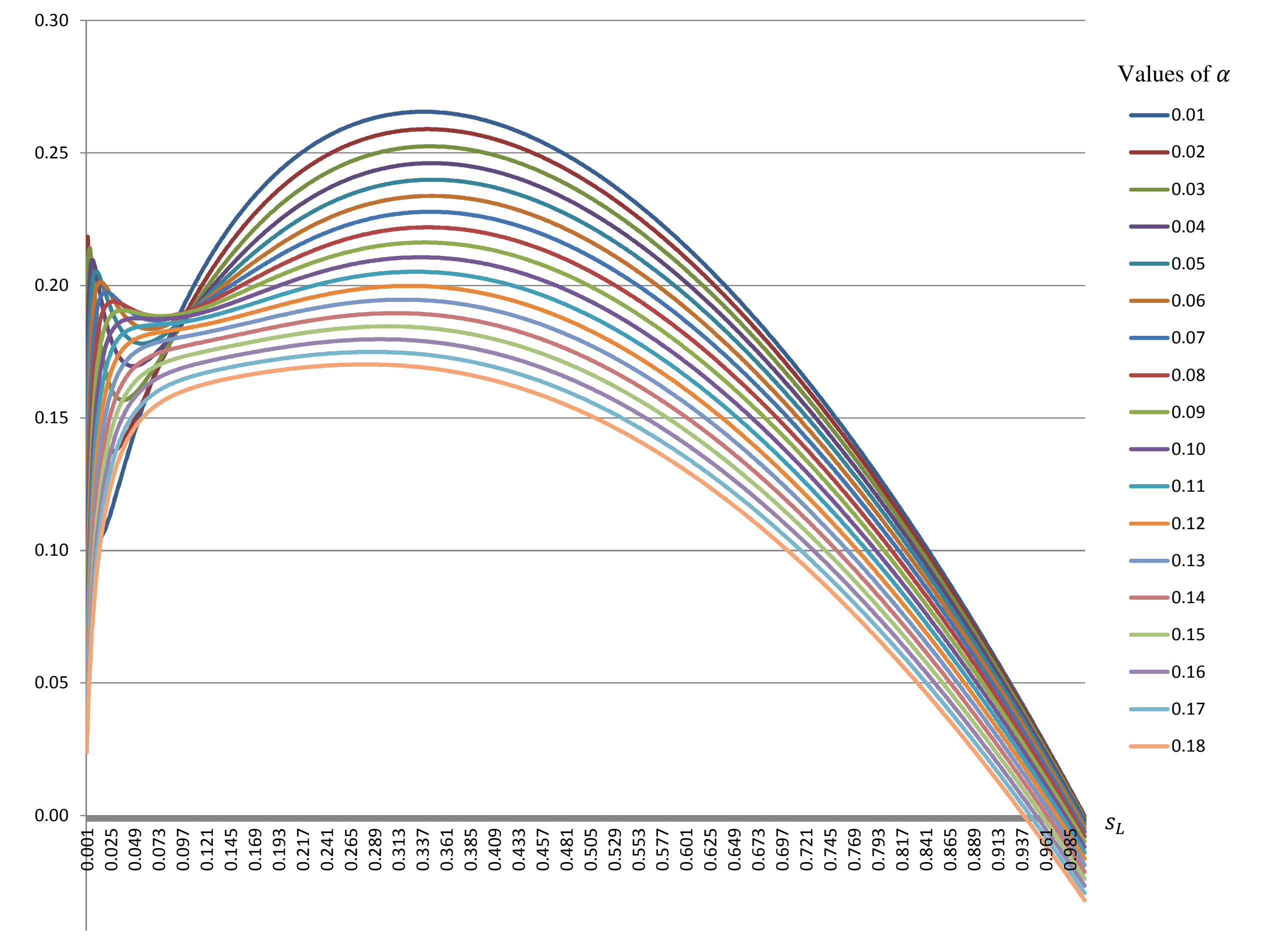
$$+ \left(\frac{\sqrt{2\alpha^2 s_L + \frac{s_L^2}{4}} - \frac{s_L}{2}}{2\alpha^2} - s_L \right)\left(\frac{dP_m}{ds_L} + \frac{\frac{1}{2} - \frac{1}{2}\sqrt{2\alpha^2 s_L + \frac{s_L^2}{4}}^{-1}\left(2\alpha^2 + \frac{s_L}{2}\right)}{2\alpha} \right)$$

where

$$\frac{dP_m}{ds_L} = P_m^{-1}\frac{1}{16\alpha^4}\left(\alpha^2(4\alpha^2 - 3) - s_L + \sqrt{2\alpha^2 s_L + \frac{s_L^2}{4}} + \frac{(s_L + 2\alpha^2)\left(2\alpha^2 + \frac{s_L}{2}\right)}{2}\sqrt{2\alpha^2 s_L + \frac{s_L^2}{4}}^{-1} \right)$$

This first order condition needs a numerical solution and presented more than one critical point for the values of (α, o) that we used. Thus, instead of computing the second order condition, we plot (A3.35) for the values under study. The result is depicted in Graph A1. We approximate the interval for the optimal value of s_L from a grid used to build the graph. Subsequently, we solve the first order condition in these restricted domains.

