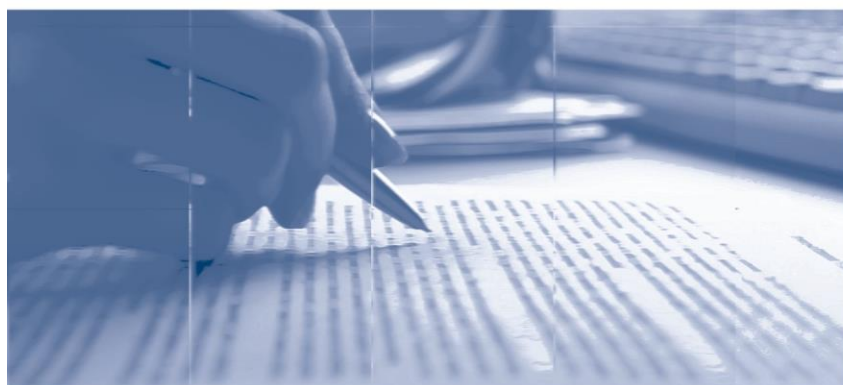**BANCO CENTRAL DO BRASIL**

A Discrete Monitoring Method for Pricing Asian Interest Rate Options

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A Discrete Monitoring Method for Pricing Asian Interest Rate Options

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Abstract

As determined by the BM&FBovespa standards, the ID index is built up discretely according to overnight DI rate. We addressed the IDI Call Option Pricing problem using this discretely compounded hypothesis in lieu of the continuous updated case found in the literature. Our method converges to the benchmark (which refers to the exact price considering the discretely compounded hypothesis). This is not possible for any short rate modeling framework which adopts IDI continuously compounded hypothesis. It is noteworthy that the benchmark prices as well as the prices under the discretely compounded hypothesis with a reasonable number of mesh points are *always* cheaper than those of the continuously compounded case. In addition, we introduced a general purpose version of a classical finite difference scheme which provides an spurious oscillation free solution of the PDE to obtain an approximate price of the Option.

Keywords: Interest Rates Derivatives, IDI Option, PDE Option Pricing, Computational Finance.
JEL Classification: C63, C65, G12.

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1   Introduction

Before the 1980s, fixed income markets were composed primarily by vanilla bonds and simple structured financial instruments. Thus, their valuations were easy and direct, done frequently via closed-form mathematical formulas (e.g. [3]). Thenceforth, markets have become sophisticated as more complex products aiming to reduce or share risks appear, complicating the pricing and hedging engines. The fast growth of financial market instruments over recent decades has spawned many challenging mathematical problems to be solved, from the underlying stochastic modeling to solutions through computational methods.

Fixed income derivatives are contracts which have payoff, contingent on the evolution of interest rates. They are traded in the equity, commodity, currency and credit markets along with hybrid derivatives engineered over the counter ([18]). The valuation of interest rate derivative contracts is a very important subject in modern financial theory and practice. The financial health of banks, governments and industrial companies are very sensitive to changes in the term structure of the interest rates. It has become mandatory nowadays to quantify and control the risk exposure to prices of interest rate associated contracts.

A large amount of academic literature has been dedicated to the pricing and hedging of such instruments. [28] introduced a Gaussian stochastic process to model the spot rate dynamics. He also developed a simple closed-form solution to compute the prices of zero-coupon bonds. [17] extended the results to options on bonds, which is automatically adapted to price interest rate caps and floors and [14] used Jamshidian's approach to express the swaption prices. Additionally, closed-form expressions have been developed to price such products based on other stochastic processes (see e.g. [4], [10]and [23]).

However, it is a hard task to extend the results and find analytical solutions to more complex structures, even in the Gaussian model. The callable bond is an example. It is a financial instrument commonly issued by banks and non-financial companies. Hence, such contracts must be priced by numerical techniques. Several computational approaches, such as Fourier methods ([6]), Monte Carlo simulation ([13]) and tree methods ([15]) can be used to price complex derivatives, but due to its efficiency in computing accurate pricing and hedging values and its flexibility in the modeling process, partial differential equations have become a very popular choice.

To improve financial engineering, we propose a new numerical finite difference method to replace the classical schemes used to solving PDEs (see e.g. [8], [11], [25]). The motivation for doing so stems from the fact that spurious oscillations

solutions occur when volatilities are low (i.e., when the Peclet number is high) and serious collateral matters appear in attempts to correct the problem. Actually, low volatilities are the range observed in interest rate markets, and unlike the classical schemes, our method covers the whole spectrum of volatilities in the interest rate dynamics.

Our method is devised as a version of the Fully Implicit method (see, e.g., [8], [25]), and extended to provide hedges along with prices. One of the modifications we introduced is inspired in a technique that appears in [19]. That method adapts to the Black-Scholes dynamics, while ours fits the interest rate derivatives with Vasicek, CIR [10] and other types of short-rate models. Our numerical scheme is second order accurate in space and consistent. Moreover, it possesses the quality of being unconditionally stable. We name it Modified Fully Implicit (Interest Rate) Method.

We show the good performance of the method, pricing a zero-coupon bond and another type of interest rate derivative security named IDI (Interbank Deposit Rate Index) option, both in the Vasicek dynamic. Namely, we perform a convergence analysis by considering both continuously compounded and daily compounded rate of interest to model the money market account and the updating of the IDI path.[1]

The ID index updating is built up discretely based on the overnight DI rate, which is an annualized rate over one day period. It is calculated and published daily, and represents the average rate of inter-bank overnight transactions [7]. Based on a martingale approach, closed form solutions to price an IDI contract are available in the literature, assuming for mathematical tractability reasons that the updating of the IDI is continuous in time. In this scenario, a one-factor model was developed in [29] to price the IDI option via the short rate dynamics as given in [28]. A multi-factor Gaussian model was developed in [1] to price the IDI option and bond prices. Also, [12] proposed to incorporate the potential changes in the targeting rates via pure jump process.

Carrying on the evaluation of our finite difference scheme, we demonstrate its advantages considering the following approaches on a pricing problem of an IDI call option with the Vasicek dynamic.

- We obtain the estimates of the prices (and hedges) according to the Modified

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[1]IDI is the shorthand of Interbank Deposit Rate Index. The IDI option is a financial option of Asian type and, as such, the payoff depends on the path followed by the short term interest rate. It presents cheaper prices than the standard options and it is less sensitive to extreme market conditions that may prevail close to the expiration day - due to random crashes or outright manipulation. So, it is commonly used by corporations to manage interest rate risk. Actually, it is a standardized derivative product traded at the Securities and Futures Exchange in the Brazilian fixed income market.

Fully Implicit method, and consider updating the IDI path discretely. This updating rule allow us to track realistically the evolution of the index and to achieve the exact pay-off representation.

- We obtain the prices via the closed form expressions given in [29], assuming a continuously compounded interest rate, which is actually an idealization for mathematical tractability.

So, our approach corresponds to obtaining approximate prices for the exact problem (with respect to the payoff) while that of [29] corresponds to obtaining an exact price for the approximate problem. The results of this comparative analysis corroborate the conjecture of Tankov and Cont [24], which asserts that, typically, the former scenario yields better results than the latter. Indeed, via numerical simulations, we observe meaningful relative discrepancies in the prices for some prescribed examples whose parameters are good representatives of the market. So, using one or other method makes a difference. Now, neither price represents a benchmark. The benchmark should correspond to a framework that models the IDI discretely and provides the exact solution for the price. However, the Modified Fully Implicit method can be refined to approach the benchmark. On the other hand, all short rate model which adopts the IDI continuously compounded hypothesis (as in [1], [12] and [29]) is obviously inconsistent with refinements with respect to the index updating, so they cannot approach the benchmark. Since the continuous updating procedure for calls boils down to a more expensive payoff than the discretely updating one, it is reasonable to expect prices to be more expensive in the former than the latter procedure. The simulations indicate more than this in fact. They show that, starting with a reasonable refined mesh, our call prices are cheaper than those of the continuous updating case of [29] and, as the mesh is refined, our prices move further downwards approaching the benchmark - and away from the prices of [29]. Analogous conclusions are obtained with a put option.

In the case of bonds, a comparative study of the continuous versus the discrete compounding interest scheme shows that the relative discrepancies between prices do not exceed 5% (in the typical range of interest rates). This is actually an expected result. However, in the case of the IDI call and put options, we find that the relative discrepancies between prices exceed 50%, when we consider the realistic discrete compound interest rate scheme (associated with the Modified Fully Implicit method) versus the continuous compounding scheme (often exploited in the literature). This immediately suggests that the former scheme is more appropriate then the latter, whenever the interest rate derivatives are more complicated than straight bonds.

Hence, the study carried out in this paper, in conjunction with the numerical

simulations performed with the above derivatives, indicate that, in fact, our method is reliable and highly competitive. It straightforwardly adapts to other interest rate derivative securities, e.g., bond options, swaptions, caps and floors, adjusting the appropriate terminal condition. Via minor changes in the functions assigned to the jump conditions, the method fits other types of path-dependent options, as well as coupon bonds, coupon bond options and callable bonds.

