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Sergio A. Lago Alves

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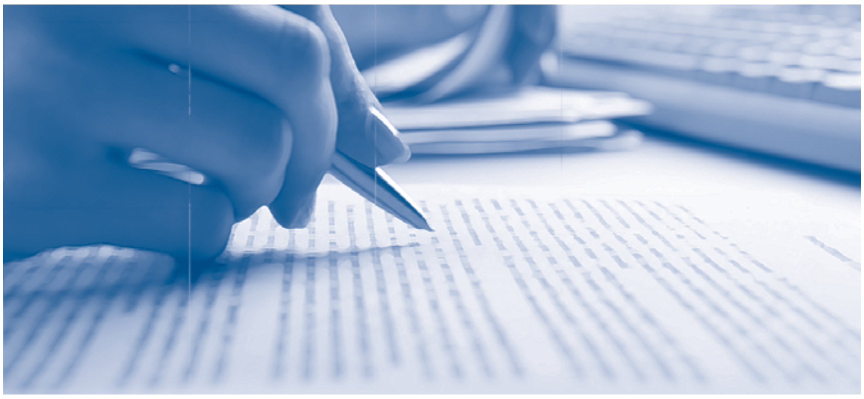

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Is the Divine Coincidence Just a Coincidence? The Implications of Trend Inflation*

Sergio A. Lago Alves†

Abstract

The Working Papers should not be reported as representing the views of the Banco Central do Brasil. The views expressed in the papers are those of the author(s) and do not necessarily reflect those of the Banco Central do Brasil.

In standard New Keynesian models, in which staggered pricing is the only nominal rigidity and shocks to preferences or technology are the only source of fluctuations, the literature has long agreed that the divine coincidence holds: the monetary authority is able to simultaneously stabilize the inflation rate and the output gap. I show that the divine coincidence holds only when the inflation rate is stabilized at exactly zero. Even a small deviation of trend (steady-state) inflation from zero generates a policy trade-off. I demonstrate this result using the model's non-linear equilibrium conditions to avoid any bias arising from log-linearization. When the model is log-linearized in line with common practice, a non-zero trend inflation gives rise to what I call an endogenous trend inflation cost-push shock. This shock enters the New Keynesian Phillips curve whenever the steady state level of inflation (trend inflation) deviates from zero. To assess the impact of trend inflation, I derive the welfare-based loss function under trend inflation and characterize optimal policy rules. Optimal policy is able to reduce the volatility of inflation and the output gap but can never fully stabilize them under non-zero trend inflation.

Keywords: Policy trade-off, Divine coincidence, Trend inflation.
JEL Classification: E31, E52

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1 Introduction[1]

The theoretical literature on the trade-off between price inflation variability and output gap variability has long agreed that the standard New Keynesian model exhibits what Blanchard and Gali (2007) called a divine coincidence: under price stickiness, any monetary policy rule that stabilizes the inflation rate, in the face of preference or technology shocks, also stabilizes the output gap. Yet in practice, most central banks perceive that they face a trade-off between stabilizing inflation and stabilizing the output gap. Hence, a puzzle arises in reconciling these two perspectives.

The literature has responded either by artificially adding shock terms to the Phillips curve (e.g. Clarida et al. (1999) and Gali (2002)) or by extending the standard New Keynesian model with additional frictions that allow the gap between output under flexible prices and the efficient output to vary over time. Those frictions take the form of time varying taxes (e.g. Woodford (2003, chap. 6), Benigno and Woodford (2005)), time varying markups (e.g. Benigno and Woodford (2005)), wage rigidities (e.g. Erceg et al. (2000) and Blanchard and Gali (2007)) and cost channels (e.g. Ravenna and Walsh (2006)), among others.

In this paper, I show that we do not need to extend the standard New Keynesian model in order to obtain a trade-off. The divine coincidence holds only in the particular case in which the monetary policy stabilizes the inflation rate at exactly zero. Using the non-linear model to avoid any bias due to log-linearizations, my main result is that it is impossible for monetary policy to simultaneously stabilize the inflation rate and the output gap except in the special cases in which either policy stabilizes the inflation rate at exactly zero or firms following an exact full indexation mechanism when not reoptimizing their price. The divine coincidence actually comes from a coincidence when the inflation rate is stabilized at zero: independently of the values of the structural parameters, a few composite parameters converge to the same level. This coincidence is sufficient to prevent the preference and technology shocks from affecting the output gap. At any other non-zero level for the stabilized inflation rate, the output gap moves in response to those shocks.

[1] The original working paper this paper is partially based on is found in Alves (2012a).

Previously, the literature assessing the divine coincidence focused on log-linearized versions of New Keynesian models. Given the fact that the benchmark approach is to log-linearize the models about a steady state equilibrium with zero inflation, it is no surprise that the literature has concluded that the monetary authority faces no trade-off in the standard New Keynesian model.

When the model is log-linearized around a steady state equilibrium with non-zero trend inflation, the main result translates into what I call the endogenous *trend inflation cost-push shock*, which ultimately depends only on the technology and preference shocks. The level of trend inflation acts as a shock amplifier: it is zero when the trend inflation is zero and increases as trend inflation rises. Therefore, policy trade-off arises even though there is no artificially added shock terms to the Phillips curve or frictions that generate time-varying gaps between output under flexible prices and the efficient output.

In order to assess the welfare effects of the trend inflation cost-push shock, I conduct a welfare analysis to characterize the distortions created by trend inflation. For that, I derive the fully-fledged welfare-based loss function under trend inflation (TIWeB),[2] obtain trend-inflation optimal policy rules, for the case in which the central bank is assigned a non-zero average inflation target, under discretion, unconditionally commitment (e.g. Damjanovic et al. (2008)), and from the timeless perspective (e.g. Woodford (1999, 2003)).

I also conduct an analysis on the gains of commitment using optimal policies that account for trend inflation and policies obtained if central banks were aware of the effects of trend inflation in the Phillips curve, but followed optimal policies based on the standard welfare-based (SWeB) loss function under zero trend inflation (Woodford (2003, chap. 6)).

Addressing welfare questions is not a simple task in models with non-zero trend inflation and staggered pricing. Even though aggregate variables have well-defined steady state levels in such equilibria,[3] there is also a steady-state dispersion of

[2] The TIWeB loss function differs from the trend inflation welfare-based loss function obtained by Coibion et al. (2012), as their approximation about the steady state with trend inflation is only partial. They actually approximate some components of the welfare function around the steady state with *flexible prices*, which is the same as the steady state with zero inflation. For this reason, Guido Ascari suggested calling their result as the hybrid welfare-based loss function.

[3] Broadly speaking, the non-stochastic steady state equilibrium is defined as the one achieved when all disturbances are fixed at their means.

relative prices. This fact is well documented in the literature on trend inflation (e.g. Ascari (2004), Ascari and Ropele (2007, 2009, 2011), Ascari and Sbordone (2013), Coibion and Gorodnichenko (2011), Coibion et al. (2012), among others).[4]

The findings of the welfare analysis are: (*i*) the trend-inflation optimal policy under unconditionally commitment dominates all other policies under trend inflation and all optimal policies based on the SWeB loss function; (*ii*) second-order approximations of the welfare function always underestimate welfare losses when average inflation exceeds zero; (*iii*) while this effect is exacerbated by always using the SWeB loss function in welfare analysis, it is mitigated by using the appropriated TIWeB loss function; and (*iv*) while the coefficient on the variability of the output gap is constant in the SWeB loss function, it varies in the TIWeB loss function to internalize the distortions created by non-zero trend inflation. For lower levels of trend inflation, this coefficient tends to fall as trend inflation rises, suggesting that fighting inflation becomes more important. This result is consistent with the findings of Ascari and Ropele (2007) for trend-inflation optimal policies under discretion: if the loss function has a constant weight on output gap volatility, determinacy and stability under rational expectations equilibria requires a decreasing upper bound for this weight as trend inflation rises.

In order to evaluate how different optimal policies perform and test whether they are able to at least mitigate the effects of the trend inflation cost-push shock, I employ simulations as in Ascari and Ropele (2007) to investigate how macro volatility and impulse responses to exogenous shocks vary with trend inflation. The analysis also compares results under optimal policies that account for trend inflation with policies obtained if central banks were aware of the effects of trend inflation in the Phillips curve, but followed optimal policies based on the SWeB loss function. The main finding is that it is the unconditional variance of the output gap rather than inflation that is most affected by whether policies are obtained using the standard loss function or the loss function that correctly incorporates trend inflation. The simulations confirm that optimal policies are able to fully stabilize the inflation rate and the output gap only when trend inflation is exactly zero. As trend inflation rises, the trade-off between inflation variability and output gap variability becomes

[4] More references on trend inflation are found in Alves (2012b), Amano et al. (2007), Cogley and Sbordone (2008), Fernandez-Corugedo (2007), Kichian and Kryvtsov (2007), and Sahuc (2006).

more and more apparent: the unconditional variances of both variables and the amplitude of their responses to shocks increase.

The remainder of the paper is organized as follows. The model is described in Section 2. The key results on policy trade-offs are derived in Section 3. The effect of trend inflation on welfare is discussed in Section 4, while Section 5 provides a welfare analysis of optimal policy under discretion and commitment. Simulations results are presented in Section 6. Section 7 summarizes the paper's conclusions.

2 The model

For simplicity, I follow Woodford (2003, chap. 4) to describe the standard new-Keynesian model with Calvo (1983) price setting and flexible wages. The economy consists of a representative infinite-lived household that consumes an aggregate bundle and supplies differentiated labor to a continuum of differentiated firms indexed by $z \in (0, 1)$, which produce and sell goods in a monopolistic competition environment.

2.1 Households

Household's workers supply $h_t(z)$ hours of labor to each firm z, at nominal wage $W_t(z) = P_t w_t(z)$, where P_t is the consumption price index and $w_t(z)$ is the real wage. Disutility over hours worked in each firm is $\upsilon_t(z) \equiv \chi h_t(z)^{1+\nu} / (1+\nu)$, where ν^{-1} is the Frisch elasticity of labor supply. The household's aggregate disutility function is $\upsilon_t \equiv \int_0^1 \upsilon_t(z)\, dz$. Consumption $c_t(z)$ over all differentiated goods is aggregated into a bundle C_t, as in Dixit and Stiglitz (1977), and provides utility $u_t \equiv \epsilon_t C_t^{1-\sigma} / (1-\sigma)$, where σ^{-1} is the intertemporal elasticity of substitution and ϵ_t is a preference shock. The household instantaneous utility is $u_t - \upsilon_t$. Aggregation and expenditure minimization relations are described by:

$$\begin{array}{ll} C_t^{\frac{\theta-1}{\theta}} = \int_0^1 c_t(z)^{\frac{\theta-1}{\theta}}\, dz & ; \; P_t^{1-\theta} = \int_0^1 p_t(z)^{1-\theta}\, dz \\ c_t(z) = C_t \left(\frac{p_t(z)}{P_t}\right)^{-\theta} & ; \; P_t C_t = \int_0^1 p_t(z)\, c_t(z)\, dz \end{array} \tag{1}$$

where $\theta > 1$ is the elasticity of substitution between goods.

Financial markets are complete and the budget constraint is $P_t C_t + E_t q_{t+1} B_{t+1} \leq B_t + P_t \int_0^1 w_t(z)\, h_t(z)\, dz + d_t$, where E_t is the time-t expectations operator, C_t is the aggregate consumption bundle, B_t is the state-contingent value of the portfolio of financial securities held at the beginning of period t, d_t denotes nominal dividend income, and q_{t+1} is the stochastic discount factor from $(t+1)$ to t.

The household chooses the sequence of C_t, $h_t(z)$ and B_{t+1} to maximize its welfare measure $\mathcal{W}_t \equiv \max E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t}(u_\tau - \upsilon_\tau)$, subject to the budget constraint and a standard no-Ponzi condition, where β denotes the subject discount factor. In equilibrium,[5] Lagrange multiplier λ_t on the budget constraint and the optimal labor supply function satisfy $\lambda_t = u'_t/P_t$ and $w_t(z) = \upsilon'_t(z)/u'_t$, where $u'_t \equiv \partial u_t/\partial C_t$ is the marginal utility to consumption, $\upsilon'_t(z) \equiv \partial \upsilon_t(z)/\partial h_t(z)$ is the marginal disutility to hours. The optimal consumption plan and the dynamics of the stochastic discount factor are described by the following Euler equations

$$1 = \beta E_t\left(\frac{u'_{t+1}}{u'_t}\frac{I_t}{\Pi_{t+1}}\right) \quad ; \; q_t = \beta\frac{u'_t}{u'_{t-1}}\frac{1}{\Pi_t} \tag{2}$$

, where $\Pi_t = 1+\pi_t$ and $I_t = 1+i_t$ are the gross inflation and interest rates at period t, which satisfies $I_t = 1/E_t q_{t+1}$, and i_t is the riskless one-period nominal interest rate.