Graph A1 – Monopolist's profit as a function of s_L



A3.6 – Results of cases ii-a and ii-b

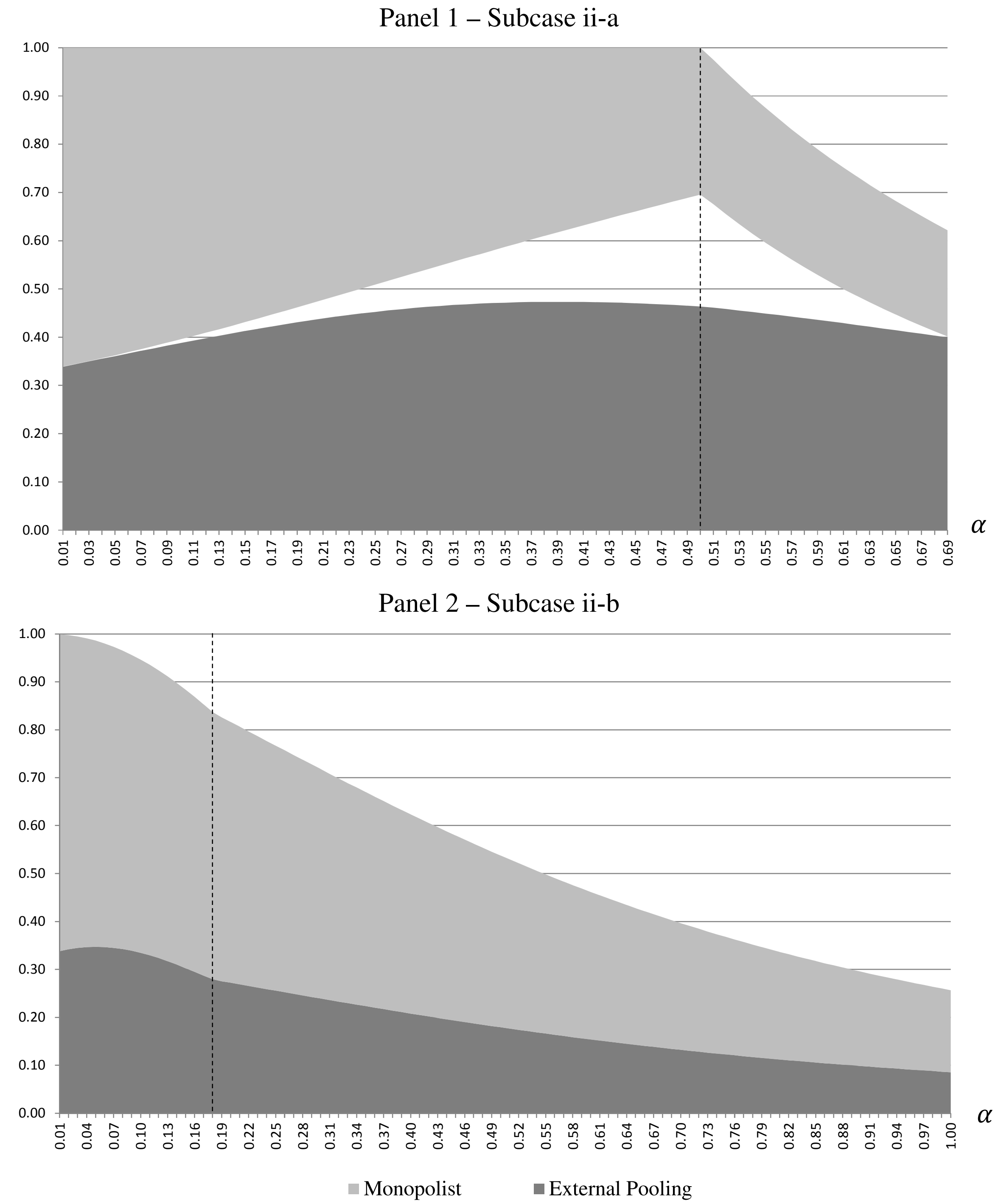
Graph 1, in the main text, is built from the comparison of maximum profits obtained by the monopolist operating in case ii-a or ii-b. It is interesting to show how the platform occupies quality spaces in an optimal fashion within each of these cases. We show that in Graph A2.

In panel 1, we show how the space of sellers who do not participate increases with α to the left of the dashed line, which corresponds to the active restriction $s_H = 1$. To the right of that line, where the restriction is not binding, that space reduces with α.

In panel 2, the region to the left of the dashed line represents situation that require the imposition of $P_m - p_m = P_{-m}$, while the region to the right is obtained from the unrestricted solution.

Graph 1 presents the results in Panel 1 of Graph A2 up until where $\alpha = 0.07$, and the ones in Panel 2 to the right of that point.

Graph A2 – Spaces occupied in case ii



Appendix 4 – Duopoly

We concentrate in this appendix all demonstrations referring to the duopoly game. It proceeds by sequentially analyzing situations to rule out those that could not support a Nash equilibrium. In this sense, we assume the (potential) existence of some equilibrium until the end, where all possible allocations are then ruled out.

A4.1 - Superposition

If more than one type of seller is present in both platforms, we say there is superposition on the sellers' side[27]. This does not mean there is multihoming. Instead, if we imagine a mass of agents concentrated on some quality value, part of them would choose one platform while the rest would choose the other one. Similarly, there might be superposition on the buyers' side.

The occurrence of superposition requires that the types of agents in question are indifferent between platforms and, in such a case, we assume that their participation in each platform is proportional to its total size. In what follows, we divide the analysis into some cases to show that superposition will not occur in equilibrium.

A4.1.1 – $m_i < m_j$

Second stage of the game

Assume some initial situation with superposition on the buyers' side. In particular, let us say that there are agents of type a_0 in both platforms. Thus, the indifference of a_0 between them means that $a_0 m_i - g(P_i + p_{ai}) = a_0 m_j - g(P_j + p_{aj})$. Assume that the expected quality in one of the platforms is inferior: $m_i < m_j$. Then, all types with $a > a_0$

[27] We define this so that one type only is not enough for superposition, in order to accommodate cases in which we find a seller's threshold between platforms and this not characterized as presenting superposition.

strictly prefer j over i, while all types with $a < a_0$ strictly prefer i over j. Thus, we obtain affirmative 2.

Affirmative 2: If the expected quality is different between platforms, there may not be superposition on the buyers' side. Furthermore, the platform with superior expected quality concentrates the high quality buyers.

Additionally, it is convenient to note that the buyers from the higher quality platform would rather trade in the lower quality one than not participating.

Still considering that case with $m_i < m_j$, we have to analyze the possibility of superposition on the sellers' side. From the buyers' side, this difference in qualities implies that in any potential equilibrium $g(P_i + p_{ai}) < g(P_j + p_{aj})$, from which it would follow that $P_i + p_{ai} < P_j + p_{aj}$.