We organize the article as follows: In Section 2 we present the motivation of the discrete daily monitoring approach and derive the analytical delta of the IDI call option. In Section 3 we present the partial differential equation that will be used to price the IDI call option and justify a coordinate transformation for the PDE. In Section 4 we revise the standard numerical discretizations commonly applied to such PDE and propose a scheme that is second order accurate and unconditional stable to convective dominant parabolic equations. A convergence study is performed numerically. Section 5 presents the pricing and hedging results, highlighting the discrepancies between the continuous and discrete updating approaches. Section 6 concludes the article.

2   The IDI option pricing problem

We consider the problem of pricing an IDI option, assuming that the ID index y accumulates discretely according to

$$y(t_n) = y(t_0) \prod_{i=1}^{n} (1 + DI(t_{i-1}))^{\frac{1}{252}}, \quad n = 1, ..., N, \tag{1}$$

where t_i denotes the end of day i and $DI(\cdot)$ assigns the DI rate, i.e., the average of the interbank rate of a one-day-period, calculated daily and expressed as the effective rate per annum. A detailed definition of the DI rate can be found in [7]. Correspondingly, the discretely monitored pay-off for the call option with maturity in $T = t_N$ is given by

$$\max\left(y(t_N) - K, 0\right). \tag{2}$$

We also suppose that the instantaneous short-term interest rate r−which shapes the DI rate, in the sense that $DI(t_i) = r(t_i)$ evolves according to Vasicek model (see [28])

$$dr(t) = a(b - r(t))dt + \sigma dW_t. \tag{3}$$

This Ornstein-Uhlenbeck stochastic process pulls the short rate to a level b at a rate a against with a normally distributed random term σdW_t, where W_t is a standard Brownian motion.

The discrete updating scheme mentioned above is consistent with reality. An idealization for mathematical tractability is to assume that the IDI index accumulates continuously according to

$$y(t) = y(0)e^{\int_0^t r_u du}, \quad t \in [0, T] \tag{4}$$

instead of (1). Correspondingly, the continuously monitored pay-off for the call option with maturity in T is given by

$$\max(y(T) - K, 0), \tag{5}$$

which stands as the counterpart of (2). Hence, concerning this important aspect, the framework we adopt here is more realistic than that usually found in the literature. Under the hypothesis of continuous compound interest rate, [29] developed a closed-form solution for the price of an ID call option with maturity in T, where the short rate also follows the Vasicek model.

It is well known that, using the above hypothesis, zero-coupon bond prices are very similar to those of the daily compounded interest. However this is not the case when dealing with assets like Asian interest rate options. The results obtained in this paper corroborate this with respect to pricing theoretical IDI options.

For later use, the price of an IDI call option with maturity in T at time t, in the continuously compounded hypothesis, is given by

$$C(r(t), y(t), t, T) = y(t)\Phi(h) - KP(r(t), t, T)\Phi(h - k) \tag{6}$$

where $N(\cdot)$ denotes the cumulative standard normal distribution function, K is the strike, $P(r(t), t, T)$ is a zero-coupon bond price, $y(t)$ is the ID index at the current time and

$$h = \frac{\frac{y(t)}{P(r(t),t,T)K} + \frac{k^2}{2}}{k} \tag{7}$$

$$k^2 = \sigma^2 \frac{(4e^{-a\tau} - e^{-2a\tau} + 2a\tau)}{2a^3}, \tag{8}$$

where the parameters σ and a are defined in equation (3) below and $\tau = T - t$. As shown in [28], the price at time t of a zero-coupon bond that pays 1 at time T is

$$P(r(t), t, T) = \alpha(t, T)e^{-\beta(t,T)r(t)}, \tag{9}$$

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where,

$$\beta(t,T) = \frac{1 - e^{-a(\tau)}}{a} \tag{10}$$

and

$$\alpha(t,T) = \exp\left[\frac{(\beta(t,T) - \tau)(a^2 b - 0.5\sigma^2)}{a^2} - \frac{\sigma^2 B(t,T)^2}{4a}\right]. \tag{11}$$

2.1 The hedging problem

We now obtain the delta of the IDI call option. Since the IDI is an index and not a physical asset that can be bought and sold, we evaluate the infinitesimal changes in the price of the IDI option with respect to a change in the whole term-structure. That is to say, our replicating portfolio will be composed by zero coupon bonds with maturity at time T and a money market account. By deriving the price $C(t,T)$ of the IDI option (as given by (6)) with respect to the price $P(t,T)$ of the zero coupon bond, we obtain

$$\begin{aligned}
\frac{\partial C}{\partial P(t,T)} &= y(t)\frac{\partial\Phi(h)}{\partial P(t,T)} - \left[K\Phi(h-k) + KP(t,T)\frac{\partial\Phi(h-k)}{\partial P(t,T)}\right] \\
&= -K\Phi(h-k).
\end{aligned} \tag{12}$$

where

$$\frac{\partial\Phi(x)}{\partial P(t,T)} = \phi(x)\frac{\partial x}{\partial P(t,T)} \quad \text{and} \quad \phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}.$$

The last equality stems from the fact that

$$\begin{aligned}
log\left(\frac{\phi(h-k)\frac{\partial(h-k)}{\partial P(t,T)}}{\phi(h)\frac{\partial(h)}{\partial P(t,T)}}\right) &= \tfrac{1}{2}[(h)^2 - (h-k)^2] \\
&= \tfrac{1}{2}[(h) + (h-k)][(h) - (h-k)] = h.k - \frac{k^2}{2} \\
&= \log\left(\frac{y(t)}{K.P(t,T)}\right) + \frac{k^2}{2} - \frac{k^2}{2}.
\end{aligned} \tag{13}$$

Hence, the delta hedge of the IDI option in the continuous compound interest scenario is given by

$$\Delta(t) = \frac{\partial C}{\partial P(t,T)} = -K\Phi(h-k). \tag{14}$$

We also develop a version of (14) to be used in our numerical approach, where

the more realistic discrete compound interest is considered. As it ought to be, this version does not require knowledge of the price values C given by (6) - which stems from the continuous compound assumption for the interest rate. So, since we intend to use the zero coupon bond to hedge the IDI option, we assume that the price of the IDI option at time t depends on the current level $y(t)$ of the index and the price $\breve{P}(t, T)$ of a bond with same maturity as that of the option. Both quantities are obtained via our numerical approach, assuming the discrete updating status (we recall that very small changes occur in the price of a bond if we switch from the continuous to the discrete updating case, so the zero coupon bond prices could have been taken from [28]. We denote this price by $\breve{u}$. We have that

$$du = \frac{\partial u}{\partial y(t)} dy(t) + \frac{\partial u}{\partial P(t, T)} dP(t, T), \tag{15}$$

so the delta hedge is given by

$$\breve{\Delta}(t) = \frac{\frac{du}{dr} - \frac{\partial u}{\partial y(t)} \frac{dy(t)}{dr}}{\frac{dP(t,T)}{dr}} = \frac{\frac{du}{dr}}{\frac{dP(t,T)}{dr}}. \tag{16}$$

3   PDE formulation

Our aim is to price a financial contract assuming that the price is a function of three variables, namely, the time t and the current values of the interest rate r and the ID index y. We assume that $u(t, r(t), y(t)) \in C^{1,2,0}(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$ and that the ID index accumulates daily. Following the steps of [5] and applying Ito's lemma (see e.g. [20]), we set up a portfolio π containing two similar contracts with different maturities, obtaining

$$d\pi_t = \frac{\partial u_1}{\partial t} dt + \frac{\partial u_1}{\partial r} dr + \frac{\sigma^2}{2} \frac{\partial^2 u_1}{\partial r^2} dt - \Delta \left(\frac{\partial u_2}{\partial t} dt + \frac{\partial u_2}{\partial r} dr(t) + \frac{\sigma^2}{2} \frac{\partial^2 u_2}{\partial r^2} dt \right) \tag{17}$$

Although we are modeling a path-dependent option, the portfolio (17) exhibits a classical shape. This is so because the stochastic differential equation for the IDI degenerates, in the sense that $dy = 0$.

We point out that the quantity given by (1) changes only at a set of discrete jump times $\Omega = (t_1, ..., t_N)$ that represent the end of the trading days.

Let the market price of risk be $\lambda = 0$. The usual no-arbitrage argument implies that the price of the IDI option $u = u(t, r, y)$ at time $t \notin \Omega$, i.e., when the IDI remains constant, is given by

$$\frac{\partial u}{\partial t} + a(b - r(t)) \frac{\partial u}{\partial r} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial r^2} = r(t)u. \tag{18}$$

Across each $t_n \in \Omega$, absence of arbitrage ensure that the price of the option is continuous ([30], [32]). This is mathematically represented by the following jump condition:

$$u(t_n - \epsilon, r, y^-) = u(t_n + \epsilon, r, y^+), \tag{19}$$

where $y^+ = y^-[(1+r)^{\frac{1}{252}}]$, and $0 < \epsilon \ll 1$. We could alternatively derive the PDE (18) by using the Discounted Feynman-Kac Theorem ([20]).