2.2 Firms

Firm $z \in (0,1)$ produce differentiated goods using the technology $y_t(z) = \mathcal{A}_t h_t(z)^\varepsilon$, where $\mathcal{A}_t$ is the aggregate technology shock and $\varepsilon \in (0,1)$. The aggregate output Y_t is implicitly defined by $P_t Y_t = \int_0^1 p_t(z)\, y_t(z)\, dz$. Using the market clearing condition $c_t(z) = y_t(z)$, $\forall z$, the definition implies that $Y_t = C_t$, and that the firm's demand function is $y_t(z) = Y_t\left(p_t(z)/P_t\right)^{-\theta}$.

With probability $(1-\alpha)$, the firm optimally readjusts its price to $p_t(z) = p_t^*$. With probability α, the firm sets the price according to $p_t(z) = p_{t-1}(z)\,\Pi_t^{ind}$, where $\Pi_t^{ind} \equiv \Pi_{t-1}^{\gamma_\pi}$ and $\gamma_\pi \in (0,1)$.[6] When optimally readjusting at period t, the price p_t^* maximizes the expected discounted flow of nominal profits $\mathcal{P}_t(z) = p_t(z)\, y_t(z) - P_t w_t(z)\, h_t(z) + E_t q_{t+1}\mathcal{P}_{t+1}(z)$, given the demand function and the

[5] As usual, equilibrium is defined as the equations describing the first order conditions of the representative household and firms, a transversality condition $\lim_{T\to\infty} E_T q_{t,T} B_T = 0$, where $q_{t,T} \equiv \Pi_{\tau=t+1}^{T} q_\tau$, and the market clearing conditions.

[6] Full indexation is the particular case in which $\gamma_\pi = 1$.

price setting structure. At this moment, the firm's real marginal cost is $mc_t^* = (1/\mu)\, X_t^{(\omega+\sigma)} \left(p_t^*/P_t\right)^{-\theta\omega}$, where $\omega \equiv (1+\nu)/\varepsilon - 1$ is a composite parameter, $\mu \equiv \theta/(\theta-1) > 1$ is the static markup parameter, $X_t \equiv Y_t/Y_t^n$ is the gross output gap, and Y_t^n is the natural (flexible prices) output, which evolves according to

$$Y_t^{n(\omega+\sigma)} = \frac{\varepsilon}{\chi\mu}\epsilon_t \mathcal{A}_t^{(1+\omega)} \tag{3}$$

Following e.g. Ascari and Sbordone (2013, Section 3) and Ascari (2004, online Appendix), the firm's first order condition can be conveniently written, in equilibrium, as follows:

$$\left(\frac{p_t^*}{P_t}\right)^{1+\theta\omega} = \frac{N_t}{D_t} \tag{4}$$

The numerator N_t and the denominator D_t functions can be written in recursive forms, avoiding infinite sums:

$$\begin{aligned} N_t &= (X_t)^{(\omega+\sigma)} + E_t n_{t+1} N_{t+1} \quad &; n_t = \alpha q_t \mathcal{G}_t \Pi_t \left(\tfrac{\Pi_t}{\Pi_t^{ind}}\right)^{\theta(1+\omega)} \\ D_t &= 1 + E_t d_{t+1} D_{t+1} \quad &; d_t = \alpha q_t \mathcal{G}_t \Pi_t \left(\tfrac{\Pi_t}{\Pi_t^{ind}}\right)^{(\theta-1)} \end{aligned} \tag{5}$$

where $\mathcal{G}_t \equiv Y_t/Y_{t-1}$ denotes the gross output growth rate. The price setting structure implies the following dynamics:

$$1 = (1-\alpha)\left(\frac{p_t^*}{P_t}\right)^{-(\theta-1)} + \alpha\left(\frac{\Pi_t}{\Pi_t^{ind}}\right)^{(\theta-1)} \tag{6}$$

3 Policy trade-off

The model is a typical example of the standard new-Keynesian framework: it has standard functional forms, only one real distortion (monopolistic competition), only one source of nominal rigidity (Calvo staggered price setting) and shocks to preferences or technology are the only source of fluctuations.

Under those circumstances, the generally accepted paradigm in the literature on optimal monetary policy is that this model has what Blanchard and Gali (2007) call the *divine coincidence* property: the monetary authority faces no trade-off in simultaneously stabilizing the (gross) inflation rate $\Pi_t \equiv 1 + \pi_t$ and the (gross) output gap $X_t \equiv 1 + x_t$, i.e. the monetary policy is able to achieve the Paretooptimal equilibrium that would occur under completely flexible wages and prices.

As I show, however, this result is only feasible if either the monetary authority stabilizes the inflation rate at exact zero percent or firms have an exact full indexation mechanism when not optimally resetting their prices. The divine coincidence no longer holds otherwise. The following theorem states the policy trade-off results for the nonlinear model. Therefore, they are robust to any log-linearized approximation and do not depend on assuming that distortions are sufficiently close to zero.

Theorem 1 *With staggered price setting* ($\alpha > 0$) *and partial indexation* ($\gamma_\pi \neq 1$), *there exists a trade-off in stabilizing the output gap* x_t *and the inflation rate* π_t *whenever the monetary policy chooses a non-zero rate for stabilizing* π_t*: it is impossible for* x_t *and* π_t *to simultaneously have zero variance if* π_t *is stabilized at* $\bar{\pi} \neq 0$. *If* $\bar{\pi} = 0$ *or* $\gamma_\pi = 1$, *there is no stabilization trade-off and the divine coincidence holds.*

Proof. It is easier to deal with the (gross) inflation rate $\Pi_t \equiv 1 + \pi_t$ and the (gross) output gap $X_t \equiv 1 + x_t$. Assume, with no loss of generality, that the monetary authority stabilizes the inflation rate at a discretionary level, i.e. $var\left(\Pi_t\right) = 0$, $\forall t$, and $\Pi_t = \bar{\Pi} \equiv 1 + \bar{\pi}$, $\forall t$. It implies that $\Pi_t^{ind} = \bar{\Pi}^{\gamma_\pi}$, $\forall t$. From (6) and (4), the optimal relative resetting price p_t^*/P_t and the ratio N_t/D_t remain at their steady state levels:

$$\frac{p_t^*}{P_t} = \left(\frac{1-\alpha}{1-\bar{\alpha}}\right)^{\frac{1}{(\theta-1)}} \quad ; \quad \frac{N_t}{D_t} = \left(\frac{1-\alpha}{1-\bar{\alpha}}\right)^{\frac{(1+\theta\omega)}{(\theta-1)}} \tag{7}$$

where $\bar{\alpha} \equiv \alpha\left(\bar{\Pi}\right)^{(\theta-1)(1-\gamma_\pi)}$. Suppose, by contradiction, that $var\left(X_t\right) = 0$, $\forall t$, i.e. X_t is stabilized at a level $\bar{X}$. It implies that the output level and the stochastic discount factor evolve, in equilibrium, according to

$$Y_t = C_t = \bar{X} Y_t^n \quad ; \; q_t = \frac{\beta}{\bar{\Pi}}\left(\frac{\epsilon_t}{\epsilon_{t-1}}\right)\left(\frac{Y_t^n}{Y_{t-1}^n}\right)^{-\sigma}$$

Plugging the last result into (5), we conclude that N_t and D_t evolve according to

$$N_t = \left(\bar{X}\right)^{(\omega+\sigma)} + \bar{\alpha}\vartheta\beta E_t\left[\left(\frac{\epsilon_{t+1}}{\epsilon_t}\right)\left(\frac{Y_{t+1}^n}{Y_t^n}\right)^{1-\sigma} N_{t+1}\right] \tag{8}$$

$$D_t = 1 + \bar{\alpha}\beta E_t\left[\left(\frac{\epsilon_{t+1}}{\epsilon_t}\right)\left(\frac{Y_{t+1}^n}{Y_t^n}\right)^{1-\sigma} D_{t+1}\right] \tag{9}$$

where $\vartheta \equiv (\bar{\Pi})^{(1+\theta\omega)(1-\gamma_\pi)}$. Equations (7) and (8) imply that:

$$D_t = \left(\frac{1-\alpha}{1-\bar{\alpha}}\right)^{\frac{-(1+\theta\omega)}{(\theta-1)}} (\bar{X})^{(\omega+\sigma)} + \bar{\alpha}\vartheta\beta E_t\left[\left(\frac{\epsilon_{t+1}}{\epsilon_t}\right)\left(\frac{Y^n_{t+1}}{Y^n_t}\right)^{1-\sigma} D_{t+1}\right] \tag{10}$$

while (9) implies that:

$$\bar{\alpha}\beta E_t\left[\left(\frac{\epsilon_{t+1}}{\epsilon_t}\right)\left(\frac{Y^n_{t+1}}{Y^n_t}\right)^{1-\sigma} D_{t+1}\right] = D_t - 1 \tag{11}$$

(*i*) Consider the case in which π_t is stabilized at a non-zero level $\bar{\pi} \neq 0$ and indexation is partial ($\gamma_\pi \neq 1$). This implies that $\bar{\Pi} \neq 1$, $\vartheta \neq 1$ and $\bar{\alpha} \neq \alpha$. Using the last two results, it is easy to verify that D_t must remain at its steady state level $\bar{D}$:

$$D_t = \bar{D} = \frac{1}{(\vartheta - 1)}\left(\vartheta - \left(\frac{1-\alpha}{1-\bar{\alpha}}\right)^{\frac{-(1+\theta\omega)}{(\theta-1)}} (\bar{X})^{(\omega+\sigma)}\right)$$

Since D_{t+1} is also fixed at $\bar{D}$, eq. (9) implies that D_t must also satisfy

$$D_t = 1 + \bar{D}\bar{\alpha}\beta E_t\left[\left(\frac{\epsilon_{t+1}}{\epsilon_t}\right)\left(\frac{Y^n_{t+1}}{Y^n_t}\right)^{1-\sigma}\right]$$

Recall that ϵ_t is the exogenous preference shock and Y^n_t is a function of the exogenous preference ϵ_t and technology $\mathcal{A}_t$ shocks. Hence, the last result describes a contradiction, for it implies that D_t should moves in response to exogenous shocks.[7] Therefore, it is impossible for x_t and π_t to simultaneously have zero variance if π_t is stabilized at $\bar{\pi} \neq 0$.

(*ii*) Consider the case in which π_t is stabilized at the zero level $\bar{\pi} = 0$, or firms follow an exact full indexation mechanism ($\gamma_\pi = 1$). This implies that $\bar{\Pi} = 1$, $\vartheta = 1$ and $\bar{\alpha} = \alpha$. Using (10) and (11), it is easy to verify that there is no contradiction in assuming that the output gap can be stabilized when the inflation rate is stabilized at $\bar{\pi} = 0$. In this case, the stabilizing level of the output gap is $\bar{X} = 1$, i.e. $\bar{x} = 0$, there is no stabilization trade-off and the divine coincidence holds. ■

Even though the theorem predicts some instances in which the monetary policy faces no trade-off, the empirical evidence strongly supports the conditions for a

[7] Of course, there is a particular case in which the distributions of ϵ_t and $\mathcal{A}_t$ are such that $E_t\left(\frac{\epsilon_{t+1}}{\epsilon_t}\right)\left(\frac{Y^n_{t+1}}{Y^n_t}\right)^{1-\sigma}$ remains fixed. I rule out this possibility as it occurs with measure zero.

policy trade-off: inflation has been systematically positive since the World War II in developed and emergent countries,[8] and empirical evidence from macro and micro data suggests that there is very small indexation on individual prices.[9] Therefore, the theorem predicts that the divine coincidence property is just a very particular case, whose necessary conditions are not supported by empirical evidence.

In light of the last theorem, a natural question is whether it is desirable to stabilize the inflation rate at a non-zero level of trend inflation $\bar{\pi}$, given that doing so would prevent the output gap from being stabilized. The literature on trend inflation, incorporating important distortions such as the zero lower bound constraint (e.g. Coibion et al. (2012)), has shown that the optimal level of trend inflation rate is positive, yet small.[10] As a consequence of Theorem 1, with a positive trend inflation rate, optimal monetary policy faces a stabilization trade off and does not fully stabilize the inflation rate or the output gap. Section 5 shows the optimal way to control the variability of inflation and output gap, given a positive trend inflation.

I do not intend to pursue the question of the optimal level of inflation and take as given a positive trend inflation rate, reflecting the practice of many central banks that have adopted or have been assigned low but positive inflation targets.[11] Even though the inflation targets are not guaranteed to be welfare-optimal,[12] I address the following question: if a central bank that is concerned about welfare is not seeking to bring the inflation rate back to its welfare optimum but has been assigned a target for trend inflation, how should alternative policies be evaluated so as to pick the best policy from the policy family that keeps average inflation equal to the target?

[8] Indeed, average annual inflation ranged between 4% to 10% in European countries, about 3-5% in the US, and higher in developing countries even after their disinflationary plans succeeded. Japan had a long period of negative inflation, from 1999 to 2005. Even though the CPI annual inflation rate averaged 3.14% from 1971 to 2008, the average from 1999 to 2005 was -0.46%. Source: Japan's Statistics Bureau.