On the other side, we know that all types of sellers for which there was superposition would be indifferent between platforms. Assuming s_0 is one such type, we obtain $P_i - p_{si} - \alpha s_0 = P_j - p_{sj} - \alpha s_0$, which implies $P_i - p_{si} = P_j - p_{sj}$. Hence, all sellers would be indifferent between platforms and there would be participation in both platforms of every type that would trade, given both platforms would admit them. Therefore, we conclude that:

Affirmative 3: $\boldsymbol{m_i < m_j}$ implies $\boldsymbol{s_{Li} < s_{Lj}}$.

First stage of the game

Even if the conditions for sellers to have the same payoff in both platforms in case they trade are observed, it still can be a dominated strategy for sellers to participate in one of the platforms. The comparison of participants' payoffs between platforms requires, rigorously speaking, a conjecture about the probability of not trading while participating. However, there are situations in which even without stating these beliefs, one platform surpasses the other in payoff. For the sellers, such a situation arises if they have assured (trading or not) in one platform a payoff at least as high as they would have if they actually traded in the other. Then, to exclude the possibility of such a situation, which could not support superposition, it must be that: $p_{si} > -(P_j - p_{sj})$ and $p_{sj} > -(P_i - p_{si})$. Adding up both expressions, we obtain: $p_{si} + p_{sj} > -(P_j + P_i) + p_{sj} + p_{si}$ and thus, $P_i + P_j > 0$. This means that if both transaction prices are null, either sellers' participation prices must be identical, or the participation in one platform will be a dominated strategy for sellers whose participation is allowed in both of them.

Conditions $P_i + p_{ai} < P_j + p_{aj}$ and $P_i - p_{si} = P_j - p_{sj}$, lead to $p_{ai} + p_{si} < p_{aj} + p_{sj}$, i.e., the platform that occupies the lower quality niche must charge a smaller amount per peer.

Now, assume an initial situation with $m_i < m_j$ and $P_k > 0$. Then, platform i could absorb all the clients of j with the following deviation[28]:

i. $P_i' + p_{ai}' = P_i + p_{ai}$,
ii. $P_i' - p_{si}' = P_i - p_{si}$,
iii. $p_{si}' \leq -P_j + p_{sj}$ (i.e., $-p_{si}' \geq P_j - p_{sj} = P_i - p_{si}$)
iv. $s_{Li}' = s_{Li}$

Condition (iii) implies that platform i pays the sellers for participating the same net amount they obtain by trading in platform j. Since the initial participation of sellers in j implies that $P_j - p_{sj} > 0$, p_{si}' is therefore negative.

With these new prices, all sellers who participate in the initial situation would go to i, without having to worry about consumers' potential responses, since the payoff in i is higher than the one in j, even if a transaction does not occur. Given that there was buyers' participation in i and that they are indifferent between this situation and the initial one, we know that the buyers who were originally in j would rather participate in i than not trading, although the expected quality is smaller.

[28] In what follows, we indicate with sign ($'$) the deviations, i.e., the strategies that could have been played. We are interested in strategies that a platform regrets not having chosen. If such "profitable" deviations exist, we can rule out the initial situation as a candidate for supporting equilibrium.

Conditions (i) and (ii) imply $p'_{ai} + p'_{si} = p_{ai} + p_{si}$. Hence, the total price per transaction is the same. Additionally, $p'_{ai} - p_{ai} = -(p'_{si} - p_{si}) = -(P'_i - P_i)$, where the last equality follows from (i). Thus, the increase in buyers' participation price must be compensated by a reduction in purchase price. This indicates a limitation of this deviation, since purchase prices may not become negative. Since (ii) implies $p'_{si} = -P_j + p_{sj} + P'_i$, the only possibility of obtaining simultaneously (ii) and (iii) is with equality in (iii) and $P'_i = 0$. Given that $P_j > 0$, it is still dominant after the deviation for sellers to participate in i.

Another possibility would be a deviation of j from the same initial situation. However, it would require quantitative analysis. By setting P'_j equal to zero and $-p'_{sj} = P_i - p_{si}$, j would become the dominant choice for sellers. The admission of more buyers, however, might require a reduction in their participation price.

Now, if in the initial situation $P_j = 0$, $P_i > 0$ and $P_i - p_{si} = P_j - p_{sj}$, the situation could not have been of superposition, since it would have been dominant for admitted sellers to participate in j. A symmetric situation would happen if $P_i = 0$ and $P_j > 0$.

Alternatively, if we originally had $P_i = P_j = 0$, there could only be superposition with $p_{si} = p_{sj}$. To analyze that situation, first, note that the participation of any seller implies $p_{si} = p_{sj} < 0$, since there is no additional value to be obtained from trade. As a consequence of the lower quality platform being cheaper for buyers, $p_{aj} > p_{ai} > 0$, where the last inequality results from participation of the platform i, i.e., from $p_{ai} \geq -p_{si}$. If $p_{ai} + p_{si} > 0$ and $p_{si} < 0$, it would be a profitable deviation for i to reduce p_{si} by an infinitesimal amount. Then, there would be an infinitesimal decrease in revenue from clients in the initial allocation and a discrete increase from absorbing the clients of j.

Still, there is the possibility of $P_i = P_j = 0$, $p_{si} = p_{sj} < 0$ and $p_{ai} + p_{si} = 0$. In that initial situation, $p_{aj} + p_{sj} > 0$ and platform j makes a profit, since it is not empty. In that case, i could implement the following profitable deviation:

i. $p'_{ai} = p_{aj}$,

ii. p'_{si} infinitesimally lower than $p_{si} = p_{sj}$;

iii. $s'_{Li} > s_{Lj}$, defined so as to allow only the number of sellers equal to the one of buyers initially in j.

Condition (iii) guarantees that there is no loss because of excess of sellers. Since originally the types with quality higher or equal to s_{Lj} participated in some proportion of i, the new lowest acceptable quality limit must be higher than that. With that deviation, the original buyers in j have a strictly positive benefit coming from the increase in quality.