To ensure uniqueness of solution we prescribed arbitrary functions to describe how the PDE must behave at the extremes of the domain. In the case of the IDI option we chose the following Neumann boundary conditions:

$$\frac{\partial u(-\infty)}{\partial r} = \frac{\partial u(+\infty)}{\partial r} = 0 \tag{20}$$

We know that the dynamics (3) allows negative and positive infinite values for r with non-zero probabilities. Hence, the conditions given by (20) ensure that an infinitesimal change in r at the boundaries does not change the value of the option. This is intuitive because the IDI option price is actually insensitive to changes in extreme negative or positive values of r. This fact can also be verified in equation (6). Latter we will revisit the issue concerning the appropriate value for the right extreme boundary when dealing with a truncated domain.

The terminal condition is the pay-off of the option, which in the case of a call, is

$$u(T, r, y) = \max(y - K, 0), \tag{21}$$

and, in the case of a put, is

$$u(T; r; y) = max(K - y, 0), \tag{22}$$

where K is the strike price and y is viewed as (2).

As happens with the Asian-Parisian stock options ([32]), we have that away from monitored times the PDE (18) has no y dependence. The terminal condition (21) implies that a set of independent one-dimensional PDEs must be solved. The IDI Option price is calculated via (18) backwards in time from the terminal condition (21) up to the first $t_n \in \Omega$. We apply, in the sequel, the jump condition to find the option value at t_n^-. Using these values as the new terminal condition we repeat the process $N + 1$ times to meet the current value of the option, where N is the cardinality of the set (Ω).

3.1 Coordinate transformations

Finding a solution to (18) is a well-known problem in physics and finance. Numerically speaking, it is inconvenient if the sign of the convective term changes and the volatility is very low, which are common facts in interest rate derivatives. Financially speaking, it is undesirable to have the same precision for all points in the grid, because we are pricing a product based on the current interest rate. So, in the same lines as in [25], we first propose a change of variable that allows us to retrieve the solution in a nonuniform grid in r, which becomes thinner in some desirable or needed region. Then we appropriately modify a finite difference scheme to overcome the drawbacks of a convective dominant PDE.

The "proximity" of the left nonzero probability boundary to the actual level of interest rates suggests that small errors at the left boundary lead to inaccurate results near the strike price, where a sharp gradient occurs in conditional derivatives. So we specified a new variable $x = ln(rd + c)$, where $d > 0$ and c are constants such that $c > -d \min(r)$.

Now, we have that

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} = \frac{\partial u}{\partial x}\left(\frac{d}{rd + c}\right) \tag{23}$$

and

$$\frac{\partial^2 u}{\partial r^2} = \left(\frac{\partial x}{\partial r}\right)^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 x}{\partial r^2}\frac{\partial u}{\partial x} = \left[\frac{d^2}{(rd + c)^2}\right]\frac{\partial^2 u}{\partial x^2} - \left[\frac{d^2}{(rd + c)^2}\right]\frac{\partial u}{\partial x}, \tag{24}$$

so we get the transformed PDE in the new coordinate x

$$\frac{\partial u}{\partial t} + \left[\frac{a(db - e^x + c)}{e^x} - \frac{\sigma^2 d^2}{2e^{2x}}\right]\frac{\partial u}{\partial x} + \left(\frac{\sigma^2 d^2}{2e^{2x}}\right)\frac{\partial^2 u}{\partial x^2} = \left(\frac{e^x - c}{d}\right)u \tag{25}$$

with the following boundary conditions:

$$\frac{\partial u(\min(x))}{\partial x} = 0 \tag{26}$$

and

$$\frac{\partial u(\max(x))}{\partial x} = \frac{(T - t)}{T}\frac{e^x}{d}\theta. \tag{27}$$

We emphasize that we could set the derivative of the right boundary in r equal to zero, as we said before, except when dealing with a very large domain. For computational cost justifications we appropriately choose $\frac{\partial u}{\partial r} = \frac{(T-t)}{T}\theta$ based on

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some model or market data θ which results in the previous boundary condition (27) in the new variable x.

The terminal condition does not depend on r directly and remains the same. The new jump condition is obtained substituting the value of r in the new coordinate system.

4 Finite difference methods

We now address the above problems via finite difference methods. We do so due to (i) uncommon features that appear in problems involving interest rate derivatives, (ii) the huge amount of works using this technique, and (iii) the lower computational effort to price and hedge options in comparison with Monte Carlo simulation methods. Unlike the Monte Carlo, finite difference methods allow us to observe the option prices considering the domain of r as a whole and provide reliable results that can be used as benchmark when there is no closed-form solution ([25]).

The finite difference method consists of the discretization of the spatial domain x over some finite interval $[x_{min}, x_{max}]$ with J points and in approximating the derivatives of the PDE by its incremental ratio Δx, which converges to the derivative as $\Delta x \to 0$. The method consists of replacing the derivatives in (25) by their numerical values at a finite number of points ([25] and [16]).

The forward, backward and central first spatial derivative is respectively approximated by

$$\partial_x^+ u_j^n = \frac{\partial u}{\partial x} + \mathcal{O}(\Delta x) = \frac{u(x_j + \Delta x, t_n) - u(x_j, t_n)}{\Delta x},$$

$$\partial_x^- u_j^n = \frac{\partial u}{\partial x} + \mathcal{O}(\Delta x) = \frac{u(x_j, t_n) - u(x_j - \Delta x, t_n)}{\Delta x} \qquad \text{and}$$

$$\partial_x^0 u_j^n = \frac{\partial u}{\partial x} + \mathcal{O}(\Delta x^2) = \frac{u(x_j + \Delta x, t_n) - u(x_j - \Delta x, t_n)}{2\Delta x},$$

where $u_j^n = u(x_j, t_n)$ and $\mathcal{O}(g(\delta))$ denotes the functions $o(\cdot)$ with the property of having $\frac{o(\delta)}{(g(\delta))} \to 0$ as $\delta \to 0$.

The central second spatial derivative is approximated by the second order stencil

$$\partial_x^- \partial_x^+ u_j^n = \frac{\partial^2 u}{\partial x^2} + \mathcal{O}(\Delta x^2) = \frac{u(x_j + \Delta x, t_n) - 2u(x_j, t_n) + u(x_j - \Delta x, t_n)}{\Delta x^2}.$$

Approximations for the forward and central first temporal derivative are:

$$\partial_t^+ u_j^n = \frac{\partial u}{\partial t} + \mathcal{O}(\Delta t) = \frac{u(x_j, t_n + \Delta t) - u(x_j, t_n)}{\Delta t}$$

and

$$\partial_t^0 u_j^n = \frac{\partial u}{\partial t} + \mathcal{O}(\Delta t^2) = \frac{u(x_j, t_n + \Delta t) - u(x_j, t_n - \Delta t)}{2\Delta t}.$$

These simple definitions allow us to construct a variety of finite difference schemes for the PDE (25). Here, Δt means the length of the time lag between n and $n + 1$ and Δx is the distance between the spatial grid points j and $j + 1$.

In what follows, we present three specific finite difference schemes that are candidates to solve the interest rate derivative pricing problem. We briefly show their limitations and disadvantages, and conclude that the Modified Fully Implicit method we provide in Section 5 eliminates the limitations mentioned above and enables us to use it without parameter restrictions.

4.1   Crank-Nicolson method

The most famous method to solve parabolic PDE is the Crank-Nicolson method, which is of order $\mathcal{O}(\Delta t^2, \Delta r^2)$. The method consists of approximating the spatial derivatives by the average[2]

$$\frac{\partial u}{\partial r} \approx \frac{1}{2} \left(\partial_r^0 u_j^{n+1} + \partial_r^0 u_j^n \right)$$

and

$$\frac{\partial^2 u}{\partial r^2} \approx \frac{1}{2} \left(\partial_r^- \partial_r^+ u_j^{n+1} + \partial_r^- \partial_r^+ u_j^n \right).$$

We applied the above method to the Vasicek type PDE (18) with terminal condition given by

$$u(r, T) = 1, \tag{28}$$

and boundary conditions given by

$$\frac{\partial u}{\partial r}(r_{min}, t) = 0 \tag{29}$$

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[2]For the sake of simplicity, we chose to deal with the original spatial variable r first rather than the transformed x.

and

$$\frac{\partial u}{\partial r}(r_{max}, t) = 0 \tag{30}$$

to solve a bond pricing problem. As can be noted from (1), the method produces an oscillating solution in the most common case of convective dominant PDE. Particularly in this example, we have maturity of the zero-coupon bond equal to five years, and $\sigma = 0.005$, $a = 0.8$, $b = 0.1$. We adopt $J = 100$ and 5 daily steps.
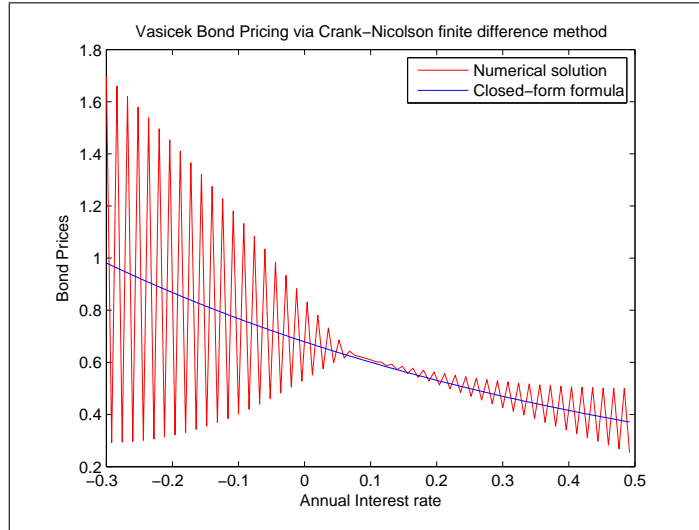

Figure 1: *Zero-coupon bond prices - Crank-Nicolson method*

Conceptually, two $(J+1) \times (J+1)-$dimensional matrices, denoted P and L are defined, as well as system

$$Pu^{n+1} = Lu^n \tag{31}$$

where P and L are $(J+1) \times (J+1)$ matrices.