[9] For instance, this evidence is found in Bils and Klenow (2004), Cogley and Sbordone (2005, 2008), Klenow and Kryvtsov (2008), Klenow and Malin (2010) and Levin et al. (2006).

[10] An excellent review of the literature on trend inflation, which includes topics on the optimal level of inflation, is found in Ascari and Sbordone (2013).

[11] While most models in the literature have zero (e.g. Woodford (2003)) or negative (e.g. **?**) inflation as the optimal level to be pursued by the monetary authority, inflation targets around the world have always been positive. As of 2011, 27 central banks have adopted inflation targeting as their framework for monetary policy, and have positive inflation targets, ranging between 2 to 5 percent in most countries. See e.g. Hammond (2011) and Rogers (2010) for the inflation targets pursued in each country.

[12] Non-optimality may arise from measurement error of empirical CPIs, political economy issues (median voter preferences, lobbies, etc.) or model uncertainty, which may include a mismatch between the preferred models used by the government and the ones used by the central bank.

An important feature of the staggered price structure under trend inflation is that there is a steady state dispersion of relative prices. Clarifying this point is important before continuing the analysis.

Under trend inflation with Calvo price setting and partial indexation ($\gamma_\pi \neq 1$), the steady state is still dynamic at the firm level. Indeed, while structural shocks are constrained to remain at their zero means, the positive trend inflation induces a stationary dispersion of relative prices. The Calvo staggered price structure implies that there is always a fraction of firms whose prices lag behind their optimal levels. As a consequence, firms adjust above the aggregate price trend when they are set to optimize. This particularity makes the individual prices and production levels to be dispersed, even under the steady state. Interestingly, the aggregate output converges to a time invariant steady state level. The individual output dispersion is such that it cancels out when aggregating.

3.1 The log-linearized model

For any variable $\mathfrak{W}_t$, the hatted (benchmark) notation $\widehat{\mathfrak{w}}_t \equiv \log\left(\mathfrak{W}_t/\overline{\mathfrak{W}}\right)$ represents its log-deviation from its steady state level $\overline{\mathfrak{W}}$ with non-zero trend inflation (Trend StSt) and the tilde notation $\widetilde{\mathfrak{w}}_t \equiv \log\left(\mathfrak{W}_t/\mathfrak{W}\right)$ represents its log-deviation from its steady state level $\mathfrak{W}$ with zero trend inflation (Zero StSt). The few cases in which Zero StSt tilde variables are used are clearly stressed.

Under flexible prices ($\alpha = 0$), the (log-deviation) real interest rate and (log-deviation) output $\hat{y}_t^n$ evolve according to the following equations:

$$\hat{r}_t^n = E_t\left[\sigma\left(\hat{y}_{t+1}^n - \hat{y}_t^n\right) - \left(\hat{\epsilon}_{t+1} - \hat{\epsilon}_t\right)\right] \quad ; \hat{y}_t^n = \tfrac{1}{(\omega+\sigma)}\left[(1+\omega)\,\widehat{\mathcal{A}}_t + \hat{\epsilon}_t\right] \tag{12}$$

Under sticky prices ($\alpha > 0$),[13] the (log-deviation) output gap $\hat{x}_t$ is defined as follows:

$$\hat{x}_t = \hat{y}_t - \hat{y}_t^n \tag{13}$$

[13]I am aware that the α degree of price rigidity is likely to endogenously decrease as the trend inflation rises. I assume, however, that the parameter remains constant for all values of trend inflation as long as it is sufficiently small (less than 5% year, for instance).

The log-linearized IS curve is:

$$\hat{x}_t = E_t\hat{x}_{t+1} - \frac{1}{\sigma}E_t\left(\hat{\imath}_t - \hat{\pi}_{t+1} - \hat{r}_t^n\right) \tag{14}$$

The (log-deviation) stochastic discount factor $\hat{q}_t$ from $(t-1)$ to t, which satisfies $\hat{\imath}_t = -E_t\hat{q}_{t+1}$, evolves according to the following dynamics:

$$\hat{q}_t = \left(\hat{\epsilon}_t - \hat{\epsilon}_{t-1}\right) - \sigma\left(\hat{y}_t - \hat{y}_{t-1}\right) - \hat{\pi}_t \tag{15}$$

The New Keynesian Phillips Curve (NKPC) under trend inflation, with indexation term $\hat{\pi}_t^{ind} = \gamma_\pi \hat{\pi}_{t-1}$, is described by the following system:

$$\begin{aligned} \left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) = &\quad \beta E_t\left(\hat{\pi}_{t+1} - \hat{\pi}_{t+1}^{ind}\right) + \bar{\kappa}\hat{x}_t + (\vartheta - 1)\,\bar{\kappa}_\varpi \beta E_t\hat{\phi}_{t+1} \\ \hat{\phi}_t = &\quad \bar{\alpha}\vartheta\beta E_t\hat{\phi}_{t+1} + \theta(1+\omega)\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) + (1-\bar{\alpha}\vartheta\beta)(\omega+\sigma)\,\hat{x}_t \\ &\quad + \hat{q}_t + \hat{\pi}_t + \left(\hat{y}_t - \hat{y}_{t-1}\right) \end{aligned} \tag{16}$$

where $\hat{\phi}_t$ is an ancillary variable with no obvious interpretation.[14] As for the composite parameters, $\vartheta \equiv \bar{\Pi}^{(1+\theta\omega)(1-\gamma_\pi)}$ is a positive transformation of the level $\bar{\pi}$ of trend inflation and $\bar{\alpha} \equiv \alpha\bar{\Pi}^{(\theta-1)(1-\gamma_\pi)}$ is the effective degree of price stickiness. They increase as trend inflation rises and are bounded by $\max(\bar{\alpha}, \bar{\alpha}\vartheta) < 1$ to guarantee the existence of an equilibrium with trend inflation. The remaining composite parameters are

$$\bar{\kappa} \equiv \tfrac{(1-\bar{\alpha})(1-\bar{\alpha}\beta\vartheta)}{\bar{\alpha}}\tfrac{(\omega+\sigma)}{(1+\theta\omega)} \quad ; \; \bar{\kappa}_\varpi \equiv \tfrac{(1-\bar{\alpha})}{(1+\theta\omega)} \quad ; \; \omega \equiv \tfrac{(1+\nu)}{\varepsilon} - 1$$

As well documented in the literature of trend inflation, the trend inflation NKPC becomes flatter ($\bar{\kappa}$ decreases) and more forward looking ($(\vartheta-1)\,\bar{\kappa}_\varpi\beta$ and $\bar{\alpha}\vartheta\beta$ increases) as trend inflation rises. It is due to the fact that $\bar{\alpha}$ and ϑ increase as trend inflation rises.

As Ascari and Ropele (2007) state, the trend inflation NKPC reduces to the usual form $\left(\tilde{\pi}_t - \tilde{\pi}_t^{ind}\right) = \beta E_t\left(\tilde{\pi}_{t+1} - \tilde{\pi}_{t+1}^{ind}\right) + \kappa\tilde{x}_t$ when the level of trend inflation

[14] In the literature on trend inflation, there are two usual ways to describe trend inflation Phillips curves: (i) with ancillary variables (e.g. Ascari and Ropele (2007)); and (ii) with infite sums (e.g. Cogley and Sbordone (2008) and Coibion and Gorodnichenko (2011)).

is zero, where κ is the coefficient on the output gap in the NKPC when $\bar{\pi} = 0$. In this case, the ancillary variable $\hat{\phi}_t$ becomes irrelevant.

Note now that one of the relevant variables for inflation dynamics is not present when trend inflation is zero: the (log-deviation) stochastic discount factor added to the inflation rate and the expected growth rate, which is in the right hand side of the law of motion of $\hat{\phi}_t$:

$$\hat{\tau}_t \equiv \hat{q}_t + \hat{\pi}_t + (\hat{y}_t - \hat{y}_{t-1})$$

Indeed, the term $q_t \mathcal{G}_t \Pi_t$ is a relevant variable for the firm's first order condition, described by system $(4) - (5)$, but becomes irrelevant when using the Zero StSt to make log-linearizations. The reason for that is simple. Note that the Trend StSt log-linearization of the firm's first order condition is described by the system:

$$\begin{aligned}
(1+\theta\omega)\,\hat{\wp}_t^* &= \hat{N}_t - \hat{D}_t \\
\hat{N}_t &= \left(\bar{X}\right)^{(\omega+\sigma)}(\omega+\sigma)\,\hat{x}_t + \bar{n}\bar{N}E_t\left(\hat{n}_{t+1} + \hat{N}_{t+1}\right) \\
\hat{D}_t &= \bar{d}\bar{D}E_t\left(\hat{d}_{t+1} + \hat{D}_{t+1}\right) \\
\hat{n}_t &= \hat{q}_t + (\hat{y}_t - \hat{y}_{t-1}) + \hat{\pi}_t + \theta\,(1+\omega)\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) \\
\hat{d}_t &= \hat{q}_t + (\hat{y}_t - \hat{y}_{t-1}) + \hat{\pi}_t + (\theta-1)\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right)
\end{aligned}$$

where $\wp_t^* \equiv p_t^*/P_t$ is the optimal resetting relative price.

As long as the level of the trend inflation is not zero, the values of $\bar{n}\bar{N}$ and $\bar{d}\bar{D}$ are different. However, in the Zero StSt, those values converge to $\alpha\beta/(1-\alpha\beta)$. In this case, the steady state level of the (gross) output gap is $X = 1$ and the whole term $E_t\tilde{\tau}_{t+1}$ is simply eliminated from the final law of motion of the optimal relative resetting price $\widetilde{\wp}_t^*$, after substituting back the dynamics of $\tilde{N}_t$, $\tilde{D}_t$, $\tilde{n}_t$ and $\tilde{d}_t$:

$$\widetilde{\wp}_t^* = \frac{(\omega+\sigma)}{(1+\theta\omega)}\tilde{x}_t + \frac{\alpha\beta}{(1-\alpha\beta)}E_t\left(\tilde{\pi}_{t+1} - \tilde{\pi}_{t+1}^{ind}\right) + \frac{\alpha\beta}{(1-\alpha\beta)}E_t\widetilde{\wp}_{t+1}^*$$

In the Trend StSt, the component $\hat{\tau}_t$ plays two important roles in explaining the inflation dynamics. First of all, it brings the traditional conflict between the income and substitution effects into the dynamics of the inflation rate $\hat{\pi}_t$, by means of the way its expectation $E_t\hat{\tau}_{t+1}$ affects $E_t\hat{\phi}_{t+1}$. Indeed, a rise in the real interest rate $(\hat{\imath}_t - E_t\hat{\pi}_{t+1})$ induces households to substitute present consumption toward future consumption. In equilibrium, this substitution effect is captured by a rise in $E_t(\hat{y}_{t+1} - \hat{y}_t)$ and a fall in the expected stochastic discount rate $E_t\hat{q}_{t+1}$. This effect is in line with what is described by equations (15), using the fact that $\hat{\imath}_t = -E_t\hat{q}_{t+1}$. As the rate rises, the expansion in future consumption leads households to raise present consumption by means of the income effect, i.e. $E_t(\hat{y}_{t+1} - \hat{y}_t)$ tends to fall. This tension becomes clearer, in equilibrium, when using $\hat{\imath}_t = -E_t\hat{q}_{t+1}$ and equations (14) and (13) to rewrite the component $E_t\hat{\tau}_{t+1}$ as follows:

$$E_t\hat{\tau}_{t+1} = E_t\left[\left(\frac{1}{\sigma} - 1\right)(\hat{\imath}_t - \hat{\pi}_{t+1}) + \frac{1}{\sigma}(\hat{\epsilon}_{t+1} - \hat{\epsilon}_t)\right]$$

If $\sigma < 1$, the substitution effect dominates: as the real interest rate $(\hat{\imath}_t - E_t\hat{\pi}_{t+1})$ increases, all being equal, the net effect is an increase in $E_t\hat{\phi}_{t+1}$, which induces the inflation rate $\hat{\pi}_t$ to rise. In this case, the effect is similar to the cost channel introduced by Ravenna and Walsh (2006). When $\sigma > 1$, the income effect dominates and the net effect is the opposite. In a nutshell, the coefficient $\left(\frac{1}{\sigma} - 1\right)$ summarizes the tension between substitution and income effects.

The second role of $\hat{\tau}_t$ is to create what I call the *trend inflation cost-push shock*, which arises endogenously as a function of the preference and technology shocks. Using (12), (15) and (13), I rewrite $\hat{\tau}_t$ as a function of the output gap and the exogenous shocks:

$$\hat{\tau}_t = (1-\sigma)(\hat{x}_t - \hat{x}_{t-1}) + \frac{(1+\omega)}{(\omega+\sigma)}\left[(1-\sigma)\left(\widehat{\mathcal{A}}_t - \widehat{\mathcal{A}}_{t-1}\right) + (\hat{\epsilon}_t - \hat{\epsilon}_{t-1})\right]$$

where the tension between the substitution and income effects, represented now by $(1-\sigma)$, splits over the growth rates of the output gap and the technology shock. As for the preference shock, the tension net effect is absent.