Summing up the results for the possibility of superposition with $m_i < m_j$, there cannot be a Nash equilibrium with:

- $P_i, P_j > 0$;
- $P_i = P_j = 0$ e
  - $p_{ai} + p_{si} > 0$, or
  - $p_{ai} + p_{si} = 0$

Then, we conclude that:

Affirmative 4: There is no Nash equilibrium with superposition and $m_i < m_j$.

A4.1.2 – $m_i = m_j$

Second stage of the game

Suppose an initial situation with $m_i = m_j$. Again, it is required from an allocation that supports a Nash equilibrium that all participants of the platform trade. If some participant does not trade, he will be regretful if the participation price is positive. If it is negative, the platform will regret having allowed his participation.

Assume $P_i - p_{ai} < P_j - p_{aj}$. Then, each buyer in j would regret to having chosen i. Thus, with this inequality, it is not possible for i to operate.

Thus, for the examination of superposition, $m_i = m_j$ implies $P_i - p_{ai} = P_j - p_{aj}$ and the participations of all types of buyers who trade are proportional to the sizes of platforms.

On its turn, on the sellers' side, superposition would imply again that $P_i - p_{si} = P_j - p_{sj}$. Since all the types of sellers who participate are present in both platforms in which they are admitted proportionally to their sizes, $m_i = m_j$ implies $s_{Li} = s_{Lj}$. Thus:

Affirmative 5: $\boldsymbol{m_i = m_j}$ implies $\boldsymbol{s_{Li} = s_{Lj}}$

First stage of the game

Again, it is necessary that $p_{si} > -(P_j - p_{sj})$ and $p_{sj} > -(P_i - p_{si})$, which imply $P_i, P_j > 0$, for the participation in one platform not to be dominated for all admitted sellers. Then, platform i could benefit, keeping the purchase price, with a deviation similar to the one proposed with $m_i < m_j$:

i. $P_i' + p_{ai}' = P_i + p_{ai}$,
ii. $P_i' - p_{si}' = P_i - p_{si}$,
iii. $p_{si}' \leq -P_j + p_{sj}$ (i.e., $-p_{si}' \geq P_j - p_{sj} = P_i - p_{si}$),
iv. $s_{Li}' = s_{Li}$

Like when we analyzed $m_i < m_j$, if originally we had, $P_j = 0$ with $P_i > 0$ and $P_i - p_{si} = P_j - p_{sj}$, the initial allocation could not be of superposition, since it would have been dominant for admitted sellers to participate in j. A symmetric situation would hold if $P_i = 0$ and $P_j > 0$.

Therefore, we must analyze the case with $P_i = P_j = 0$. In that situation, $p_{si} = p_{sj} < 0$ and, consequently $p_{ai} = p_{aj} > 0$. If $p_{si} + p_{ai} > 0$, it would be possible for one platform to absorb the other. For example, i could deviate choosing:

i. $p'_{ai} = p_{ai} = p_{aj}$,
ii. p'_{si} infinitesimally lower than $p_{si} = p_{sj}$,
iii. $s'_{Li} = s_{Lj}$.

Given that, the participation in i would become dominant for all sellers with quality higher than s_{Lj}, and buyers would keep their original utilities, since neither the expected quality nor their participation price would be altered.

Finally, if $P_i = P_j = 0$, $p_{si} = p_{sj} < 0$ and $p_{ai} + p_{si} = 0$, it must be that $p_{aj} + p_{sj} = 0$, i.e., the profit of both platforms is null. Then, one of the platforms, for example i, could obtain profit by implementing this deviation:

i. $p'_{ai} > p_{aj}$,
ii. p'_{si} infinitesimally lower than $p_{si} = p_{sj}$,
iii. $s'_{Li} > s_{Lj}$, defined to allow at least as many buyers as the amount of sellers.

With conditions (ii) and (iii), all sellers with quality superior to s'_{Li} have as a dominant strategy to participate in i. The expected quality of the sellers in j would be given by $m'_j = \left(s'_{Li} + s_{Lj}\right)/2$.

Call the best seller and the worst buyer from the original situation s_H and a_L, respectively. Since the reduction of p_{si} in (ii) is infinitesimal, in the limit s_H will not change. The increase in (iii), on the other hand, needs to be discrete, because otherwise, in the limit, $m'_i = m_i$, which would not support the discrete increase in (i).

It is worth noticing that, for the initial situation to have an equilibrium on the second stage of the game, it would be necessary for a_L to be indifferent between participating or not. Were a_L to strictly prefer participation, there would be some nonparticipating buyer who would prefer it too.

We need to show that there is a deviation (iii) which allows the discrete increase in (i) without the possibility of generating a loss for i with more buyers and sellers.

Given (ii) (iii), all buyers who are sure to trade in i prefer it strictly to j at the original participation prices. Then, there is some deviation (i) that preserves this order of preferences. The buyers who are sure to trade are those in $[1-(s_H - s'_{Li}), 1]$. Take some buyer a_0 in that interval. Then, $a_0 m_j - p_{aj}^2 = a_{0m_i} - p_{ai}^2$ implies $a_0 m'_j - p_{aj}^2 < a_0 m'_i - p_{ai}^2$. It suffices to choose some $p'_{ai} > p_{ai}$ that preserves inequality $a_0 m'_i - p''^2_{ai} > a_0 m'_j - p_{aj}^2$.

Summing up the results for the possibility of superposition with $m_i = m_j$, there cannot be a Nash equilibrium with:

- $P_i, P_j > 0$;
- $P_i = P_j = 0$
  - $p_{ai} + p_{si} > 0$, or
  - $p_{ai} + p_{si} = 0$

A4.1.3 – Conclusion about platform superposition in duopoly

As we have shown, there cannot be a Nash equilibrium with superposition with either $m_i = m_j$ or $m_i < m_j$. The same, by symmetry goes for $m_i > m_j$. Thus, we obtain affirmative 6.

Affirmative 6: There cannot be superposition in a Nash equilibrium in duopoly.

The main importance of affirmative 6 is to restrict the candidates to Nash equilibrium to situations without superposition.