D. Duffy listed in [11] some drawbacks of the Crank-Nicolson methods in finance. An important drawback is that the resulting tridiagonal matrix P for the PDE (25) reads as

$$P = \text{diag}\left[-\frac{\mu_j}{4\Delta r} + \frac{S_j}{2\Delta r^2}; \ \frac{1}{\Delta t} - \frac{S_j}{\Delta r^2} - r; \ \frac{\mu_j}{4\Delta r} + \frac{S_j}{2\Delta r^2}\right], \tag{32}$$

$$L = \text{diag}\left[\frac{\mu_j}{4\Delta r} - \frac{S_j}{2\Delta r^2}; \ \frac{1}{\Delta t} + \frac{S_j}{\Delta r^2}; \ -\frac{\mu_j}{4\Delta r} - \frac{S_j}{2\Delta r^2}\right] \tag{33}$$

where μ and S, are respectively

$$\mu = \frac{1}{2}\left[a(b - r_j) + a(b - r_{j+1})\right] \tag{34}$$

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and

$$S \;=\; \frac{\sigma^2}{2}. \tag{35}$$

That is to say, the mean-reversion feature of the Vasicek model run into difficulties in assuring the negativeness of the off-diagonal entries of the matrix P when volatility dominates the convective term or when the short-term rate is lower than the mean b. This implies having some negative entries of P^{-1}, which results in spurious oscillations (see [31]). As a result, we can see that the Crank-Nicolson solution

$$u^{n+1} = \frac{1}{\Delta t} P^{-1}(Lu^n) \tag{36}$$

is not positive-preserving. Thus, we see that the method is not adequate to estimate prices in the fixed-income scenario.

4.2   Up-wind method

To produce oscillations free solutions, a common way-out is to approximate the first order spatial derivative of the Crank-Nicolson method by the upwind scheme:

$$\frac{\partial u}{\partial r} = \begin{cases} \partial_r^+ u_j^n & \text{se} \quad \frac{a(b-r)}{2\Delta r} \geq \frac{\sigma^2}{2\Delta r^2} \\ \partial_r^- u_j^n & \text{se} \quad \frac{a(b-r)}{2\Delta r} < \frac{\sigma^2}{2\Delta r^2}, \end{cases}$$

which simply means adding an artificial volatility

$$\pm \frac{a(b-r)\Delta r}{2} \tag{37}$$

to $\frac{\sigma^2}{2}-$ the coefficient of $\frac{\partial^2 u}{\partial r^2}$. This technique eliminates possible negative values in the off-diagonal entries from the P-matrix. Another numerical treatment based on flux limiters can be found in [26] and [33].

There are two main problems with this strategy. The first one is that the numerical solution is now first-order accurate in time and space. Consequently it has slower convergence rates. The second one is that the solution would eventually be mischaracterized due to the numerical diffusion introduced above, which is of order $\mathcal{O}(\Delta r)$.

We can compare the solutions to the PDE (25) as given by the Crank-Nicolson method (Figure 1), the Up-wind strategy (Figure 2) and analytically (Figure 2). Figure 3 shows indeed that the method produces spurious oscillations free solutions for any time to maturity.

However, inspection of Figure 2 shows that, in order to prevent the spurious

scenario, the numerical diffusion introduced in the up-wind scheme mischaracterizes the solution. Again, we see that this approach is not adequate for pricing in the interest-rate market.
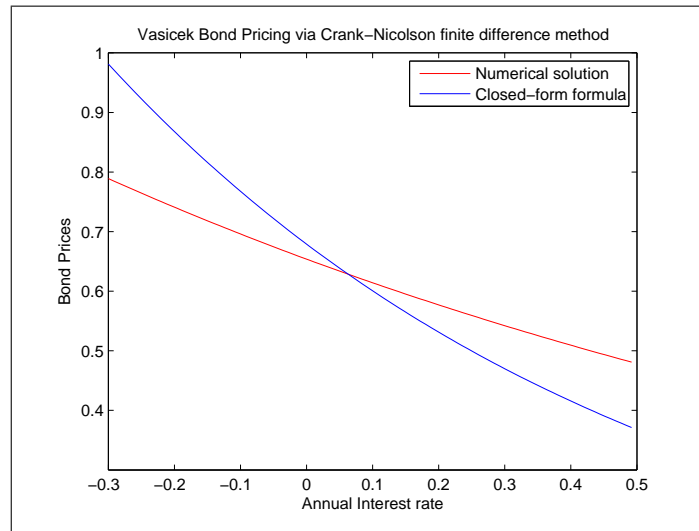

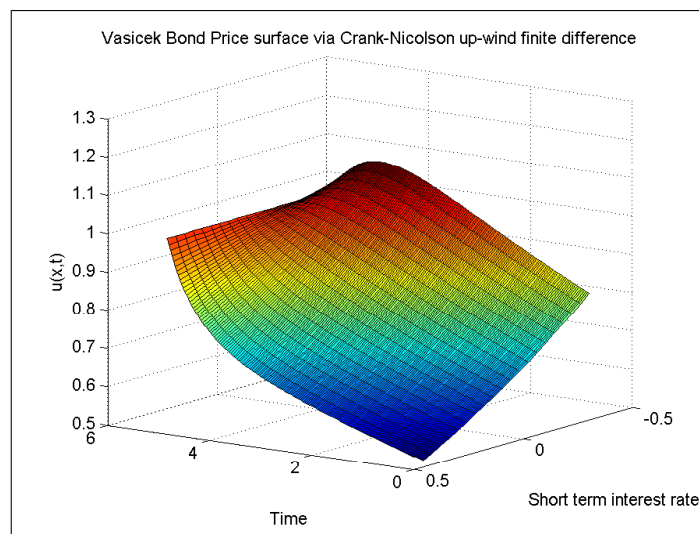Figure 2: *Zero-coupon bond prices - Up wind method*

Figure 3: *Zero-coupon bond prices surface - Up wind method*

4.3 Fully Implicit method

The use of the fully implicit scheme faces identical problems as those of the Crank-Nicolson P-matrix. We define μ and S in the transformed variable x as

$$\mu = \left(\frac{a(db - e^x + c)}{e^x} - \frac{\sigma^2 d^2}{2e^{2x}} \right) \tag{38}$$

17

and

$$S = \frac{\sigma^2 d^2}{2e^{2x}}. \tag{39}$$

The discretization of the PDE (25) in this case is:

$$\partial_t^- u_j^{n+1} + \mu_j \partial_x^0 u_j^{n+1} + S_j \partial_x^+ \partial_x^- u_j^{n+1} = \left(\frac{e^{x_j} - c}{d}\right) u_j^{n+1}. \tag{40}$$

This leads us to $(J+1) \times (J+1)-$dimensional matrix P that governs the system

$$u^{n+1} = \frac{1}{\Delta t} P^{-1}(u^n + D). \tag{41}$$

The vector D has $(J-1)$ zero entries, and is either zero for Neumann boundary conditions or stems from the Dirichlet boundary conditions. The above fully implicit discretization is of order $\mathcal{O}(\Delta t, \Delta x^2)$. Again, the matrix P has a tridiagonal form with entries

$$0 \geq \left[\frac{\mu_j}{2\Delta x} - \frac{S_j}{\Delta x^2}\right], \tag{42}$$

$$0 < \left[\frac{1}{\Delta t} + \frac{2S_j}{\Delta x^2} + \frac{e^{x_j} - c}{d}\right] \quad \text{and} \tag{43}$$

$$0 \geq \left[-\frac{\mu_j}{2\Delta x} - \frac{S_j}{\Delta x^2}\right], \tag{44}$$

for any choice of $a, b, \sigma \geq 0$ and $-\frac{c}{d} < r$. Figure 4 exhibits the spurious oscillating solution for a zero-coupon bond price with maturity in two years and parameters $a = 0.8$, $b = 0.1$ and $\sigma = 0.005$.

It is now convenient to introduce the following definitions (see [2], [31] and [21]).