Therefore, the NKPC system can be rewritten in terms of the output gap as the only demand variable:

$$\begin{aligned}
\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) &= \beta E_t\left(\hat{\pi}_{t+1} - \hat{\pi}_{t+1}^{ind}\right) + \bar{\kappa}\hat{x}_t + (\vartheta - 1)\bar{\kappa}_{\varpi}\beta E_t\hat{\varpi}_{t+1} + \widehat{\mathfrak{u}}_t \\
\hat{\varpi}_t &= \bar{\alpha}\vartheta\beta E_t\hat{\varpi}_{t+1} + \theta(1+\omega)\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) \\
&\quad + (1-\bar{\alpha}\vartheta\beta)(\omega+\sigma)\hat{x}_t + (1-\sigma)(\hat{x}_t - \hat{x}_{t-1})
\end{aligned} \tag{17}$$

$$\widehat{\mathfrak{u}}_t = \bar{\alpha}\vartheta\beta E_t\widehat{\mathfrak{u}}_{t+1} + (\vartheta - 1)\beta E_t\hat{\xi}_{t+1}$$

$$\hat{\xi}_t = \bar{\kappa}_{\varpi}\tfrac{(1+\omega)}{(\omega+\sigma)}\left[(1-\sigma)\left(\widehat{\mathcal{A}}_t - \widehat{\mathcal{A}}_{t-1}\right) + (\hat{\epsilon}_t - \hat{\epsilon}_{t-1})\right]$$

where $\hat{\varpi}_t$ is an ancillary variable with no obvious interpretation, $\hat{\xi}_t$ is an aggregate shock term that collects the effects of the technology shock $\widehat{\mathcal{A}}_t$ and the utility shock $\hat{\epsilon}_t$, and $\widehat{\mathfrak{u}}_t$ is the endogenous *trend inflation cost-push shock*, which ultimately depends only on the technology and preference shocks.

The effect of $\hat{\varpi}_t$ on the inflation dynamics is to make it even more forward looking. This is due to the fact that the coefficients $(\vartheta - 1)$ on $E_t\hat{\varpi}_{t+1}$, in the first equation, and $\bar{\alpha}\vartheta\beta$ on $E_t\hat{\varpi}_{t+1}$, in the second equation, increase as trend inflation rises. The aggregate shock term $\hat{\xi}_t$ collects the effects of the exogenous shocks $\widehat{\mathcal{A}}_t$ and $\hat{\epsilon}_t$. The trend inflation cost-push shock $\widehat{\mathfrak{u}}_t$ amplifies, by means of $(\vartheta - 1)$ and the coefficient $\bar{\alpha}\vartheta\beta$ on $E_t\widehat{\mathfrak{u}}_{t+1}$, the effect of the aggregate shock $\hat{\xi}_t$ and transmits it through the inflation dynamics.

Notice, however, that the effects of $\hat{\varpi}_t$ and $\widehat{\mathfrak{u}}_t$ on the inflation dynamics becomes irrelevant as trend inflation converges to zero.

Using the equations in system (17), it is easy to verify the predictions of Theorem 1. Stabilizing the inflation rate around any level $\bar{\pi}$ implies that $\hat{\pi}_t = \hat{\pi}_t^{ind} = 0 \;\; \forall t$. In this case, the output gap must satisfy

$$\begin{aligned}\hat{x}_t \;\; &= \;\; \bar{\alpha}\vartheta\beta E_t\hat{x}_{t+1} - (\vartheta - 1)\frac{\bar{\kappa}_{\varpi}}{\bar{\kappa}}\beta E_t\left[(1-\bar{\alpha}\vartheta\beta)(\omega+\sigma)\hat{x}_{t+1} + (1-\sigma)(\hat{x}_{t+1} - \hat{x}_t)\right] \\ &\quad -\frac{1}{\bar{\kappa}}\left(\widehat{\mathfrak{u}}_t - \bar{\alpha}\vartheta\beta E_t\widehat{\mathfrak{u}}_{t+1}\right)\end{aligned}$$

When $\vartheta = 1$, i.e. when the trend inflation is exactly zero ($\bar{\pi} = 0$) or when firms have exactly full indexation ($\gamma_\pi = 1$), the equation simplifies to $\tilde{x}_t = \alpha\beta E_t\tilde{x}_{t+1}$, whose solution is $\tilde{x}_t = 0 \;\; \forall t$. Therefore, the divine coincidence holds when $\vartheta = 1$. When $\vartheta \neq 1$, i.e. when the trend inflation is not zero ($\bar{\pi} \neq 0$) and when firms have partial indexation ($\gamma_\pi \neq 1$), the solution for the output gap depends on the data generating process of $\widehat{\mathfrak{u}}_t$. When the inflation rate is stabilized, the general solution is $\hat{x}_t = a\left(\widehat{\mathfrak{u}}_t - \bar{\alpha}\vartheta\beta E_t\widehat{\mathfrak{u}}_{t+1}\right)$, where a is an undetermined coefficient.[15] In any solution,

[15] In the simplest case in which $\widehat{\mathfrak{u}}_t$ is a pure white noise, the solution for the undetermined coefficient is $a = -\frac{1}{\bar{\kappa}b}$, where $b \equiv 1 - (\vartheta - 1)\beta\frac{\bar{\kappa}_{\varpi}}{\bar{\kappa}}(1-\sigma)$. In this case, the output gap evolves according to $\hat{x}_t = a\widehat{\mathfrak{u}}_t$ when the inflation rate is stabilized.

the output gap oscillates with $\widehat{\mathfrak{u}}_t$ and cannot be stabilized. Therefore, the divine coincidence does not hold when $\vartheta \neq 1$.

4 Welfare analysis

Following, I present a pair of equations describing the evolution of the aggregate disutility υ_t to labor supplied to all firms as a function of aggregate variables only. This result is important because it allows the fully-fledged derivation of the trend inflation welfare-based (TIWeB) loss function as a second order approximation of the (negative) true welfare function around the steady state with trend inflation.

Let $\mathcal{P}_t$ denote a measure of aggregate relative prices:

$$\mathcal{P}_t^{-\theta(1+\omega)} \equiv \int_0^1 \left(\frac{p_t(z)}{P_t}\right)^{-\theta(1+\omega)} dz \tag{18}$$

Using the Calvo (1983) price setting structure, I am able to derive the laws of motion of $\mathcal{P}_t$. The result is general and independent of any level of trend inflation. The following system describes the evolution of the aggregate disutility function υ_t and the aggregate relative price $\mathcal{P}_t$:

$$\upsilon_t = \frac{\chi}{1+\nu}\left(\frac{Y_t}{\mathcal{A}_t}\right)^{(1+\omega)} \mathcal{P}_t^{-\theta(1+\omega)} \tag{19}$$

$$\mathcal{P}_t^{-\theta(1+\omega)} = (1-\alpha)\left(\wp_t^*\right)^{-\theta(1+\omega)} + \alpha\left(\frac{\Pi_t}{\Pi_t^{ind}}\right)^{\theta(1+\omega)} \mathcal{P}_{t-1}^{-\theta(1+\omega)} \tag{20}$$

where again $\wp_t^* \equiv p_t^*/P_t$ is the optimal resetting relative price.

The way I derive the law of motion of $\mathcal{P}_t$ is very similar to how e.g. Yun (2005), Ascari (2004), and Schmitt-Grohe and Uribe (2006) derive relevant price dispersion variables for aggregate output, employment, or resource constraint in their models.

The price setting structure implies that the dispersion of relative prices rises as the inflation to indexation ratio $\left(\Pi_t/\Pi_t^{ind}\right)$ increases (see e.g. Woodford (2003, chap. 6)). Eq. (20) implies that $\mathcal{P}_t$ is negatively related to the inflation ratio, and consequently to price dispersion, and that υ_t increases as price dispersion rises. The distortion created by price dispersion on the aggregate disutility function is well explained by the employment cycling and labor supply smoothing effects, defined in Graham and Snower (2004): as price dispersion increases with trend inflation, the dispersion of production and hours also increases. If the individual marginal disutility to work rises with hours, hours dispersion leads to an increasing aggregate disutility to work as trend inflation rises.

Consider now the welfare function $\mathcal{W}_t$, which evolves according to the Bellman-shaped equation $\mathcal{W}_t = u_t - v_t + \beta E_t \mathcal{W}_{t+1}$. An important issue regards its curvature. In the Trend StSt, $\mathcal{W}_t$ decreases fast and becomes highly concave as the inflation rate rises, as also shown in Coibion et al. (2012). For illustration, using a calibration in line with accepted values for the USA in quarterly frequency (described below), figure 1 depicts the steady state levels of the welfare function $\bar{\mathcal{W}}$, its second derivative $\partial^2 \bar{\mathcal{W}} / \partial \bar{\pi}^2$, the aggregate output $\bar{Y}$ and the natural output $\bar{Y}^n$ as the annual trend inflation rises. Since $\bar{Y}^n$ does not vary with trend inflation, I normalize its level to 1. In this case, $\bar{Y}$ is the same as the gross output gap $\bar{X}$. All steady state levels (barred variables) are described in the appendix.

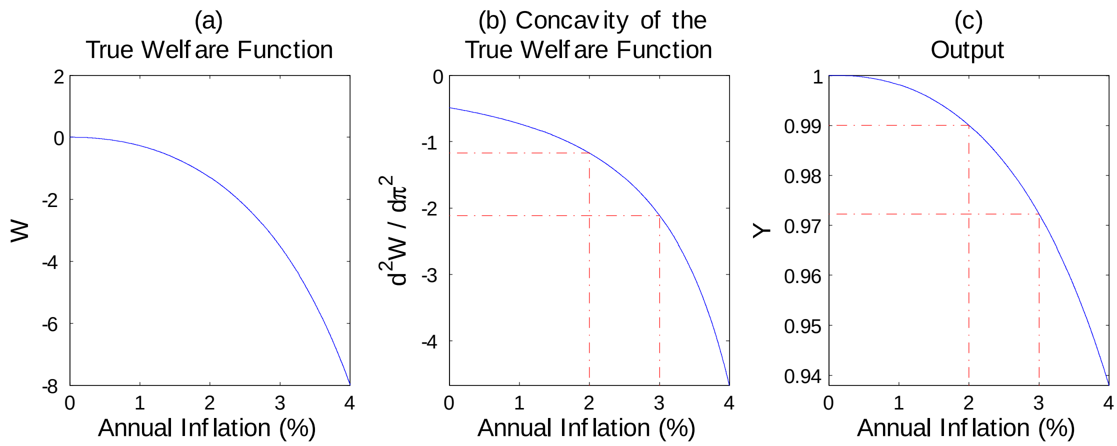

Figure 1: Steady state levels

As in Cooley and Prescott (1995), I set the subject discount factor at $\beta = 0.99$ and the elasticity to hours at the production function at $\varepsilon = 1 - 0.36$. I set $\alpha = 0.60$ as the degree of price stickiness, which is consistent with micro evidence[16] and recent macro evidence.[17] Using the central estimates[18] obtained by Smets and Wouters

[16] Nakamura and Steinsson (2008), using micro data from 1988 to 2005, estimate the median duration between price changes at roughly 4.5 months (including sales) and 10 months (excluding sales). Those median durations τ_m are consistent with $\alpha = 0.63$ and $\alpha = 0.81$ in quarterly fequency, using $\tau_m = -\log(2) / \log(\alpha)$.

[17] Cogley and Sbordone (2008), for instance, report $\alpha = 0.588$ as their median estimate), while Smets and Wouters (2007) report $\alpha = 0.65$ as the mode estimate, using the full sample period from 1966:1 to 2004:4.

[18] More specifically the modes of the posterior distribution.

(2007), I set the degree of indexation at $\gamma_\pi = 0.22$, the reciprocal of the elasticity of intertemporal substitution at $\sigma = 1.39$ and the reciprocal of the Frisch elasticity at $\nu = 1.92$. The latter is consistent with micro evidence (e.g. Chetty et al. (2011)). Based on the median estimates of Cogley and Sbordone (2008) and Ascari and Sbordone (2013), I set the elasticity of substitution at $\theta = 9.5$ which implies a monopolistic static markup of $\mu = 1.12$. In this paper, I refer to this parameter set as the benchmark calibration.

The concavity of the welfare function increases as trend inflation rises. For instance, the curvature at the 2 percent annual trend inflation is 2.4 times as large as the curvature at the Zero StSt. At the 3 percent annual trend inflation, it is 4.3 times as large. It implies that: (*i*) second order approximations around the Zero StSt underestimate the appropriate curvature of the welfare function when the inflation rate is actually oscillating around a positive value; and (*ii*) such second order approximations do not internalize the large welfare loss achieved when trend inflation is positive. When policy is meant to keep inflation at a positive level, as in an inflation targeting framework, those findings suggest that a better policy evaluation is obtained when the approximation is done around the level of trend inflation.