A4.2 – Allocations without superposition

A4.2.1 – Continuity

Second stage of the game

As we have defined, client types who are indifferent between platforms would distribute between both of them, proportionally to their sizes. Therefore, in cases without superposition, types should not be indifferent between platforms, but for a threshold type.

We should ask, however, if agent intervals occupied by platforms are necessarily continuous. Take, for instance, two types of sellers, s_0 and s_1, such that $s_1 > s_0$, who both participate in platform i. Then, if participation of at least one of them is allowed in j, $P_i - p_{si} - \alpha s > P_j - p_{sj} - \alpha s$ for all sellers. Additionally, $P_i - p_{si} - \alpha s_0 > P_i - p_{si} - \alpha s_1 > 0$. This implies that sellers in $[s_0, s_1]$ participate in i, since the platform will have no incentive in excluding any seller in this interval, given that it admits s_0. On the other hand, if s_0 and s_1 are not admitted in j, the types between them should also be excluded from that platform. Even so, we know that $P_i - p_{si} - \alpha s_1 > 0$, implying that all types in $[s_0, s_1]$ prefer to participate in i over not participating. Since i does not have any incentive to exclude any seller with quality higher than s_0 (given the admittance of this type), all sellers in $[s_0, s_1]$ should participate in i. The continuity of participant seller intervals imply that expected qualities are different between platforms.

Affirmative 7: In each platform, participant sellers form continuous subsets of $[0, 1]$. Without superposition, this implies $m_i \neq m_j$.

From this affirmative, we see that the distribution of buyers between platforms is given by what we studied while analyzing superposition on the buyers' side with $m_i \neq m_j$. Thus, they will form continuous intervals, with one type of buyer indifferent between platforms.

A4.2.2 – Digressions about mechanisms available to platforms

We assume that $m_i < m_j$ in the initial situations we propose. Then, we define a_{Lk} as the quality of the worst buyer participant of platform k. Therefore, platform i will have buyers in $\left[a_{Li}, a_{Lj}\right]$, while j has those in $\left[a_{Lj}, 1\right]$. As for the sellers, we define s_{Lk} and s_{Hk} as, respectively, the lowest and the highest quality value for a seller in k. Thus, the participant sellers of platforms are $[s_{Lk}, s_{Hk}]$, where $s_{Hi} \leq s_{Lj}$.

Given prices P_k, p_{ak} and p_{sk}, platforms' profits in a potential equilibrium situation (with $a_{Hk} - a_{Lk} = s_{Hk} - s_{Lk}$) are given by:

$$\Pi_k^D = [a_{Hk} - a_{Lk}](p_{ak} + p_{sk})$$

On the buyers' side, $m_i < m_j$ implies in any potential equilibrium that $g(P_i + p_{ai}) < g(P_j + p_{aj})$, from which we know that $P_i + p_{ai} < P_j + p_{aj}$. On the sellers' side, $P_i - p_{si} < P_j - p_{sj}$, otherwise the sellers in j would participate in i. From these two inequalities alone, it is not possible to know the ordering between $p_{ai} + p_{si}$ and $p_{aj} + p_{sj}$.

First stage of the game

Assume $x, z \in \{i, j\}$, with $x \neq z$. Assume, further, that platform z earns positive profits in some initial situation and that $P_z > 0$. Then, x could guarantee a profit in the limit equal to the one obtained initially by z, taking its place, by implementing:

i. $P'_x = 0$

ii. $p'_{ax} = p_{az} + P_z$;

iii. $p'_{sx} = -(P_z - p_{sz})$;

iv. Participation thresholds: $s'_{Lx} = s_{Lz}$ and $s'_{Hx} = s_{Hz}$

Note that, although it is not in the interest of the platform to set a maximum limit for the quality of participant sellers, this possibility is allowed in the game and simplifies the argument. Since the platform actually pays the sellers (in net value), a situation in which it is dominant for sellers to participate in x, and with excess sellers in that platform, may not support a Nash equilibrium, because x would like to exclude some participant buyers, while keeping the rest, for who it would be dominant to stay. Thus, it would be possible to design a deviation at least as profitable as the one proposed by setting only an inferior threshold in such a way that there would be no excess sellers.

Continuing, suppose that in the initial situation $P_z = 0$, but platform z still earns a profit. In this case, we would find $p_{sz} < 0$ and $p_{az} > 0$. Then, x could guarantee a payoff in the limit as high as the one of z, by taking its place, implementing:

i. $P'_x = 0$

ii. $p'_{ax} = p_{az}$;

iii. p'_{sx} infinitesimally lower than p_{sx};

iv. Participation thresholds: $s'_{Lx} = s_{Lz}$ e $s'_{Hx} = s_{Hz}$.

As a result, in any allocation candidate to support a Nash equilibrium, no platform must earn a profit higher than the other one.

Affirmative 8: There cannot be a Nash equilibrium in which platforms earn different profit levels.

Then, we consider the possibility of equilibrium in a situation with trade occurring in both platforms, but generating null profits for them. In that case, $p_{ak} + p_{sk} = 0$ and $P_k \geq 0$. Any one of the platforms, which we will call z, could earn a profit with deviation:

i. $P_z' = 0$;

ii. $p_{az}' > p_{aj} + P_j$;

iii. p_{sz}' infinitesimally lower than – $(P_j - p_{sj})$;

iv. $s_{Lz}' > s_{Lj}$, defined to allow the participation of at least as many buyers as there will be participant sellers.

A4.2.3 – Both platforms with equal positive profits

For simplicity, we will consider cases with $p_{ak} = 0$ and $P_k > 0$[29].