Definition 1 *A matrix whose off-diagonal entries are less than or equal to zero is called $\mathbb{Z}$-matrix. Formally:*

$$\mathbb{Z}^{n \times n} = \{Q = (q_{ij}) \in \mathbb{R}^{n \times n} : q_{ij} \leq 0, i \neq j\}.$$

Proposition 1 *If a matrix $Q \in \mathbb{Z}^{n \times n}$, then the following assertions are equivalent to "Q is a non-singular M-matrix".*

- Q has all positive diagonal elements and there exists a positive diagonal matrix W such that QW is strictly diagonally dominant;
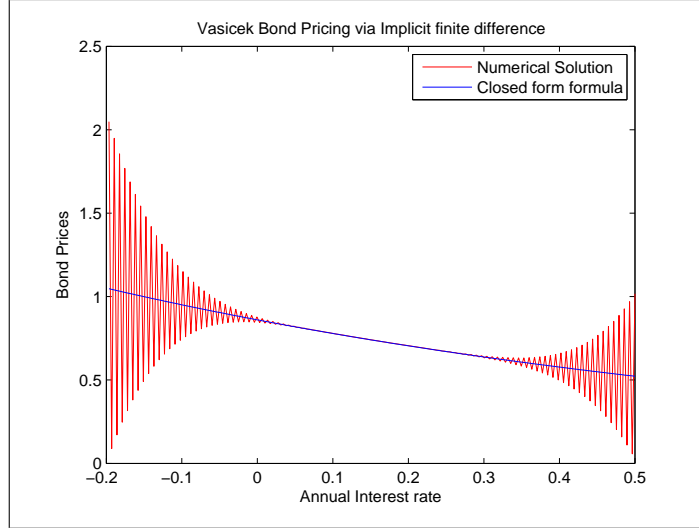
Figure 4: *Zero-coupon bond prices - fully implicit method*

- Q is inverse-positive, that is, A^{-1} exists and $A^{-1} \geq 0$;
- Q is positive stable, that is, the real part of each eigenvalue of Q is positive.

Therefore, we cannot guarantee that the matrix P of the PDE (40) is an M-matrix $\forall$ σ, b and a, $\Delta x \lll 1$. In the case of a non-M-matrix, there are some negative entries of P^{-1} leading to an oscillatory solution. So again, as in the cases of the previous methods, the fully implicit scheme does not lead to reliable pricing estimates of interest-rate derivatives.

5 Modified Fully Implicit method

To overcome the restrictions of the fully implicit method, we introduce a function $f = f(\sigma, b, a, c, x, \Delta r)$ appropriately chosen, given by

$$f = \frac{1}{2\delta} \left(\frac{a(b+c+1)}{e^x} + \frac{d^2\sigma^2}{2e^{2x}} + 1 \right), \tag{45}$$

which, in conjunction with a new reaction term prescribed as

$$G_j = \left(\frac{e^{x_j} - c}{d} \right) u_j^{n+1} + 4f u_j^{n+1} - 2f(u_{j-1}^{n+1} + u_{j+1}^{n+1}), \tag{46}$$

yields the modified version of the PDE (40), namely

$$\partial_t^- u_j^{n+1} + \mu_j \partial_x^0 u_j^{n+1} + S_j \partial_x^+ \partial_x^- u_j^{n+1} = G_j \tag{47}$$

and the corresponding system of equations

$$u^{n+1} = \frac{1}{\Delta t}\bar{P}^{-1}(u^n + D).$$ (48)

In this case $\bar{P} = (\bar{p}_{ij})$ turns out to be such that

$$\bar{p}_{j,j-1} \leq 0,$$ (49)

$$\bar{p}_{j,j} > 0 \qquad \text{and}$$ (50)

$$\bar{p}_{j,j+1} \leq 0.$$ (51)

Moreover,

$$\bar{p}_{j,j} > \sum_{j \neq i} |\bar{p}_{i,j}|,$$ (52)

so that $\bar{P}$ becomes an M-matrix.

A similar idea is suggested in [19] for the stock-options case. The following proposition shows that the modified version of the fully implicit scheme - given by (40) with its right side replaced by (46) - is in fact free of spurious oscillations.

Theorem 1 *The matrix $\bar{P}$ satisfies inequalities* (49), (50) *and* (51) *and is strictly diagonally dominant.*

Proof Relying on the modified version of the fully implicit method associated with PDE (25), and bearing in mind that μ and S are given by (38) and (39), it follows that

$$0 \geq \left[\frac{\mu_j}{2\Delta x} - \frac{S_j}{\Delta x^2} - 2f_j\right],$$ (53)

$$0 < \left[\frac{1}{\Delta t} + \frac{2S_j}{\Delta x^2} + \frac{e^{x_j} - c}{d} + 4f_j\right] \qquad \text{and}$$ (54)

$$0 \geq \left[-\frac{\mu_j}{2\Delta x} - \frac{S_j}{\Delta x^2} - 2f_j\right];$$ (55)

for any choice of $a, b, \sigma \geq 0$, $-\frac{c}{d} < r$ and some $\delta \ll 1$.

From proposition 1, $\bar{P}$ is an M-matrix, so that $\bar{P}^{-1} \geq 0$. So, the solution u provided by the finite difference scheme (47) is positivity-preserving: negative prices are precluded.

5.1 Stability

We show that u is stable and a non-increasing function in $t \in [0, T]$ (simulation results are provided in Figures 5 and 12 ahead). First, let us state an auxiliary lemma and a result of conditional stability.

Lemma 1 *Assume that Q is diagonally dominant by rows and set $\alpha = \min_k(|q_{kk}| - \sum_{j \neq k} |q_{kj}|)$. Then $\|Q^{-1}\|_\infty < \frac{1}{\alpha}$.*

Proof See [27].

Proposition 2 *Under the (very) mild condition*

$$0 < \frac{2S}{\Delta x^2} + \frac{e^x - c}{d} + 4f - \frac{\mu}{\Delta x}. \tag{56}$$

the solution u is stable. So, it is spurious oscillations free and we say that u is conditionally stable.

Proof The left side of

$$\frac{1}{\Delta t} \frac{1}{\left| \frac{1}{\Delta t} + \frac{2S}{\Delta x^2} + \frac{e^x - c}{d} + 4f - \frac{\mu}{\Delta x} \right|} < 1 \tag{57}$$

is an upper bound for the spectral radius of the iteration matrix $\left(\frac{1}{\Delta t} \bar{P}^{-1} \right)$. So, (57) suffices to render u stable Moreover, (56) implies (57).

We have that (56) expresses no interplay between Δx and Δt. Hence, we can say that the method is unconditionally stable with respect to Δt, whenever Δx satisfies (56). Actually, this fact strongly suggests that the method is unconditionally stable. We performed several computational tests using a variety of parameters and, in all of them, (56) was satisfied by a huge margin. Thus, for all practical means, we can assert that the method is *unconditionally stable*. Additional support for the assertion of unconditional stability is provided by the strong sufficient condition

$$0 < \frac{2S}{\Delta x^2} + r \quad \text{iff} \quad \min r \leq r < 0. \tag{58}$$

To see that (58) indeed implies (56), notice that S and $4f - \mu/\Delta x$ are nonnegative numbers and (56) is always satisfied whenever $r \geq 0$ (also recall that $r = (e^x - c)/d$). Note that the negative values of r - which typically occur in the Vasicek scenario - are the focus in (58). Expression (58) also writes

$$\frac{\sigma}{\Delta x} > (r + \frac{c}{d})\sqrt{-r} \quad \text{iff} \quad \min r \leq r < 0, \tag{59}$$

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where we use the fact that $S = \frac{\sigma^2}{2(r+\frac{c}{d})^2}$ (see (39)). If we consider the extended range $-\frac{c}{d} \leq r < 0$, we can easily derive the point of maximum of the right side of (59) given by $r^* = -\frac{c}{3d}$, so (59) becomes

$$\Delta x < \frac{\sigma}{2\left(\frac{c}{3d}\right)^{3/2}} \tag{60}$$

This (strong) sufficient condition is very simple and only depends on σ and the parameters used to adjust the negative range of r, namely, c and d. It clearly shows that low volatilities are more difficult to handle. Again, we entered several values of σ, c and d, and the values allowed for Δx were far beyond the usually required ones.

Proposition 3 *The solution of* (48) *satisfies the maximum principle.*

Proof Applying the sup-norm $\| \cdot \|_\infty$, using Lemma 1 and the conditional stability property of u, we have

$$
\begin{aligned}
\|u^{n+1}\|_\infty &= \frac{1}{\Delta t}\|\bar{P}^{-1}u^n\|_\infty \\
&\leq \frac{1}{\Delta t}\frac{1}{\left|\frac{1}{\Delta t} + \frac{2S}{\Delta x^2} + \frac{e^x - c}{d} + 4f - \frac{\mu}{\Delta x}\right|}\|u^n\|_\infty \\
&\leq \|u^n\|_\infty
\end{aligned}
$$

5.2 Consistency and convergence

Theorem 2 *The Modified Fully Implicit method associated with PDE* (47) *is of order of accuracy* $\mathcal{O}(\Delta t, \Delta x^2)$.