Finally, notice that the steady state output sharply falls below the aggregate output under flexible prices as trend inflation rises above zero, as documented in Yun (2005), Ascari (2004), Ascari and Ropele (2009) and Ascari and Sbordone (2013). This effect clearly shows the distortion caused by high trend inflation. When the level of trend inflation is 2 percent, the output gap is open at about 1 percent. I recognize that the degree of price rigidity is likely to endogenously decrease as trend inflation rises, and hence part of the inflation distortion may not be as high as the pictures suggest. However, empirical evidence for the US suggests that a Calvo parameter larger than 0.60 is consistent with annual trend inflation of 3 percent.[19] Therefore, my theoretical assessments may be quite reasonable in the inflation range I consider.

In this stylized model, the optimal level of inflation is positive, but very close to zero. A small positive level of trend inflation $\bar{\pi}$ can minimize the distortion caused

[19] From 1984 to 2004, the CPI annual inflation rate averaged 3.05% in the US. Using data from the same period, Smets and Wouters (2007) estimate the Calvo parameter at 0.73.

by monopolistic competition,[20] but also brings more distortions, as described in the next section. For illustration, figure 2 depicts the levels of trend inflation that maximize the steady state level of the welfare function $\bar{\mathcal{W}}$ as the static markup μ rises from 1 to 1.3, using the benchmark calibration.

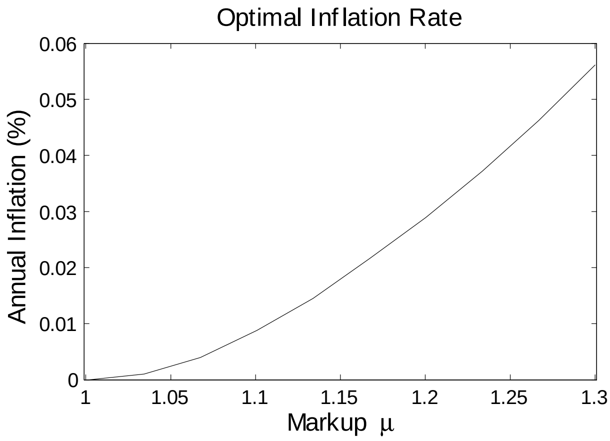

Figure 2: Markup and Inflation

Therefore, it is optimal to prevent the output gap and inflation to be both stabilized by keeping the trend inflation at the optimal (positive) value. Of course, using a more comprehensive model augmented with important distortions (e.g. the zero lower bound constraint), as done by Coibion et al. (2012), is necessary to justify higher levels of optimal inflation rates.

4.1 Distortions

In order to cope with the distortion caused by trend inflation, I expand Woodford's analysis on the efficient output under the Zero StSt (see Woodford (2003, chap. 6)). Consider a central planner who chooses the prices and the output level to maximize the welfare. The optimal solution clearly imposes every firm to produce the same efficient level Y_t^{ef}, which implies that all prices are the same, i.e. $\mathcal{P}_t = 1$. Thus the solution is $\partial v_t^{ef}/\partial Y_t = \partial u_t^{ef}/\partial Y_t$. This result implies that the efficient output evolves according to $Y_t^{ef} = \mu^{1/(\omega+\sigma)} Y_t^n$.

[20] When the level of trend inflation is positive, but sufficiently small, the average markup (as defined in King and Wolman (1996)) decreases as trend inflation rises. See also Ascari and Sbordone (2013, fn. 39) for a similar conclusion.

In particular, the steady state levels of consumption utility and labor disutility depend on the steady state aggregates as follows:

$$\bar{u} = \frac{\bar{Y}^{1-\sigma}}{1-\sigma} \quad ; \bar{\upsilon} = \frac{1}{1+\omega}\bar{Y}^{(1+\omega)}\left(\bar{Y}^{ef}\right)^{-(\omega+\sigma)}\bar{\mathcal{P}}^{-\theta(1+\omega)} \tag{21}$$

Ignoring the indirect effect of $\bar{Y}$ on $\bar{\mathcal{P}}$, the steady state level of the marginal rate of substitution can be roughly approximated by the ratio of the derivatives of $\bar{\upsilon}$ and $\bar{u}$ with respect to $\bar{Y}$, i.e. $\bar{\upsilon}_Y/\bar{u}_Y$. Note that it is not the same as the steady state level of the ratio of the derivatives of υ_t and u_t with respect to Y_t. However, this approximation makes it easier to understand the first distortion component in this model economy:

$$\frac{\bar{\upsilon}_Y}{\bar{u}_Y} = \mathcal{E}_\mu \mathcal{E}_{\bar{\pi}} \quad ; \mathcal{E}_\mu \equiv \frac{1}{\mu} \quad ; \mathcal{E}_{\bar{\pi}} \equiv \frac{(1-\bar{\alpha}\beta\vartheta)}{(1-\bar{\alpha}\beta)}\frac{(1-\bar{\alpha})}{(1-\bar{\alpha}\vartheta)} \tag{22}$$

where again $\vartheta \equiv \bar{\Pi}^{(1+\theta\omega)(1-\gamma_\pi)}$ is a positive transformation of the level $\bar{\pi}$ of trend inflation, and $\bar{\alpha} \equiv \alpha\bar{\Pi}^{(\theta-1)(1-\gamma_\pi)}$ is the effective degree of price stickiness.

Following Woodford (2003, chap. 6), let $\bar{\Phi}_y \equiv 1 - \bar{\upsilon}_Y/\bar{u}_Y$ denote the inefficiency degree of the steady state output. Under the Trend StSt, the term $\mathcal{E}_\mu$ is driven by the monopolistic competition distortion alone, while the second term $\mathcal{E}_{\bar{\pi}}$ is driven by the non-zero trend inflation. Note that the second term collapses to unity when $\bar{\pi} = 0$. Considering only the Zero StSt, Woodford (2003, chap. 6) defines the inefficiency degree as $\Phi_y = 1 - \mathcal{E}_\mu$.

The second component of distortionary effects of the non-zero trend inflation is explained as follows. There is steady state dispersion of relative prices and production under positive trend inflation, and the indirect effect $\partial\bar{\mathcal{P}}/\partial\bar{Y}$ omitted in the previous computation of $\bar{\upsilon}_Y/\bar{u}_Y$ captures this additional distortion source. In this regard, I define $\bar{\Phi}_\vartheta \equiv (\vartheta - 1)$ as an additional inefficiency parameter to measure how much of the gross trend inflation is above one.

As it turns out, tracking those two inefficiency parameters ($\bar{\Phi}_y$ and $\bar{\Phi}_\vartheta$) is sufficient for deriving the TIWeB loss function. Since efficiency requires both parameters to be zero, it is interesting to assume them to be small enough, as first order disturbance terms.[21] With such an assumption, linear terms multiplied by $\bar{\Phi}_y$ and $\bar{\Phi}_\vartheta$

[21] Woodford (2003, chap. 6) models the distortion variable Φ_y as a first order disturbance term,

become of second-order importance and the TIWeB loss function is useful for policy analysis using the log-linearized structural equations.

4.2 The welfare-based loss function under trend inflation

In this section, I present the trend inflation welfare-based (TIWeB) loss function, derived as the second order log-approximation of the True loss function.

Before showing the main result, I clarify the reasons why my approach to (second-order) approximate the aggregate disutility $\upsilon_t \equiv \int_0^1 \upsilon_t(z)\,dz$ differs from the one in Woodford (2003, chap. 6). His approach consists of two parts. In the first, he approximates the integrand $\upsilon_t(z)$, which is proportional to $(p_t(z)/P_t)^{-\theta(1+\omega)}$, around the Zero StSt. This procedure is possible because there is no price dispersion in this steady state. Next, he integrates the approximated result and uses the Calvo price setting structure to rewrite the variance of (log) relative prices as a function of the squared inflation rate.

Under the Trend StSt, this approach is not possible because the individual relative prices do not converge in the steady state. In order to fully approximate the aggregate disutility under the Trend StSt, avoiding the problem arising from the steady state dispersion of relative prices, I use the equations described in system $(19)-(20)$. The first equation describes the evolution of υ_t as a function of aggregate variables only, such as $\mathcal{P}_t$, which captures the effect of the dispersion of relative prices. The second equation provides the law of motion of $\mathcal{P}_t$. Therefore, I derive the TIWeB loss function as follows:

Proposition 1 *The true welfare function is (second-order) approximated as*

$$\mathcal{W}_t = -\frac{1}{2}\bar{\mathcal{V}} E_t \sum_{\tau=0}^{\infty} \beta^{\tau} \bar{\mathcal{L}}_{t+\tau} + \overline{tip}_t^{\mathcal{W}} \tag{23}$$

where

$$\bar{\mathcal{L}}_t \equiv \left(\hat{\pi}_t - \hat{\pi}_t^{ind} + \bar{\phi}_\pi\right)^2 + \bar{\mathcal{X}}\left(\hat{x}_t - \bar{\phi}_x\right)^2 \tag{24}$$

is the trend inflation welfare-based (TIWeB) loss function, $\overline{tip}_t^{\mathcal{W}}$ *stands for terms independent of policy at period* t, $\bar{\phi}_\pi$ *and* $\bar{\phi}_x$ *are constants that depend on the in-*

in order to derive the SWeB loss function.

efficiency parameters $\bar{\Phi}_{\vartheta}$ and $\bar{\Phi}_y$, and $\bar{\mathcal{V}}$ corrects for the aggregate reduction in the welfare when trend inflation increases. Those composite parameters are defined as follows:

$$\bar{\phi}_{\pi} \equiv \frac{(1-\bar{\alpha})}{(1-\bar{\alpha}\vartheta)(1+\theta\omega)}\bar{\Phi}_{\vartheta} \quad ; \bar{\phi}_x \equiv \frac{1}{(\omega+\sigma)}\bar{\Phi}_y \quad ; \bar{\mathcal{V}} \equiv \frac{(\omega+\sigma)}{\bar{\mathcal{X}}}\bar{Y}^{1-\sigma} \quad ; \bar{\mathcal{X}} \equiv \frac{(1-\bar{\alpha})}{(1-\bar{\alpha}\vartheta)}\frac{\bar{\kappa}}{\theta} \tag{25}$$

The proof is shown in the appendix, in which I use the assumption that $\bar{\Phi}_y$ and $\bar{\Phi}_{\vartheta}$ are first-order parameters in order to cope with linear terms in the approximation.

As expected, the scale parameter $\bar{\mathcal{V}}$ is proportional to the steady state level of the consumption utility, in equilibrium, by means of $\bar{Y}^{1-\sigma}$. The critical effect of trend inflation appears in the coefficient $\bar{\mathcal{X}}$ on the output gap term in (24). In the SWeB loss function, this coefficient is given by $\mathcal{X} \equiv \kappa/\theta$, where κ is the coefficient on the output gap in the NKPC when $\bar{\pi} = 0$. Let $\mathcal{R} \equiv \bar{\mathcal{X}}/\mathcal{X}$ denote the coefficient ratio of the output gap volatility. Figure 3 plots $\mathcal{R}$ as the annual trend inflation rate increases for the benchmark calibration and for the cases $\alpha = 0.50$ and $\alpha = 0.70$.

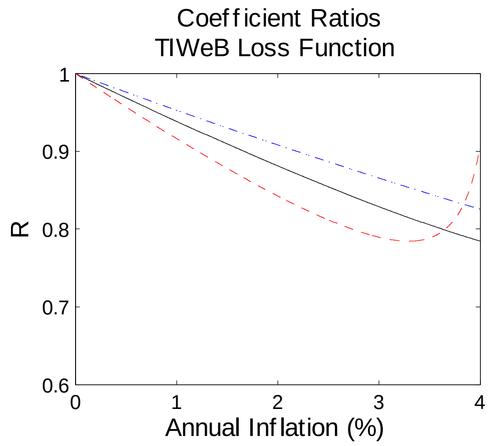

Figure 3: Coefficient Ratios (R)

Note: $\alpha = 0.50$ *(dashdotted blue),* $\alpha = 0.60$ *(black),* $\alpha = 0.70$ *(dashed red).*

Note that $\mathcal{R}$ tends to decrease rapidly as trend inflation rises, and this effect is amplified if the degree of price stickiness α is larger. At the 3 percent trend inflation ($\alpha = 0.60$), in particular, the TIWeB relative weight of the output gap is about 80 percent of the coefficient suggested by the Woodford's standard welfare-based (SWeB) loss function.

The key factor is the increasing concavity of the true welfare function in the steady state. The TIWeB loss function fully internalizes the large welfare loss due to inflation volatility under positive trend inflation. Hence, the monetary authority optimally places less weight into fighting volatility of the output gap, as opposed to fighting inflation volatility. This result is also consistent with the findings of Ascari and Ropele (2007). Studying optimal discretionary policy under trend inflation with a loss function with constant coefficient $\mathcal{X}$ on the output gap volatility, they find a decreasing upper bound for the central bank preference coefficient $\mathcal{X}$. The upper bound decreases as trend inflation rises to guarantee determinate equilibria with rational expectations.