The equality of positive profits implies:

$$(a_{Lj} - a_{Li})p_{si} = (1 - a_{Lj})p_{sj} > 0 \quad \text{(D1)}$$

Secondly, we have the two conditions of supply and demand equilibrium within each platform. In platform j: $1 - a_{Lj} = s_{Hj} - s_{Lj}$ ∴

$$s_{Hj} = 1 - a_{Lj} + s_{Lj} \quad \text{(D2)}$$

In platform i:$a_{Lj} - a_{Li} = s_{Hi} - s_{Li}$ ∴

$$s_{Hi} = a_{Lj} - a_{Li} + s_{Li} \quad \text{(D3)}$$

Next, we know that platform j will be strictly preferred by all sellers to i, i.e., $P_j - p_{sj} > P_i - p_{si} > 0$, and that $m_i < m_j$ implies $P_i < P_j$. On the other hand, a_{Lj} should be indifferent between platforms: $m_i a_{Lj} - P_i^2 = m_j a_{Lj} - P_j^2$.

Substituting m_i and m_j into this last equation:

$$\frac{(s_{Li}+s_{Hi})}{2} a_{Lj} - P_i^2 = \frac{(s_{Lj}+s_{Hj})}{2} a_{Lj} - P_j^2 \therefore$$

[29] The advantage of this is to simplify the profit calculation. There is always a strategy which ex ante guarantees the payoff for the sellers (even if they do not trade) and that generates the same payoffs for all agents, with $P_k' = 0, p'_{ak} = P_k, p_{sk}' = -(P_k - p_{sk})$.

$$a_{Lj} = 2\frac{\left(P_j^2 - P_i^2\right)}{s_{Lj} + s_{Hj} - s_{Li} - \left(1 - a_{Lj} + s_{Lj}\right)} \quad \text{(D4)}$$

On the other hand, a_{Li} must be indifferent between participating in i and not participating[30]. Then:

$$\frac{(s_{Li} + s_{Hi})}{2} a_{Li} - P_i^2 = 0 \therefore$$
$$a_{Li} = 2P_i^2 / (s_{Li} + s_{Hi}) \quad \text{(D5)}$$

Considering the sellers' side, either s_{Hj} is indifferent between participation in j and not participating, or $s_{Hj} = 1$. For the first possibility:

$$P_j - p_{sj} - \alpha s_{Hj} = 0 \therefore$$
$$p_{sj} = P_j - \alpha s_{Hj} \quad \text{(D6)}$$

Equations (D5) and (D6) will be used to recover p_{sk}. We proceed as in the monopoly section, using a_{Lk} as the decision variable.
For the second possibility, $s_{Hj} = 1$ and $P_j - p_{sj} - \alpha > 0$ implies $P_j - p_{sj} - \alpha s_{Lj} > 0$, i.e., all sellers prefer strictly participation in j to not participating and, from $P_j - p_{sj} > P_i - p_{si} > 0$, also to participating in i. Therefore, such an allocation could not support a Nash equilibrium, since j could benefit from a deviation such as:

i. $P'_j = 0$;
ii. $p'_{aj} = P_j$;
iii. p'_{sj} infinitesimally larger than $-\left(P_j - p_{sj}\right)$;
iv. $s'_{Lj} = s_{Lj}$.

In platform i, the set of sellers can be $[s_{Li}, s_{Hi}]$, with $s_{Hi} < s_{Lj}$, or $[s_{Li}, s_{Hj}[$, with $s_{Hi} = s_{Lj}$. That is to say, in case there is no interval of sellers who choose not to

[30] Were a_{Li} to strictly prefer paticipation in i, either there would be some excluded buyer who would rather participate than staying out, or $a_{Li} = 0$, which implies $P_i = 0$ and, thus, participation of no seller with $s > 0$ in i.

participate between the sets of participant sellers, the set of sellers in i will not include its superior boundary, given that all sellers prefer j to i. In both cases, s_{Hi} must be indifferent between participation in i and nonparticipation.[31] Thus:

$$P_i - p_{si} - \alpha s_{Hi} = 0 \therefore$$
$$p_{si} = P_i - \alpha s_{Hi} \qquad \text{(D7)}$$

We wish to know if some situation observing (D1) through (D7) could support a Nash equilibrium. We will show that the answer is no. The strategy is to maximize profit for both platforms using (D2) through (D7) and then, showing that these maximum profits are different, which violates (D1).

Substituting (D3) into (D5), we obtain:

$$a_{Li} = 2P_i^2 / \left(s_{Li} + \left(a_{Lj} - a_{Li} + s_{Li}\right)\right) \therefore$$
$$P_i^2 = \frac{a_{Li}\left(a_{Lj} - a_{Li} + 2s_{Li}\right)}{2} \therefore$$
$$P_i\left(a_{Li}, a_{Lj}, s_{Li}\right) = \sqrt{\frac{a_{Li}\left(a_{Lj} - a_{Li} + 2s_{Li}\right)}{2}} \qquad \text{(D8)}$$

Substituting (D2), (D3) and (D8) into (D4):

$$a_{Lj} = 2\frac{\left(P_j^2 - \frac{a_{Li}\left(a_{Lj} - a_{Li} + 2s_{Li}\right)}{2}\right)}{s_{Lj} + \left(1 - a_{Lj} + s_{Lj}\right) - s_{Li} - \left(a_{Lj} - a_{Li} + s_{Li}\right)} \therefore$$

$$P_j\left(a_{Li}, a_{Lj}, s_{Li}, s_{Lj}\right) = \sqrt{\frac{1}{2}\left(a_{Lj} + 2a_{Li}a_{Lj} + 2a_{Li}s_{Li} - 2a_{Lj}s_{Li} + 2a_{Lj}s_{Lj} - a_{Li}^2 - 2a_{Lj}^2\right)} \qquad \text{(D9)}$$

[31] This is because, in the first case, were s_{Hi} to strictly prefer i to nonparticipation, there would be some seller not admitted in j who would also like to participate in i and is not participating. In the second case, if s_{Hi} preferred strictly i to nonparticipation, this will also be true for all other sellers, and i could benefit from a deviation similar to the one just described for j.