Proof The Taylor series for (46) boils down to

$$\frac{(e^{x_j} - c)}{d}u(t_n, x_j) + \frac{(e^{x_j} - c)}{d}\Delta t\frac{\partial u}{\partial t}(\xi_t, x) - f(\Delta x)^2\left(\frac{\partial^2 u}{\partial x^2}(t, \xi_x') + \frac{\partial^2 u}{\partial x^2}(t, \xi_x'')\right),$$

where $t_n \leq \xi_t \leq t_{n+\Delta t}$, $x_j \leq \xi_x' \leq x_{j+\Delta x}$ and $x_{j-\Delta x} \leq \xi_x'' \leq x_j$. It follows that (46) has order of accuracy $\mathcal{O}(\Delta t, \Delta x^2)$. In turn, relying on the definitions of $\partial_t^- u_j^{n+1}$, $\partial_x^0 u_j^{n+1}$ and $\partial_x^- \partial_x^+ u_j^{n+1}$, it follows that the left side of (40) also has order of accuracy $\mathcal{O}(\Delta t, \Delta x^2)$.

The Lax Theorem states that if a finite difference scheme is consistent (e.g., in the sense of proposition 2) and stable, then it is convergent ([25]).

6 Numerical results

In this section, we address two pricing problems in the Vasicek dynamic. The first aims to demonstrate the good performance and the properties of the Modified Fully Implicit method as described in Section 5, addressing a zero-coupon bond and the IDI option. The second aims a comparative analysis addressing the prices of the IDI option according to our approach and that of [29].

6.1 Convergence study

Assuming a continuous compound interest rate, we calculate the discrepancy between the price $P(r_j, 0, T)$ of the bond given by the closed-form expression (9) and the price u_j^0 given by the Modified Fully Implicit method with the prescribed terminal condition of a zero-coupon bond. The solutions are respectively represented by the dashed dark line and red line in Figure 5. The error measure we adopt is

$$\epsilon = \sqrt{\frac{1}{N} \sum_{j=1}^{N} (u_j^0 - P(r_j, 0, T))^2}, \tag{61}$$

where the subscript j assigns the spatial grid of the interest rates.

In the simulations we use $\Delta t = 0,00099206$ (four time-steps per day) and $\delta = 11 \times 10^{-4}$. Table 1 illustrates discrepancies for solutions with 200, 400, 600 and 800 spatial grid points and parameters set as $a = 0.1$, $b = 0.1$ and $\sigma = 0.02$ in a 1-year zero-coupon bond price problem. Columns 2 and 3 show that if $\epsilon \to 0$ then $\Delta x \to 0$.

To numerically estimate the order of convergence of the method, let us find q such that

$$\epsilon \leq C\Delta x^q, \tag{62}$$

for constant C.

The $(log \times log)$ plot of Figure 6 stems from Table 1 and shows that $q = 2.03$ in the domain of interest $r \in (-0.25, 0.65)$ implied by $x \in (-5, -0.01)$, $c = 0.3$ and $d = 1.1$.

We also tested the convergence rate of the method using the IDI option assuming daily updating. Since in this case the limit value of the price is not available, we look at ratios of differences between u_J^0 computed for different J's, given by

$$q = \log_2 \left[\frac{u_J^0 - u_{\frac{J}{2}}^0}{u_{\frac{J}{2}}^0 - u_{\frac{J}{4}}^0} \right]. \tag{63}$$

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For $r_j = 0.1$ and starting with $J = 1600$ grid points we obtained $q = 2.02$. To confirm the performance of the method for the case where the limit value of the price is not available, we replicated the above procedure for the zero-coupon bond (we did not use the prices given by the closed-form expression (9)). Again, a consistent rate was obtained, namely $q = 1.997$.

All the results above corroborate the early consistency analysis and the method's good performance.

As a lead-in to the next section, where our main results appear, we consider two zero-coupon bond pricing problems, where the sole difference between them is adopting a daily compounded interest in one problem and a continuously compounded interest in the other. Again the Modified Fully Implicit method is applied, now in conjunction with the algorithm described at the end of Section 3. The comparison can be seen in Figure 5 where the blue line refers to the daily compounded case and the red line (afore-mentioned) stands for the continuously compounded case. In this particular example, the relative discrepancy (defined in the same lines as in equation (65) below) did not exceed 10% in the whole interest rate domain. In contrast to this, we will see that the prices in the IDI case differ significantly if one adopts the continuous or the discrete updating scheme.

The small discrepancies we found here are in fact a well known result when interest rates are deterministic. However, we believe this is the first time this result is observed for stochastic interest rates following the Vasicek dynamic.

We remind that, in the zero-coupon bond case, the discrete compound yields can be straightforwardly obtained from the continuous compounding case. However, this is not the case when dealing with complex types of interest rate derivatives (e.g., callable bonds). For these types of derivatives, the PDE technique using discretely compounded interest rates can indeed be helpful.

Table 1: Modified Fully Implicit method spatial convergence rate

N	Δx	ϵ	$-\log(\Delta x)$	$-\log(\epsilon)$
200	0.02482	0.01295	1.60509	1.88770
400	0.01244	0.00327	1.90504	2.48572
600	0.00830	0.00142	2.08078	2.84924
800	0.00623	0.00078	2.20553	3.10777

6.2   Pricing

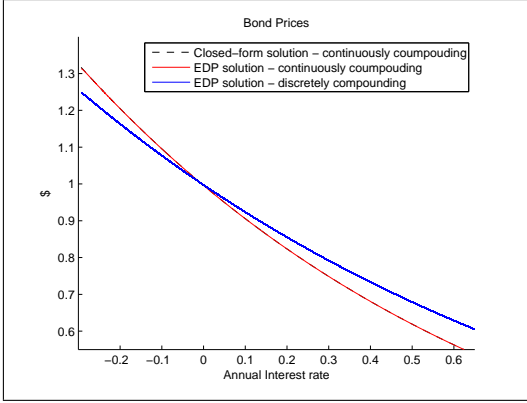We compare the prices of IDI call options under the Vasicek model, considering the following approaches:

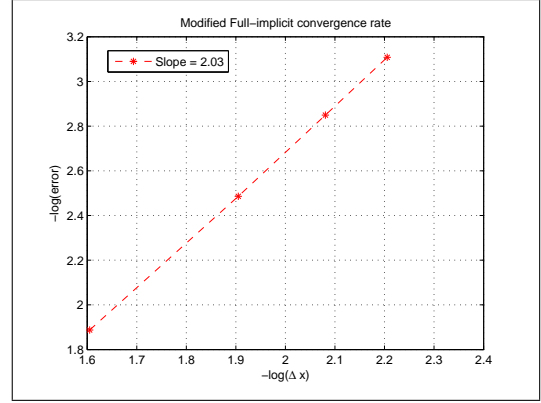Figure 5: *Zero-coupon bond price*

Figure 6: *Convergence rate analysis*

- We obtain the estimates of the prices according to the Modified Fully Implicit method and the coordinate transformation mentioned, and consider updating the IDI path discretely. This updating rule allow us to track realistically the evolution of the index and to achieve the exact pay-off representation.

- We solve the closed form expressions given in [29] for prices, assuming a continuously compounded interest rate, which is actually an idealization for mathematical tractability.

The numerical results of cases I, II, III and IV are summarized in Figures 7, 8, 9, 10, respectively, where we set $a = 0.1265$, $b = 0.0802$ and $\sigma = 0.0218$ in the Vasicek model. This calibration stemmed from the Brazilian overnight interest rate data from 2002 to 2014 and was produced via the General Method of Moments ([9]). The initial value of the IDI is 100.000 points. The Modified Fully Implicit method is used with 800 grid points for the ID index and a spatial mesh of 400 grid points for the interest rate. We use 5 steps per day with a daily jump condition at the last step, which satisfies the mild stability conditions required. Cases I and II (resp. cases III and IV) refers to a call option (resp. put option). Cases I and III (resp. II and IV) have maturity in 252 days (resp. 504 days) and strike K=109.550 points (resp. K=122.000 points). In the discretely compounding approach, we use the terminal condition given by (22) (the option's payoff) to solve the prices of the put (in lieu of that given by (21) of the call). For the continuous compound approach, we use the put-call parity to produce the prices Π of the put, namely,

$$\Pi(r_j, t, T) = C(r_j, t, T) + K.P(r_j, t, T) - y(0), \tag{64}$$

where $y(0) = 100.00$, $P(r_j, t, T)$ is the zero-coupon price given by (9) and $C(r_j, t, T)$ is the IDI call option price given by (6). Case V is summarized in Figure 11, where we set $a = 0.2$, $b = 0.1$ and $\sigma = 0.1$ in the Vasicek model and a short maturity of

20 days. We set several refinements in this case for spatial mesh sizes with 50; 150; 250; 400 and 600 points.