In the TIWeB loss function, however, $\mathcal{R}$ starts to rise when the combination of higher levels of inflation trend and larger degrees of price stickiness[22] starts to be critical, as depicted by the dashed red line ($\alpha = 0.70$). Indeed, output deteriorates so much as trend inflation rises that there are welfare gains from avoiding larger output gap volatility by allowing greater fluctuations in the inflation rate around trend.

4.3 The *static* and the *stochastic* wedges

I identify two main wedges driven between the True loss function and any approximated assessment: the *static wedge* – defined as the difference between the True and approximated loss functions, when evaluated in equilibria where the (gross) inflation rate remains fixed at a certain level Π, the remaining endogenous variables are fixed at the levels consistent with Π and the exogenous shocks are fixed at their means; and the *stochastic wedge* – defined as the extra wedge arising in the difference of the (unconditional) expected values of the loss functions in a stochastic equilibrium: this is a direct application of the Jensen's inequality.

Under the Zero StSt, the log-linearized aggregate demand and supply curves and the SWeB loss function are:

$$\begin{aligned} &\tilde{x}_t = E_t\tilde{x}_{t+1} - \tfrac{1}{\sigma}E_t\left(\tilde{\imath}_t - \tilde{\pi}_{t+1} - \tilde{r}^n_t\right) &&;\ \tilde{\Delta}\tilde{\pi}_t = \beta E_t\tilde{\Delta}\tilde{\pi}_{t+1} + \kappa\tilde{x}_t \\ &\mathcal{W}_t = -\tfrac{1}{2}\mathcal{V}E_t\sum_{\tau=0}^{\infty}\beta^\tau\tilde{\mathcal{L}}_{t+\tau} + \widetilde{tip}^{\mathcal{W}}_t &&;\ \tilde{\mathcal{L}}_t = \left(\tilde{\pi}_t - \tilde{\pi}^{ind}_t\right)^2 + \mathcal{X}\left(\tilde{x}_t - \phi_x\right)^2 \end{aligned} \tag{26}$$

[22] In the model, this combinatin is reflected in the composite parameters $\bar{\alpha}$ and ϑ.

where $\tilde{\Delta}\tilde{\pi}_t \equiv \left(\tilde{\pi}_t - \tilde{\pi}_t^{ind}\right)$, $\tilde{\pi}_t^{ind} = \gamma_\pi \tilde{\pi}_{t-1}$, $\tilde{r}_t^n = \hat{r}_t^n$, and

$$\phi_x \equiv \frac{1}{\theta(\omega+\sigma)} \quad ; \mathcal{V} \equiv \frac{(\omega+\sigma)}{\mathcal{X}} \left(\frac{\varepsilon}{\chi\mu}\right)^{\frac{1-\sigma}{\omega+\sigma}} \quad ; \mathcal{X} \equiv \frac{\kappa}{\theta} \tag{27}$$

In order to compute the static wedge of the SWeB loss function, the nominal interest rate is adjusted to keep the (gross) inflation rate fixed at a certain level Π when satisfying the Zero StSt log-linearized equations. The SWeB static wedge is computed as $\bar{\mathcal{W}} - \mathcal{W}_0$, where

$$\begin{array}{ll} \mathcal{W}_0 = -\frac{1}{2}\frac{\mathcal{V}}{(1-\beta)}\mathcal{L}_0 + tip_0^{\mathcal{W}} & ; \pi_0 = \log(\Pi) \\ \mathcal{L}_0 = (1-\gamma_\pi)^2 \pi_0^2 + \frac{\kappa}{\theta}(x_0 - \phi_x)^2 & ; x_0 = \frac{(1-\gamma_\pi)(1-\beta)}{\kappa}\pi_0 \end{array} \tag{28}$$

Analogously, the static wedge of the TIWeB loss function on trend inflation $\bar{\Pi}$ is computed as $\bar{\mathcal{W}} - \mathcal{W}_{\bar{\pi}}$, where

$$\begin{array}{ll} \mathcal{W}_{\bar{\pi}} = -\frac{1}{2}\frac{\bar{\mathcal{V}}}{(1-\beta)}\mathcal{L}_{\bar{\pi}} + tip_{\bar{\pi}}^{\mathcal{W}} & ; \pi_{\bar{\pi}} = \log\left(\Pi/\bar{\Pi}\right) \\ \mathcal{L}_{\bar{\pi}} = \left[(1-\gamma_\pi)\pi_{\bar{\pi}} + \bar{\phi}_\pi\right]^2 + \bar{\mathcal{X}}\left(x_{\bar{\pi}} - \bar{\phi}_x\right)^2 & ; x_{\bar{\pi}} = \mathfrak{c}_{\bar{\pi}}\frac{(1-\gamma_\pi)(1-\beta)}{\bar{\kappa}}\pi_{\bar{\pi}} \end{array} \tag{29}$$

where

$$\mathfrak{c}_{\bar{\pi}} \equiv \frac{(1-\bar{\alpha}\vartheta\beta)(1-\beta)-\theta\bar{\kappa}(\mathfrak{c}_2-\mathfrak{c}_1)}{\mathfrak{c}_2(1-\bar{\alpha}\vartheta\beta)(1-\beta)} \quad ; \mathfrak{c}_1 \equiv 1 - \frac{(\vartheta-1)\beta\bar{\kappa}_\varpi(1-\sigma)}{\bar{\kappa}} \quad ; \mathfrak{c}_2 \equiv 1 + \frac{(\vartheta-1)\beta\bar{\kappa}_\varpi(\omega+\sigma)}{\bar{\kappa}}$$

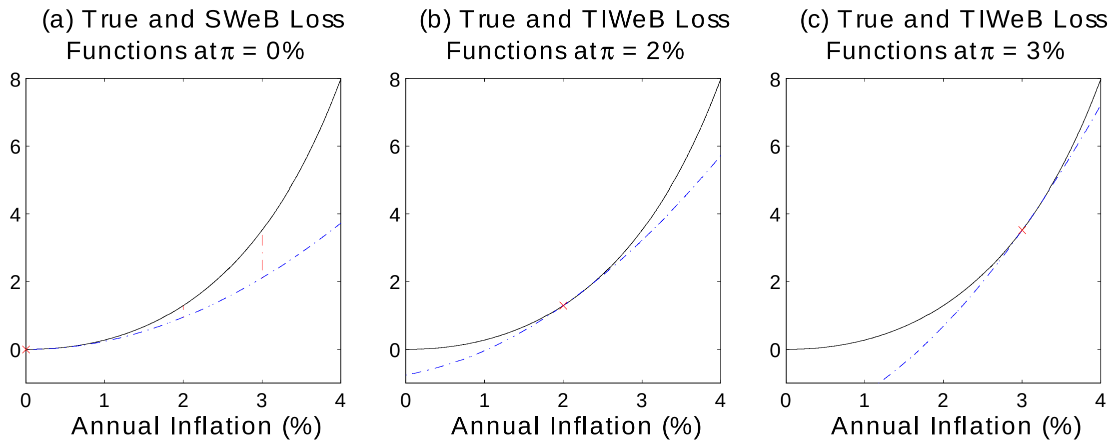

Figure 4: Loss functions

Note: True (black), SWeB and TIWeB (dashdotted blue), Static Wedge(dotted red).

Using the benchmark calibration, panel (a) from figure 4 compares the True and the SWeB loss functions, showing the static wedges as the (red) vertical dashdotted lines. For simplification, the values of the loss functions at the zero inflation rate are normalized to zero. Note that both metrics agree pretty well both in level and curvature at the vicinity of the zero trend inflation (up to about $\bar{\pi} = 1$). After that, the curvature of the true loss function increases fast, while the SWeB curvature remains constant. Therefore, static wedges increase as trend inflation rises.

The SWeB has a constant and small curvature $\partial^2 SWeBLoss/\partial\bar{\Pi}^2$ relatively to the curvature of the True loss function $\partial^2 TrueLoss/\partial\bar{\Pi}^2$. The curvature ratio (True over SWeB) jumps from 1.0 at $\bar{\pi} = 0$, to 2.6 at $\bar{\pi} = 2$ and 4.7 at $\bar{\pi} = 3$. Therefore, the stochastic wedge of the SWeB loss function is relevant. Unfortunately, there are no closed form expressions for the expected True loss function and for the stochastic wedge. Due to the model's nonlinearity, the stochastic wedge may only be obtained by simulating the model's higher order approximations. That is the reason why it is not shown in figure 4.

The main message from the figure is that the approximation about the Zero StSt becomes very quickly a bad approximation of the True Welfare Function as soon as we move slightly from the zero trend inflation. This fact parallels the results in the trend inflation literature, covered in the literature survey done by Ascari and Sbordone (2013), that shows that the log-linearized Phillips curve about the Zero StSt becomes very poor as soon as trend inflation moves slightly from zero.

Regarding the TIWeB loss function, its static wedge is zero at the specific trend inflation $\bar{\pi}$ used to approximate the True loss function. Because it also has a constant curvature, the static wedge increases in the neighborhood of $\bar{\pi}$. However, the TIWeB loss wedges are smaller than the SWeB loss ones in the vicinities of each specific trend inflation $\bar{\pi}$. Panels (b) and (c) from figure 4 depict the performances of two TIWeB loss functions, approximated at $\bar{\pi}_a = 2$ and $\bar{\pi}_b = 3$.

5 Optimal policies

Assume the central bank implements inflation targeting by keeping the unconditional mean of the inflation rate at the central target $\bar{\pi}$, or $E\pi_t = \bar{\pi}$. When log-linearizing around the inflation target, $E\hat{\pi}_t = 0$.

I derive trend inflation optimal policies rules under commitment and discretion. The former policy rules are derived for both unconditional (e.g. Damjanovic et al. (2008)) and timeless (e.g. Woodford (2003)) perspectives. In all cases, the welfare-concerned central bank minimizes the Lagrangian problem formed by the discounted sum of the TIWeB loss function, subject to the IS, to the trend inflation NKPC curves and to $E\hat{\pi}_t = 0$.

A problem arises in deriving optimal policies under discretion and from the timeless perspective because the expectation operator considered in both welfare problems is conditioned on the information set at the initial period t, while the expectation operator of the targeting objective is unconditional.[23] A possible solution to this problem is to assume that the Ergodic Theorem holds: $E\hat{\pi}_t = \lim_{T\to\infty} \frac{1}{T+1} \sum_{\tau=0}^{T} \hat{\pi}_\tau$. This limiting sum can be reasonably approximated by $(1-\beta)\sum_{\tau=0}^{\infty} \beta^\tau \hat{\pi}_\tau$, provided that the subject discount factor β is sufficiently close to unity. The inflation targeting objective is then approximated by $(1-\beta)\sum_{\tau=0}^{\infty} \beta^\tau \hat{\pi}_\tau = 0$, which implies $E_t (1-\beta)\sum_{\tau=0}^{\infty} \beta^\tau \hat{\pi}_{t+\tau} = 0$. That is what I use instead of $E\hat{\pi}_t = 0$.

Let Λ denote the Lagrange multiplier of the inflation target constraint. In equilibrium achieved under each type of optimal policy, Λ is obtained by imposing the inflation targeting constraint $E\hat{\pi}_t = 0$ into the equilibrium system described by the IS, the NKPC and the first order conditions. Since the model is linear, it is easy to verify that the inflation targeting constraint requires Λ to offset all constant terms in the equations. Therefore the following proposition describes the optimal policies under inflation targeting, using the TIWeB loss function.