Platform i's profit is:

$$\Pi_i^D = (a_{Lj} - a_{Li})p_{si}$$

Substituting (D7) and then (D3):

$$\Pi_i^D = (a_{Lj} - a_{Li})\left(P_i(a_{Li}, a_{Lj}, s_{Li}) - \alpha(a_{Lj} - a_{Li} + s_{Li})\right)$$

A requirement of a Nash equilibrium is maximum profit. The first order condition for s_{Li} is:

$$\frac{\partial \Pi_i^D}{\partial s_{Li}} = (a_{Lj} - a_{Li})\left(\frac{\partial P_i(a_{Li}, a_{Lj}, s_{Li})}{\partial s_{Li}} - \alpha\right) = 0$$

where $\frac{\partial P_i(a_{Li}, a_{Lj}, s_{Li})}{\partial s_{Li}} = \frac{a_{Li}}{2P_i}$

Thus,

$$P_i = \frac{a_{Li}}{2\alpha} \qquad (\text{FOC}s_{Li})$$

Substituting into (D8), we obtain:

$$\frac{a_{Li}}{2\alpha} = \sqrt{\frac{a_{Li}(a_{Lj} - a_{Li} + 2s_{Li})}{2}}$$

Taking squares from both sides and using $a_{Li} > 0$ (since $P_i > 0$), we can isolate s_{Li}:

$$s_{Li} = \frac{(1 + 2\alpha^2)a_{Li} - 2\alpha^2 a_{Lj}}{4\alpha^2} \qquad (\text{D10})$$

The first order condition for a_{Li} is:

$$\frac{\partial \Pi_i^D}{\partial s_{Li}} = -\left(P_i - \alpha(a_{Lj} - a_{Li} + s_{Li})\right) + (a_{Lj} - a_{Li})\left(\frac{\partial P_i(a_{Li}, a_{Lj}, s_{Li})}{\partial a_{Li}} + \alpha\right) = 0$$

where $\frac{\partial P_i(a_{Li}, a_{Lj}, s_{Li})}{\partial a_{Li}} = \frac{a_{Lj} - 2a_{Li} + 2s_{Li}}{4P_i}$

Thus,

$$-P_i + \alpha a_{Lj} - \alpha a_{Li} + \alpha s_{Li} + \left(a_{Lj} - a_{Li}\right)\left(\frac{a_{Lj} - 2a_{Li} + 2s_{Li}}{4P_i} + \alpha\right) = 0 \qquad (\text{FOC}a_{Li})$$

Substituting (FOCs_{Li}) and (D10), and using $a_{Li} > 0$, we obtain:

$$a_{Li} = \frac{\left(1+4\alpha^2\right)}{2(1+2\alpha^2)} a_{Lj} \qquad (\text{D11})$$

Substituting (D11) into (D10):

$$s_{Li} = \frac{a_{Lj}}{8\alpha^2} \qquad (\text{D12})$$

In platform j, profit is given by:

$$\Pi_j^D = \left(1 - a_{Lj}\right) p_{sj}$$

Substituting sequentially (D6) and (D2):

$$\Pi_j^D = \left(1 - a_{Lj}\right)\left(P_j\left(a_{Li}, a_{Lj}, s_{Li}, s_{Lj}\right) - \alpha\left(1 - a_{Lj} + s_{Lj}\right)\right)$$

Therefore, the first order condition for s_{Lj} is given by:

$$\frac{\partial \Pi_j^D}{\partial s_{Lj}} = \left(1 - a_{Lj}\right)\left(\frac{\partial P_j\left(a_{Li}, a_{Lj}, s_{Li}, s_{Lj}\right)}{\partial s_{Lj}} - \alpha\right) = 0$$

where $$\frac{\partial P_j\left(a_{Li}, a_{Lj}, s_{Li}, s_{Lj}\right)}{\partial s_{Lj}} = \frac{a_{Lj}}{2P_j}$$

Then:

$$P_j = \frac{a_{Lj}}{2\alpha} \qquad (\text{FOC}s_{Lj})$$

Substituting (FOCs_{Lj}) into (D9) and then, substituting (D11) and (D12), after some algebra:

$$a_{Lj} = \frac{\left(1+2\alpha^2\right)^2 8\alpha^2}{28\alpha^2 + 48\alpha^4 + 32\alpha^6 + 5}\left(1 + 2s_{Lj}\right) \qquad (\text{D13})$$

The first order condition for a_{Lj}, is:

$$\frac{\partial \Pi_j^D}{\partial a_{Lj}} = -\left(P_j\left(a_{Li}, a_{Lj}, s_{Li}, s_{Lj}\right) - \alpha\left(1 - a_{Lj} + s_{Lj}\right)\right)$$

$$+\left(1 - a_{Lj}\right)\left(\frac{\partial P_j\left(a_{Li}, a_{Lj}, s_{Li}, s_{Lj}\right)}{\partial a_{Lj}} + \alpha\right) = 0$$

where $$\frac{\partial P_j\left(a_{Li}, a_{Lj}, s_{Li}, s_{Lj}\right)}{\partial a_{Lj}} = \frac{1+2a_{Li}-2s_{Li}+2s_{Lj}-4a_{Lj}}{4P_j}$$

Then:

$$-\left(P_j - \alpha + \alpha a_{Lj} - \alpha s_{Lj}\right) + \left(1 - a_{Lj}\right)\left(\frac{1+2a_{Li}-2s_{Li}+2s_{Lj}-4a_{Lj}}{4P_j} + \alpha\right) = 0$$

(FOCa_{Lj})

Substituting (FOCs_{Lj}), (D12) and (D11), after some algebra:

$$s_{Lj} = \frac{-4\left(1+2\alpha^2\right)\alpha^2+\left(10\alpha^2+16\alpha^4+3\right)a_{Lj}^2+\left(-8\alpha^4+2\alpha^2+1\right)a_{Lj}}{8(1+2\alpha^2)\alpha^2} \qquad \text{(D14)}$$

The system constituted of (D13) and (D14) produces, as solutions:

$$a_{Lj} = \frac{\left(8\alpha^2+3\right)\left(4\alpha^2+8\alpha^4+1\right)}{2(1+2\alpha^2)(10\alpha^2+16\alpha^4+3)} \qquad \text{(D15)}$$

$$s_{Lj} = \frac{136\alpha^2+536\alpha^4+1152\alpha^6+1280\alpha^8+512\alpha^{10}+15}{32\alpha^2(10\alpha^2+16\alpha^4+3)(2\alpha^2+1)^3} \qquad \text{(D16)}$$