The first thing to notice from the numerical data is that, unlike the zero-coupon scenario previously mentioned, the relative discrepancy between prices obtained from the approaches under concern are not negligible at all, even with low volatilities (cases I - IV) and short maturities (case V). The relative discrepancy between prices is here defined as

$$\theta_j = \frac{(\eta(r_j, 0, T) - u_j^0)}{u_j^0}, \tag{65}$$

where η stands for C or Π. So, for $r_j = 10\%$, we have $\theta_j = 45.88\%$, $\theta_j = 50.96\%$, $\theta_j = 54.8\%$ and $\theta_j = 38.6\%$ for cases I to V, in that order. We recall that in the zero-coupon scenario, such relative difference did not exceed 10%. So, using one or other method makes a difference. Notice that neither price represents a benchmark - which should correspond to a framework that models the IDI discretely and provides the exact solution for the price. However, the Modified Fully Implicit method can be refined to approach the benchmark. On the other hand, any short rate modeling framework which adopts the IDI continuously compounded hypothesis - which is the case of [29] - is obviously inconsistent with refinements with respect to the index updating, so they cannot approach the benchmark. Recalling that the discrete updating procedure for calls (resp. puts) boils down to a cheaper (resp. more expensive) payoff than the continuous updating one, we expect prices to be cheaper (resp. more expensive) in the former than the latter procedure, for a reasonable mesh refinement. Figures 7 to 11 show this indeed. Figure 11 shows the downward movement of the prices as the spatial mesh sizes are refined in a sequence of $100, 150, 250, 400$ and 600 points, leading the solutions towards cheaper call option prices, which actually represent the benchmark. So, even with reasonably refined meshes, our call prices are cheaper than those of the continuous updating case of [29]. As the mesh is refined, our prices move further downwards, approaching the benchmark and simultaneously move further away from the prices of [29].

With a view to showing how prices evolve in time, Figure 12 provides the prices of an IDI call option considering a sequence of 8 time changes. Again, we change the problem parameters, now to $a = 0.5$, $b = 0.1$, $\sigma = 0.05$ and $T = 40$. We do this to show that the method is not biased toward any specific parametrization. The time sequence starts from $t = 0$ and ends at $t = 40$ days - where prices actually coincide with the option's payoff, in this case with strike $K = 100.900$. As above, the Modified Fully Implicit method is used with 800 grid points for the ID index and a spatial mesh of 400 grid points for the interest rate. The example gives another

indication of fitness of the method, in that a sort of rotation of the solution surface is observed as t varies. This is an intrinsic feature of Asian-style options in any market (see [22]).
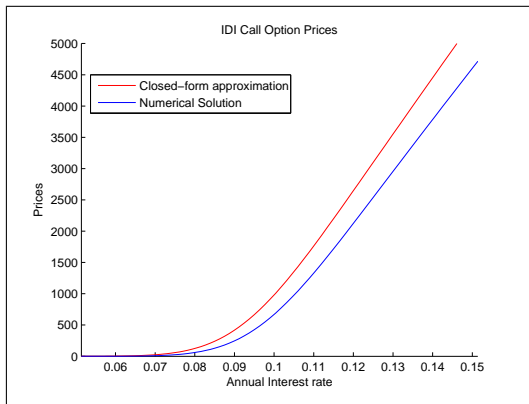

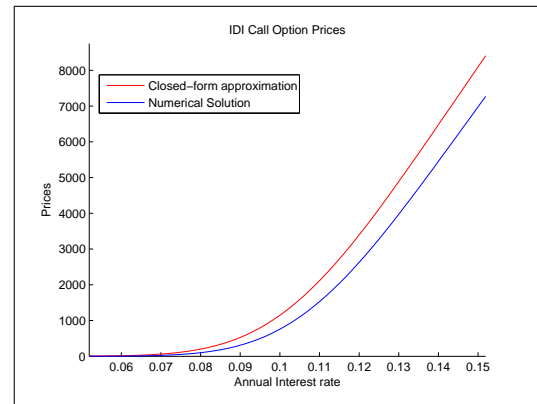Figure 7: *IDI call option prices Relative discrepancy* $= 45.88\%$ *at* $r = 10\%$

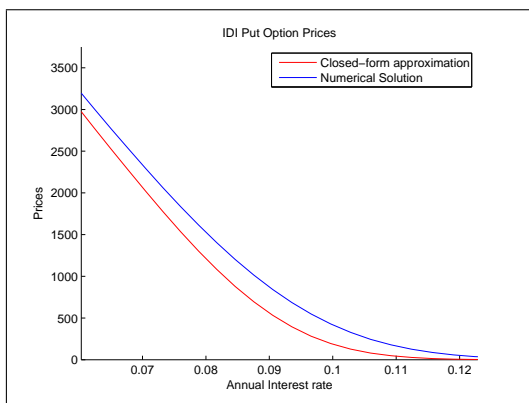Figure 8: *IDI call option prices Relative discrepancy* $= 50.96\%$ *at* $r = 10\%$

Figure 9: *IDI put option prices Relative discrepancy* $= 54.8\%$ *at* $r = 10\%$
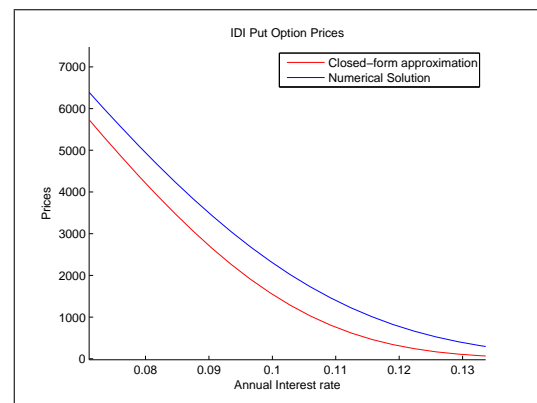

Figure 10: *IDI call option prices Relative discrepancy* $= 38.62\%$ *at* $r = 10\%$
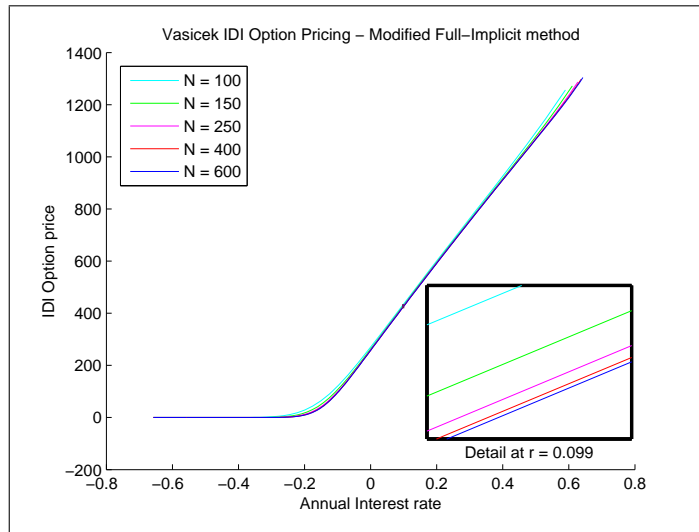
Figure 11: *IDI call option prices with meshes of 100, 150, 250, 400 and 600 points: Strike= 100900, $\sigma = 0.1, a = 0.2, b = 0.1, \ T = 20 \ \ days,$*
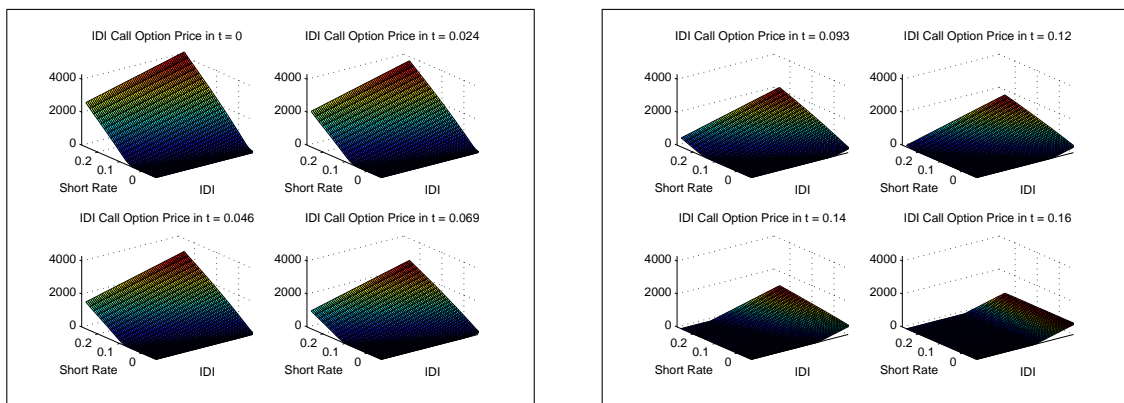

Figure 12: *How the prices of an IDI call option evolve in time (using the Modified Fully Implicit scheme) $IDI(0) = 100000, \ Strike= 100900, \ a = 0.5 \ b = 0.1, \ \sigma = 0.05, \ T = 40 \ days$*

6.3 Hedging

Before we present the main result of this section, which is summarized in Figure 16, it is worth addressing some aspects - with respect to a certain replicating portfolio and the hedging error - which are inherent to the interest rate scenario and differ much from those of stock markets. The motivation for doing so is that not much information of this kind is found in the fixed income literature. Also, parallel to the conclusions we obtained in Section 6.2, it gives us a glimpse on how the results may be misleading using the classical approach. In what follows in this section, we set the initial IDI value $IDI(0) = 100.000$, strike $K = 109500$, the Vasicek parameters $a = 0.1265$, $b = 0.0802$ and $\sigma = 0.0218$, and maturity in $T = 252$ days.