Proposition 2 *When a welfare-concerned central bank targets $\bar{\pi}$ as the inflation target and follows the recommendations of the TIWeB loss function, the optimal policies under unconditional commitment, timeless perspective commitment and dis-*

[23] Posing and solving the Lagrangian problem this way would require dealing with ratios such as $f(\varkappa_{t+j}|\mathcal{I}_t)/f(\varkappa_{t+j})$, where $f(\cdot)$ is the density function of the random variable $\varkappa_{t+j}$ and $\mathcal{I}_t$ is the information set at period t. The ratios never cancel out in the first order conditions.

cretion are described by the following targeting rules:

$$
\begin{aligned}
&\textit{Unconditional:}\\
&0=\left(\hat{\pi}_t-\hat{\pi}_t^{ind}\right)+\tfrac{1}{\mathfrak{c}_1}\tfrac{\bar{\mathcal{X}}}{\bar{\kappa}}\left(\hat{x}_t-\beta\hat{x}_{t-1}\right)-\tfrac{(\mathfrak{c}_2-\mathfrak{c}_1)}{\mathfrak{c}_1}\tfrac{\bar{\mathcal{X}}}{\bar{\kappa}}\tfrac{[\mathfrak{c}_3-\beta(1-\bar{\alpha}\beta\vartheta)L]}{(\mathfrak{c}_1-\mathfrak{c}_4L)}\hat{x}_{t-1}\\
&\textit{Timeless:}\\
&0=\left(\hat{\pi}_t-\hat{\pi}_t^{ind}\right)+\tfrac{1}{\mathfrak{c}_1}\tfrac{\bar{\mathcal{X}}}{\bar{\kappa}}\left(\hat{x}_t-\hat{x}_{t-1}\right)-\tfrac{(\mathfrak{c}_2-\mathfrak{c}_1)}{\mathfrak{c}_1}\tfrac{\bar{\mathcal{X}}}{\bar{\kappa}}\tfrac{[\beta\mathfrak{c}_3-\beta(1-\bar{\alpha}\beta\vartheta)L]}{(\beta\mathfrak{c}_1-\mathfrak{c}_4L)}\hat{x}_{t-1}\\
&\textit{Discretion:}\\
&0=\left(\hat{\pi}_t-\hat{\pi}_t^{ind}\right)+\tfrac{\bar{\mathcal{X}}}{\bar{\kappa}}\hat{x}_t+\tfrac{\bar{\mathcal{X}}}{\bar{\kappa}}\tfrac{\gamma_\pi\beta}{(1-\gamma_\pi\beta L^{-1})}\hat{x}_t
\end{aligned}
\tag{30}
$$

where L is the lag operator, L^{-1} is the expected lead operator, and the composite parameters are defined as follows:

$$
\begin{aligned}
&\mathfrak{c}_1\equiv 1-(\vartheta-1)\beta\tfrac{\bar{\kappa}_{\varpi}}{\bar{\kappa}}(1-\sigma) &&;\ \mathfrak{c}_2\equiv 1+(\vartheta-1)\beta\tfrac{\bar{\kappa}_{\varpi}}{\bar{\kappa}}(\omega+\sigma)\\
&\mathfrak{c}_3\equiv\theta\bar{\kappa}\mathfrak{c}_1+(1-\bar{\alpha}\beta\vartheta) &&;\ \mathfrak{c}_4\equiv\mathfrak{c}_1-(1-\bar{\alpha}\beta\vartheta)\,\mathfrak{c}_2
\end{aligned}
\tag{31}
$$

The proof closely follows the standard steps used in the literature (e.g. Damjanovic et al. (2008) and Woodford (2003)), and hence are not shown here.

In particular, the parameters $\mathfrak{c}_1$, $\mathfrak{c}_2$ and ϑ converge to 1 as trend inflation approaches zero. Therefore, the standard targeting rules (Damjanovic's unconditional and Woodford's timeless) are retrieved in the vicinities of the zero inflation steady state. Whenever there is indexation ($\gamma_\pi \neq 0$) in price setting, the optimal rule under discretion is future-dependent, i.e. the central bank takes the (given) expectations of inflation and output gap in consideration when implementing the optimal policy. If $\gamma_\pi = \varepsilon = \nu = 0$, the trend inflation targeting rule under discretion has the same form as the one derived by Ascari and Ropele (2007). The only difference is the use of the microfounded weight $\bar{\mathcal{X}}$ in (30).

An interesting property of the trend inflation optimal policies under unconditional and timeless perspective commitment is that the targeting rules are more history-dependent than the ones derived under the Zero StSt. Multiplying the first expression by $(\mathfrak{c}_1 - \mathfrak{c}_4 L)$, or the latter by $(\beta\mathfrak{c}_1 - \mathfrak{c}_4 L)$, we realize that the rules depend on the second lag of the output gap and on the first lag of inflation even when there is no price indexation. Also important, the persistency implied by both targeting rules increases as trend inflation rises.

It is important to stress that imposing the inflation targeting constraint $E\hat{\pi}_t = 0$ prevents the optimal policy under discretion to have an equivalent of the inflation bias property held by the standard optimal policy under discretion. Imposing the constraint prevents the bias even under zero trend inflation.

5.1 Gains from commitment

As in Ascari and Ropele (2007), I assess the gains from using the commitment optimal policies against alternative monetary policies, as trend inflation rises from 1 percent to 4 percent.[24] In order simplify the evaluation, I use a slightly modified version of the TIWeB loss function:

$$\bar{\mathcal{L}}_t^v \equiv V\left(\hat{\pi}_t\right) + \bar{\mathcal{X}} V\left(\hat{x}_t\right) \tag{32}$$

where $V(\cdot)$ stands for the unconditional variance operator. Note that $\bar{\mathcal{L}}_t^v$ is not the same as $\bar{\mathcal{L}}_t$, but gives us similar information.

The first exercise is to test if the results of Damjanovic et al. (2008) extends under trend inflation, i.e. if the TIWeB optimal policy under unconditional commitment (TIu) dominates the one under timeless perspective commitment (TIt). Using the benchmark calibration, table 1 reports $V_{\mathsf{i}}\left(\hat{\pi}_t\right)$, $V_{\mathsf{i}}\left(\hat{x}_t\right)$, $\bar{\mathcal{L}}_{\mathsf{i}t}^v$ for $\mathsf{i} \in \{\mathsf{TIu}, \mathsf{TIt}\}$ and the gain $\mathcal{G}_{\mathsf{i}t}^v \equiv \left(\bar{\mathcal{L}}_{\mathsf{i}t}^v / \bar{\mathcal{L}}_{\mathsf{TIu}t}^v - 1\right)$ from using the unconditional commitment policy over optimasl policy i. All values are displayed in percentage notation.

Confirming the results of Damjanovic et al. (2008), the optimal policy under unconditional commitment slightly dominates the one under timeless perspective commitment. Moreover, the gains increase as trend inflation rises. Since β is typically very close to 1, both optimal policy rules imply almost indistinguishable dynamics and unconditional moments. In this sense, the gains are expected to be small. Due to this dominance, I use only the TIu optimal policy when assessing the gains from commitment.

Using the benchmark calibration, the TIWeB optimal policy under discretion (TId) ceases to be consistent with stability and determinacy in trend inflation equilibria with rational expectations at a very low level of trend inflation. This fact is

[24] When the trend inflation is at zero percent, the endogenous trend inflation cost push shock is zero. In this case, the model is not disturbed by the exogenous shocks.

Table 1: Gains of Commitment: TI Timeless and TI Unconditional

$\bar{\pi}$	TIt			TIu			*Gain* (%)
	$V(\hat{\pi}_t)$	$V(\hat{x}_t)$	$\bar{\mathcal{L}}^v_{\mathsf{TIt}t}$	$V(\hat{\pi}_t)$	$V(\hat{x}_t)$	$\bar{\mathcal{L}}^v_{\mathsf{TIu}t}$	$\mathcal{G}^v_{\mathsf{TIt}t}$
1	0.002	0.183	0.002	0.002	0.183	0.002	0.125
2	0.009	0.607	0.011	0.009	0.610	0.011	0.210
3	0.022	0.954	0.025	0.022	0.962	0.025	0.292
4	0.042	0.888	0.045	0.042	0.899	0.045	0.330

Note: TIWeB timeless perspective commitment (TIt), TIWeB unconditional commitment (TIu), unconditional variance of $\widehat{\mathfrak{w}}_t$ $\left(V\left(\widehat{\mathfrak{w}}_t\right)\right)$, variance-based loss from using optimal policy i $\left(\bar{\mathcal{L}}^v_{\mathsf{i}t}\right)$, gain from using optimal policy TIu over optimal policy i $(\mathcal{G}^v_{\mathsf{i}t})$. Standard calibration.

the first evidence towards the gain of commitment under trend inflation, and is basically the consequence of using non-linear functional forms for the disutility function ($\nu > 0$) and for the production function ($\varepsilon < 1$). Therefore, I use the alternative calibration in which the reciprocal of the Frisch elasticity is reduced to $\nu = 0.50$ (consistent with macro evidence) in the exercises below.

Table 2 reports the unconditional variances, the losses and the gain from using the TIu optimal policy under commitment over the TId optimal policy under discretion. As before, the gains increase as trend inflation rises. However, they are of two orders of magnitude greater than before.

Table 2: Gains of Commitment: TI Commitment and TI Discretion

$\bar{\pi}$	TId			TIu			*Gain* (%)
	$V(\hat{\pi}_t)$	$V(\hat{x}_t)$	$\bar{\mathcal{L}}^v_{\mathsf{TId}t}$	$V(\hat{\pi}_t)$	$V(\hat{x}_t)$	$\bar{\mathcal{L}}^v_{\mathsf{TIu}t}$	$\mathcal{G}^v_{\mathsf{TId}t}$
1	0.002	0.127	0.003	0.001	0.127	0.002	49.410
2	0.010	0.482	0.012	0.005	0.487	0.008	52.465
3	0.023	1.004	0.028	0.013	1.030	0.018	55.456
4	0.043	1.612	0.050	0.024	1.679	0.031	58.319

Note: TIWeB discretion (TId), TIWeB unconditional commitment (TIu), unconditional variance of $\widehat{\mathfrak{w}}_t$ $\left(V\left(\widehat{\mathfrak{w}}_t\right)\right)$, variance-based loss from using optimal policy i $\left(\bar{\mathcal{L}}^v_{\mathsf{i}t}\right)$, gain from using optimal policy TIu over optimal policy i $(\mathcal{G}^v_{\mathsf{i}t})$. Alternative calibration.

Finally, I compare the use of the trend inflation optimal policy under unconditional commitment against both standard optimal policies (unconditional commitment (Su) and discretion (Sd)), i.e. the ones that the monetary authority would obtain if she considered instead the SWeB loss function as the appropriate metrics for welfare optimization in a trend inflation equilibrium characterized by the equations described in Section 3.1. Those optimal policies are retrieved by substituting $\mathcal{X}$ for $\bar{\mathcal{X}}$ in system (30).

Table 3 reports the unconditional variances and the losses from using the Su and Sd optimal policies, under the alternative calibration, and computes the gains from using the TIu optimal policy. For computing the gains, I use the same statistics reported for the TIu optimal policy in table 2. Again, the gains increase as trend inflation rises. As expected, the TIWeB optimal policy under commitment dominates the SWeB optimal policies under commitment and discretion.

Table 3: Gains of Commitment: TI Commitmen , S Commitment and S Discretion

$\bar{\pi}$	Sd			Su			*Gains* (%)	
	$V(\hat{\pi}_t)$	$V(\hat{x}_t)$	$\bar{\mathcal{L}}^v_{\mathsf{Sd}t}$	$V(\hat{\pi}_t)$	$V(\hat{x}_t)$	$\bar{\mathcal{L}}^v_{\mathsf{Su}t}$	$\mathcal{G}^v_{\mathsf{Sd}t}$	$\mathcal{G}^v_{\mathsf{Su}t}$
1	0.002	0.115	0.003	0.001	0.117	0.002	50.408	0.087
2	0.010	0.392	0.012	0.006	0.415	0.008	54.545	0.405
3	0.025	0.727	0.028	0.014	0.801	0.018	58.587	0.931
4	0.047	1.027	0.051	0.027	1.176	0.032	62.343	1.620

Note: SWeB discretion (Sd), SWeB unconditional commitment (Su), TIWeB unconditional commitment (TIu), unconditional variance of $\widehat{\mathfrak{w}}_t$ $\left(V\left(\widehat{\mathfrak{w}}_t\right)\right)$, variance-based loss from using optimal policy i $\left(\bar{\mathcal{L}}^v_{\mathsf{i}t}\right)$, gain from using optimal policy TIu over optimal policy i $(\mathcal{G}^v_{\mathsf{i}t})$. Alternative calibration. The statistics for optimal policy TIu are the same as in table 2.

The main message from table 3 is that policy recommendations under non-zero trend inflation must be based on the TIWeB loss function.

6 Simulations

In this section, I study the dynamics implied by trend inflation optimal policies. Paralelling the analysis done in Section 5.1, this section also compares the results implied by trend inflation optimal policies with the ones that the monetary authority would obtain if she considered instead the SWeB loss function as the appropriate metrics for welfare optimization in a trend inflation equilibrium characterized by the equations described in Section 3.1. Finally, the analysis compares the results with the ones obtained when the monetary authority is assigned a inflation target $\bar{\pi} \geq 0$ to pursuit and implements monetary policy according to the generalized Taylor rule:

$$\hat{\imath}_t = 0.86\hat{\imath}_{t-1} + (1 - 0.86)\left[2.20 E_t \hat{\pi}_{t+1} + 1.56\left(\hat{y}_t - \hat{y}_{t-1}\right)\right] \tag{33}$$

where the parameters are set at the central estimates of the Mixed Taylor rule, the preferred specification estimated by Coibion and Gorodnichenko (2011).[25] Based on the results in the last section, I do not use the optimal policy under timeless commitment in the simulations.

The analysis is done in terms of volatility schedules and impulse responses to shocks as trend inflation rises, paralleling the exercises done by Ascari and Ropele (2007). I assume that the structural shocks are independent, serially uncorrelated and normally distributed, i.e. $\hat{\epsilon}_t \overset{iid}{\sim} N\left(0, \sigma_\epsilon^2\right)$ and $\widehat{\mathcal{A}}_t \overset{iid}{\sim} N\left(0, \sigma_\mathcal{A}^2\right)$. I use the benchmark calibration when assessing the results for optimal policies under commitment, and the alternative calibration ($\nu = 0.50$) when assessing the results for optimal policies under discretion. Computing volatility schedules are done in terms of relative standard deviations, i.e. the unconditional standard deviations of the endogenous variables divided by the unconditional standard deviation of the aggregate shock $\hat{\xi}_t$ (see eq. (17)). Using relative measures avoids scale problems due to the amplitude of the shocks and to the calibrated parameter values.[26] Relative standard deviations are also invariant with respect to which shock is hitting the economy, i.e. either the technology shock alone, the preference shock alone or both.