Substituting (D15) in (D11) and (D12):

$$a_{Li} = \frac{\left(8\alpha^2+3\right)\left(4\alpha^2+8\alpha^4+1\right)\left(1+4\alpha^2\right)}{4(1+2\alpha^2)^2(10\alpha^2+16\alpha^4+3)} \qquad \text{(D17)}$$

$$s_{Li} = \frac{\left(8\alpha^2+3\right)\left(4\alpha^2+8\alpha^4+1\right)}{16\alpha^2(1+2\alpha^2)(10\alpha^2+16\alpha^4+3)} \qquad \text{(D18)}$$

Substituting (D17) into ($FOCs_{Li}$) and (D15) into ($FOCs_{Lj}$):

$$P_i = \frac{(8\alpha^2+3)(4\alpha^2+8\alpha^4+1)(1+4\alpha^2)}{8\alpha(1+2\alpha^2)^2(10\alpha^2+16\alpha^4+3)} \quad \text{(D19)}$$

$$P_j = \frac{(8\alpha^2+3)(4\alpha^2+8\alpha^4+1)}{4\alpha(1+2\alpha^2)(10\alpha^2+16\alpha^4+3)} \quad \text{(D20)}$$

Combining (D2) and (D3) with (D15) through (D17):

$$s_{Hi} = \frac{(6\alpha^2+1)(8\alpha^2+3)(4\alpha^2+8\alpha^4+1)}{16\alpha^2(2\alpha^2+1)^2(10\alpha^2+16\alpha^4+3)} \quad \text{(D21)}$$

$$s_{Hj} = \frac{124\alpha^2+424\alpha^4+672\alpha^6+384\alpha^8+15}{32\alpha^2(10\alpha^2+16\alpha^4+3)(2\alpha^2+1)^3} \quad \text{(D22)}$$

Substituting (D19) through (D22) into (D6) and (D7), we obtain:

$$p_{si} = \frac{(8\alpha^2+3)(4\alpha^2+8\alpha^4+1)}{16\alpha(2\alpha^2+1)(10\alpha^2+16\alpha^4+3)} \quad \text{(D23)}$$

$$p_{sj} = \frac{(12\alpha^2+16\alpha^4+3)(12\alpha^2+24\alpha^4+32\alpha^6+3)}{32\alpha(2\alpha^2+1)(10\alpha^2+16\alpha^4+3)} \quad \text{(D24)}$$

Computing platform profits, we obtain:

$$\Pi_i^D = \frac{(8\alpha^2+3)^2(4\alpha^2+8\alpha^4+1)^2}{64\alpha(2\alpha^2+1)^3(10\alpha^2+16\alpha^4+3)^2} \quad \text{(D25)}$$

$$\Pi_j^D = \frac{(12\alpha^2+16\alpha^4+3)^2(12\alpha^2+24\alpha^4+32\alpha^6+3)}{64\alpha(2\alpha^2+1)^3(10\alpha^2+16\alpha^4+3)^2} \quad \text{(D26)}$$

Manipulating the numerators of both expressions, it is possible to show that for any $\alpha > 0$, we obtain $\Pi_j^D > \Pi_i^D$, which violates (D1).

The solutions found from (D15) through (D26), guarantee $P_j > P_i$, $P_j - p_{sj} > P_i - p_{si}$ and $1 > a_{Lj} > a_{Li} > 0$, in addition to $s_{Hj} > s_{Lj}$ and $s_{Hi} > s_{Li} > 0$. The conditions that might require the imposition of restrictions for the values of α are $s_{Hj} \leq 1$ and $s_{Lj} \geq s_{Hi}$, analogously as in the monopoly case. However, we do not extend the analysis any further, given that it is possible to grasp from the case under study without such restrictions that there is no mechanism that would make maximum profits equal between platforms.

A4.2.3 – Both platforms with zero profits

Finally, we consider an initial situation in which both platforms make zero profits and at least one of them, $z = \{i, j\}$, operates, i.e., contains a positive and equal masses of trading buyers and sellers, which correspond to the subgame equilibrium in the second stage of the game. This situation implies that $p_{az} + p_{sz} = 0$. Then, z could obtain a positive profit with deviation:

i. $P'_z = 0$,
ii. p'_{az} infinitesimally higher than $P_z + p_{az}$,
iii. $p'_{sz} = -(P_z - p_{sz})$,
iv. s'_{Lz} Infinitesimally higher than s_{Lz}, defined to obtain equal number of sellers and buyers.

Note that with this deviation, z keeps all its original sellers, with (iii). (iv) guarantees that $m'_z \geq m_z$. Therefore, the increase of p_{az}, in (ii), results in the exclusion of an infinitesimal number of buyers (the second stage of the game originally being in subgame equilibrium means that a_{Lz} is indifferent between z and his "second" best option). Therefore, with the proposed deviation, the platform guarantees a positive profit, because $p'_{az} + p'_{sz} > 0$.

It remains to analyze a situation in which no platform operates. However, it could not support a Nash equilibrium, given that one of them does not operate, the other could guarantee a positive profit by allowing the participation of any seller and setting:

$$p_{az} = \frac{\alpha + \alpha\sqrt{1 - 8\varepsilon^2 - 8\varepsilon\alpha} + 2\varepsilon}{2(1 + 2\alpha^2)}$$

$$p_{sz} = -\frac{\alpha + \alpha\sqrt{1 - 8\varepsilon^2 - 8\varepsilon\alpha} + 4(1 + \alpha^2)\varepsilon}{2(1 + 2\alpha^2)}$$

In this case, $p_{az} + p_{sz} = \varepsilon$ and there is always some $\varepsilon > 0$ for which there is equilibrium in the second stage with participation of buyers and sellers. This is guaranteed, because taking the limit of these prices when $\varepsilon \to 0$, we obtain the pooling equilibrium presented in section 3.3.

Therefore, we conclude that, for general values of α there is no Nash equilibrium in the simultaneous platform duopoly game.