So, we create a discrete (one daily rebalance) self financing delta hedging strategy based on the zero coupon bond, in the short position of the IDI call option. Figures 13 and 14 address the continuous updating of the ID index. The classical tools are used to build the strategy, namely the prices given by equation (6) and the deltas according to (14). Figure 13 and the left panel of Figure 14 illustrate one realization of the hedging strategy.

The left panel of Figure 13 shows that the values of the delta-derived replicating portfolio tracks very well the option prices over time. The right panel shows the delta values while the left panel of figure 14 shows the borrowings in the bank account. The delta in the negative field means that the trader must sell $|\Delta|$ bonds whose unit value is $P(t,T)$ and deposit the proceeds in a bank account earning the risk-free DI rate. This is the opposite to what occurs in other interest rate derivatives (e.g., bond options), since there the issuer must settle his / her liability by delivering an asset - the bond - while in the IDI option he / she must deliver money in cash.

The right panel of Figure 14 shows the hedging error, which denotes the difference between the portfolio value at the expiration time and the payoff of the IDI call. This error stems from the discreteness of the hedging strategy. The histogram in the figure was generated from 10.000 Monte Carlo simulations. Its mean is approximately zero and, due to the discreteness of the strategy, the trader will have to deal with gains and losses that have an approximate normal distribution to settle his / her liability. The good performance observed is an expected result, since the closed form expressions of [29] corresponds to this exact modeling framework, i.e., the Vasicek model and a continuously compounded ID index. In contrast, the use of the closed form expressions when a discretely updating of the index is adopted - and this is actually the real life situation - produces a relative hedging error around 40% which is in accordance with the experiments of Section 6.2. This is a very large error indeed; traders are usually aware of this and perform some compensations in the option prices. Figure 15 illustrates these aspects.
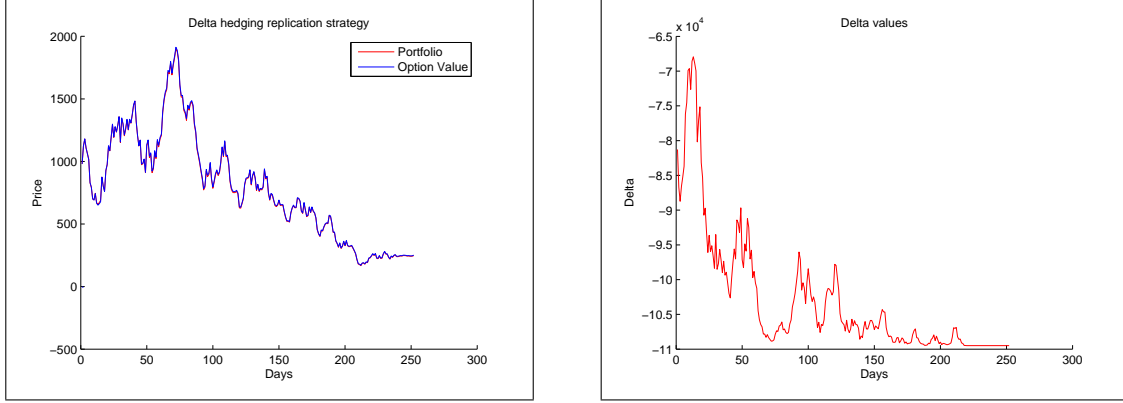
Figure 13: *Continuous updating of the IDI & [29]: Replicating portfolio and delta values*
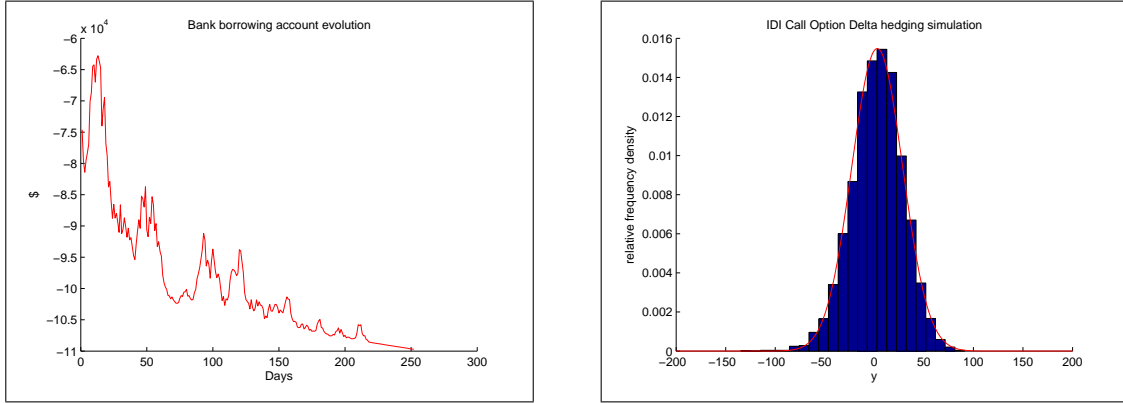

Figure 14: *Continuous updating of the IDI & [29]: bank account and hedging error*

Let us now return to the main subject of the section, which refers to the comparison of the deltas obtained according to the modeling frameworks described at the beginning of Section 6.2, i.e.,

- Under the assumption of a discrete updating of the ID index, we use the Modified Fully Implicit method with a spatial mesh size of 400 points to obtain the price estimates u of the IDI option, in conjunction with equation (16) to calculate the delta. A fourth order accurate central finite difference scheme was used to obtain numerically the derivatives for equation (16).

- We use the closed form expression (equation (6)) for the prices and calculate the delta according to (14), in which case a continuous updating scheme is assumed.

Figure 16 illustrates the analytical and numerical deltas with respect to the short-term rate at time t. The discrepancies exhibited strongly suggest that the updating of cumulative interest rate indexes should be treated realistically when pricing and hedging options.
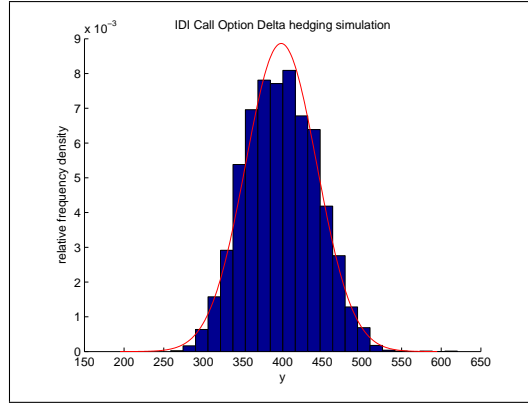
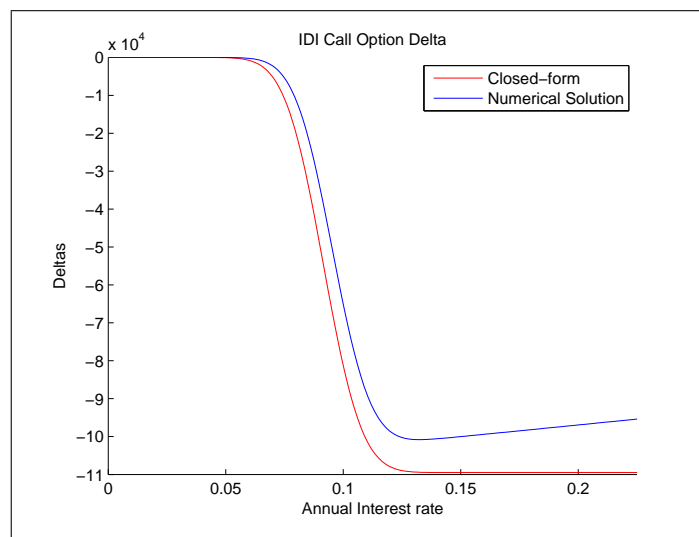Figure 15: *IDI call option hedging error: discrete updating of the IDI & [29]*

Figure 16: *IDI call option deltas*

7 Conclusions

We provide a new numerical finite difference method for pricing and hedging derivatives in the fixed income markets. The Modified Fully Implicit (interest rate) method - as we call it, is unconditionally stable and consistent, and at the same time exhibits high accuracy in obtaining estimates of prices and hedges. These qualities are preserved in the whole spectrum of volatilities that occurs in the interest rate dynamics.

We benefited from the good results that the numerical method gives, allowing the updating procedure of the interest to be discrete - which in fact is the realistic approach - rather than continuous. The results that we have obtained suggest that this scheme, which corresponds to obtaining estimates of prices for the 'exact' problem (referring to the discrete updating procedure of Asian options), is more

efficient than that of obtaining 'exact' prices - via closed form expressions - for the approximate problem (referring to the continuous updating procedure).

We considered the Vasicek model in this work. The method however can be adapted to other types of models (e.g., CIR and Sandmann-Sondermann) with modifications of modest proportions. A complementary study on this subject matter is under way.

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