It is important to note that the results obtained using SWeB optimal policies parallel those obtained by Ascari and Ropele (2007). The authors make exercises similar to mine, with rising levels of trend inflation, except for using a loss function with constant coefficient for the output gap variability, which mimics the SWeB loss function. It means that the loss function does not internalize the distortions created by rising trend inflation, and possibly places more weight on fighting output gap

[25] The authors find that reacting to the observed output growth rather than to the level of the output gap has two major advantages: (*i*) it has more stabilizing properties when the trend inflation is not zero; and (*ii*) it is empirically more relevant. To test the empirical dominance, they estimate the generalized Taylor rule using Greenbook forecasts prepared for each meeting of the Federal Open Market Committee (FOMC) as real-time measures of expected inflation, output growth, and the output gap. This approach is advantageous because it avoids any extra assumption on how the FED's expectations are formed.

[26] The standard deviation of $\hat{\xi}_t$ actually slightly decreases as trend inflation rises. In order to check whether that variation was affecting my results, I tested different normalization measures that were close enough to $sd\left(\hat{\xi}_t\right)$ and were invariant with respect to trend inflation. The different normalization measures were $sd\left(\hat{\xi}_t^{\bar{\pi}^*}\right)$, where $\hat{\xi}_t^{\bar{\pi}^*} \equiv \left(\bar{\kappa}_\varpi^{\bar{\pi}^*}/\bar{\kappa}_\varpi\right)\hat{\xi}_t$, $\bar{\kappa}_\varpi^{\bar{\pi}^*} \equiv \left(1 - \bar{\alpha}_{\bar{\pi}^*}\right)/\left(1 + \theta\omega\right)$, and $\bar{\alpha}_{\bar{\pi}^*}$ is the effective degree of price rigidity when the trend inflation is fixed at invariant levels $\bar{\pi}^* \in \{0, 1, 2, 3, 4, 5\}$. As it turns out, the shape and levels of the volatility schedules are robust to the different normalization measures.

volatility than it should.

Panels (a) and (b) of figure 5 depict the relative standard deviations of the annualized inflation rate $4\hat{\pi}_t$, the output gap $\hat{x}_t$ and the aggregate output $\hat{y}_t$, for optimal policies under unconditional commitment based on the TIWeB (black) and SWeB (blue) loss functions, as the annual level of trend inflation rises. Panels (c) and (d) of figure 5 depict the relative standard deviations of the output gap $\hat{x}_t$, output $\hat{y}_t$ and annualized inflation rate $4\hat{\pi}_t$, when the monetary authority follows the Taylor rule.

There are important lessons from figure 5. While the volatility of the inflation rate monotonically increases as trend inflation rises, the volatility of the output gap reverts its rising trend and starts to decrease if the level of trend inflation is sufficiently large. The reversion is due to the fact that the coefficients of $(\hat{x}_t - \hat{x}_{t-1})$ and $\hat{x}_{t-1}$ in the trend inflation targeting rules (eq. (30)) are decreasing functions of the level of trend inflation. This behavior induces the optimal volatilities of the output gap to decrease as trend inflation rises past a critical level (about 3.5 percent under the benchmark calibration).[27]

Optimal policies based on the SWeB loss function have the same reversion property, but deliver smaller output gap volatility and slightly higher inflation volatility. The reson for that is that the fixed Zero StSt coefficient $\mathcal{X}$ is smaller than the Trend StSt coefficient $\bar{\mathcal{X}}$ and does not react to distortions created by the level of trend inflation as $\bar{\mathcal{X}}$ does.

The main lesson is confirming that the central bank faces a policy trade off as trend inflation rises: it is only possible to completely offset the volatilities of both inflation and output gap when the level of trend inflation is zero. In a nutshell, the divine coincidence ceases to hold as soon as the level of trend inflation departs from zero, as shown in Section 3. Even though the volatilities of both the inflation rate and the output gap are not completely offset at non-zero levels of trend inflation, the monetary authority is still able to reduce significantly those volatilities when compared to volatilities implied by following a Taylor rule.

Panels (a) and (b) of figure 6 depict the relative standard deviations of the

[27] The coefficients increase with trend inflation because their denominators have the composite parameter $\bar{\kappa}$, i.e. the coefficient of the output gap in the Phillips curve, which decreases as trend inflation rises. If the optimal policies had a fixed coefficient κ instead of $\bar{\kappa}$, the output gap volatility would monotonically increase as trend inflation rises.

annualized inflation rate $4\hat{\pi}_t$, the output gap $\hat{x}_t$ and the aggregate output $\hat{y}_t$, for optimal policies under unconditional commitment based on the TIWeB (black) and SWeB (blue) loss functions, as the annual level of trend inflation rises. Panels (c) and (d) of figure 6 depict the relative standard deviations for the output gap $\hat{x}_t$, output $\hat{y}_t$ and annualized inflation rate $4\hat{\pi}_t$, when the monetary authority follows an optimal policy under discretion. Now, all relative volatilities are obtained using the alternative calibration ($\nu = 0.50$). As expected, following the optimal policy under commitment allows for smaller relative volatilities. Again, the divine coincidence holds only when the level of trend inflation is zero.

Figure 7 shows the relative responses of the annualized inflation rate $4\hat{\pi}_t$ and the output gap $\hat{x}_t$ to the preference shock $\hat{\epsilon}_t$ as trend inflation rises, using the benchmark calibration, for the cases in which the monetary policy follows the Taylor rule or the optimal policies under unconditional commitment based on the TIWeB and SWeB loss functions. By relative responses, I mean that they are divided by the unconditional standard deviation of the aggregate shock $\hat{\xi}_t$. Based on the definition of the aggregate shock $\hat{\xi}_t$ in (17), responses to the technology shock $\widehat{\mathcal{A}}_t$ look identical to responses to the preference shock $\hat{\epsilon}_t$ whenever $\sigma < 1$, or perfectly symmetrical when $\sigma > 1$.[28] Due to this redundancy, I am not showing the responses to the technology shock.

Panels (a1) and (a2) show the responses when the monetary policy follows the optimal policy under unconditional commitment based on the TIWeB loss function. Note that a positive preference shock has a negative impact on the endogenous trend inflation cost-push shock $\widehat{\mathrm{u}}_t$. That is this way because the aggregate shock hits the trend inflation cost-push shock by means of its expected value $E_t\hat{\xi}_{t+1}$. It implies that the inflation rate is hit by a negative shock and falls on impact, as shown in panel (a1). For lower levels of trend inflation, the optimal policy under unconditional commitment brings back the aggregate price level to the vicinities where the price trend was before the shock. For that reason, inflation has to remain positive for a few quarters. Under higher levels of trend inflation, the third term of the targeting rule (eq. (30)) becomes relevant and prevents the optimal policy to bring back the aggregate price level to the vicinities where the price trend was

[28]The reason for this dependency on the value of σ is described in Section 3.1.

before the shock.[29] In any case, the higher is the level of the trend inflation, the more amplified is the effect of the shock. In general, the output gap has to increase to push the inflation rate to positive values, as shown in panel (a2). Recall, from figure 5, that the volatility of the output gap starts to fall when the level of the trend inflation is slightly higher than 3.5 percent. This fact is reflected in the response of the output gap when the level of inflation trend hits about four percent: it is about the same as the response at the three-percent level of trend inflation.

Panels (b1) and (b2) show the responses when the monetary policy follows the optimal policy under unconditional commitment based on the SWeB loss function. The qualitative dynamics are the same as the ones shown in panels (a1) and (a2). The amplitude of the responses, however, are different. While the amplitude of the responses of the inflation rate are slightly greater than ones obtained with the TIWeB optimal policy, the amplitude of the responses of the output gap are much smaller. This is due to the fact that the SWeB loss function has a constant weight on output volatility, while the weights decrease as trend inflation rises in the TIWeB loss function. That is, the TIWeB loss function is more hawkish against higher levels of trend inflation.

As expected, there is no response when using optimal policies at the zero level of trend inflation.

Panels (c1) and (c2) show the responses when the monetary policy follows the Taylor rule. The main results are that there are non-zero responses even when the level of trend inflation is zero, and the amplitude of the responses are of one order of magnitude greater than the ones obtained using the optimal policies. In particular, inflation responses are always positive because this particular form of the Taylor rule responds to $E_t\hat{\pi}_{t+1}$ instead of $\hat{\pi}_t$. Since the main point of this exercise is to ilustrate how the amplitudes increase when not using trend inflation optimal policies, I am not comparing the responses obtained from Taylor rules with different forms.

[29] Note that this effect would also happen if the monetary authority followed the trend inflation optimal policy under timeless commitment.

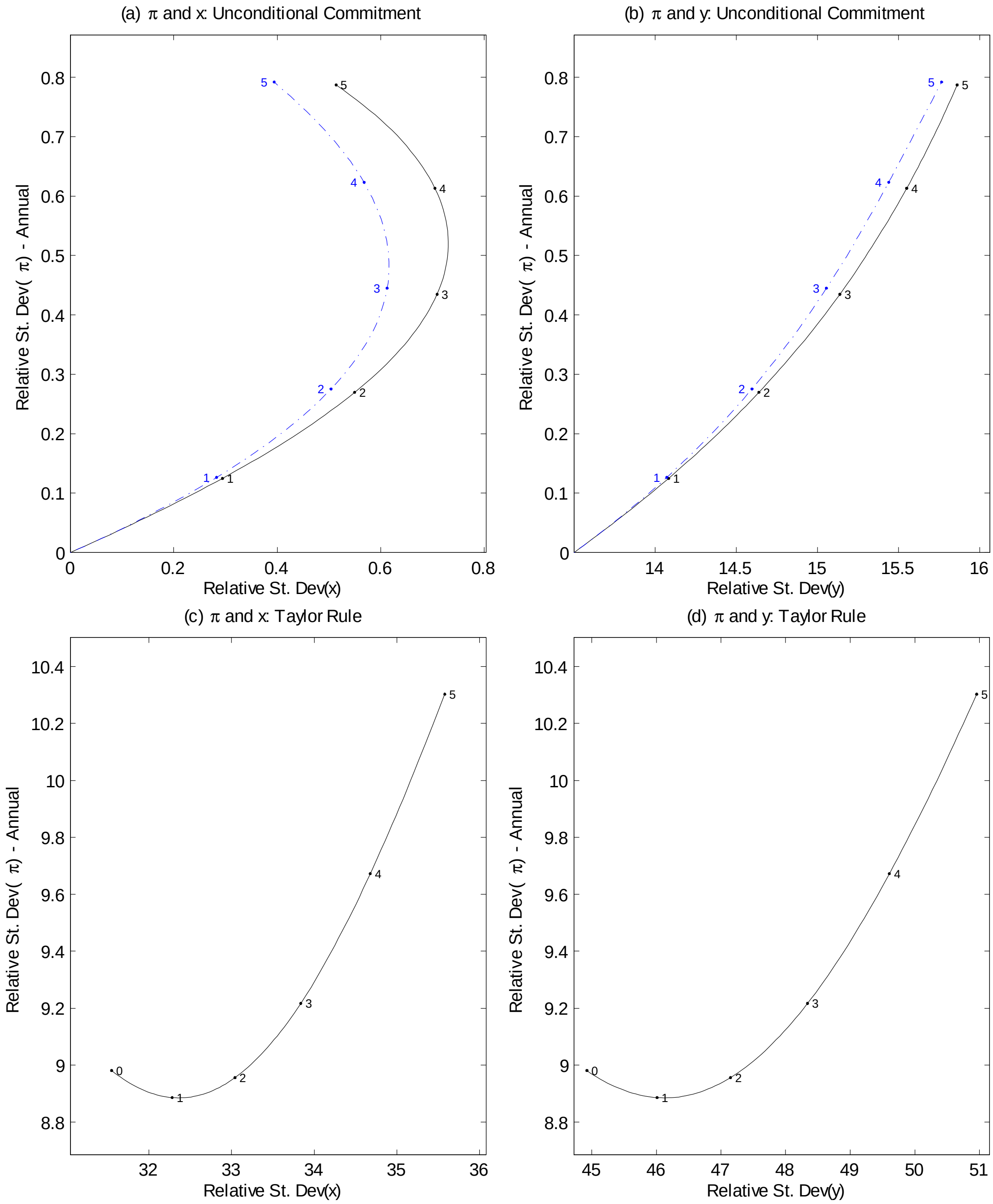

Figure 5: Relative volatilities: unconditional commitment and Taylor rule

Note: *Optimal policy based on the TIWeB loss function (black), optimal policy based on the SWeB loss function (dashdotted blue), volatility pairs at specific levels of trend inflation (dots). Panels (a) and (b): optimal policy under commitment. Panels (c) and (d): Taylor rule.*
Results were obtained using the benchmark calibration.

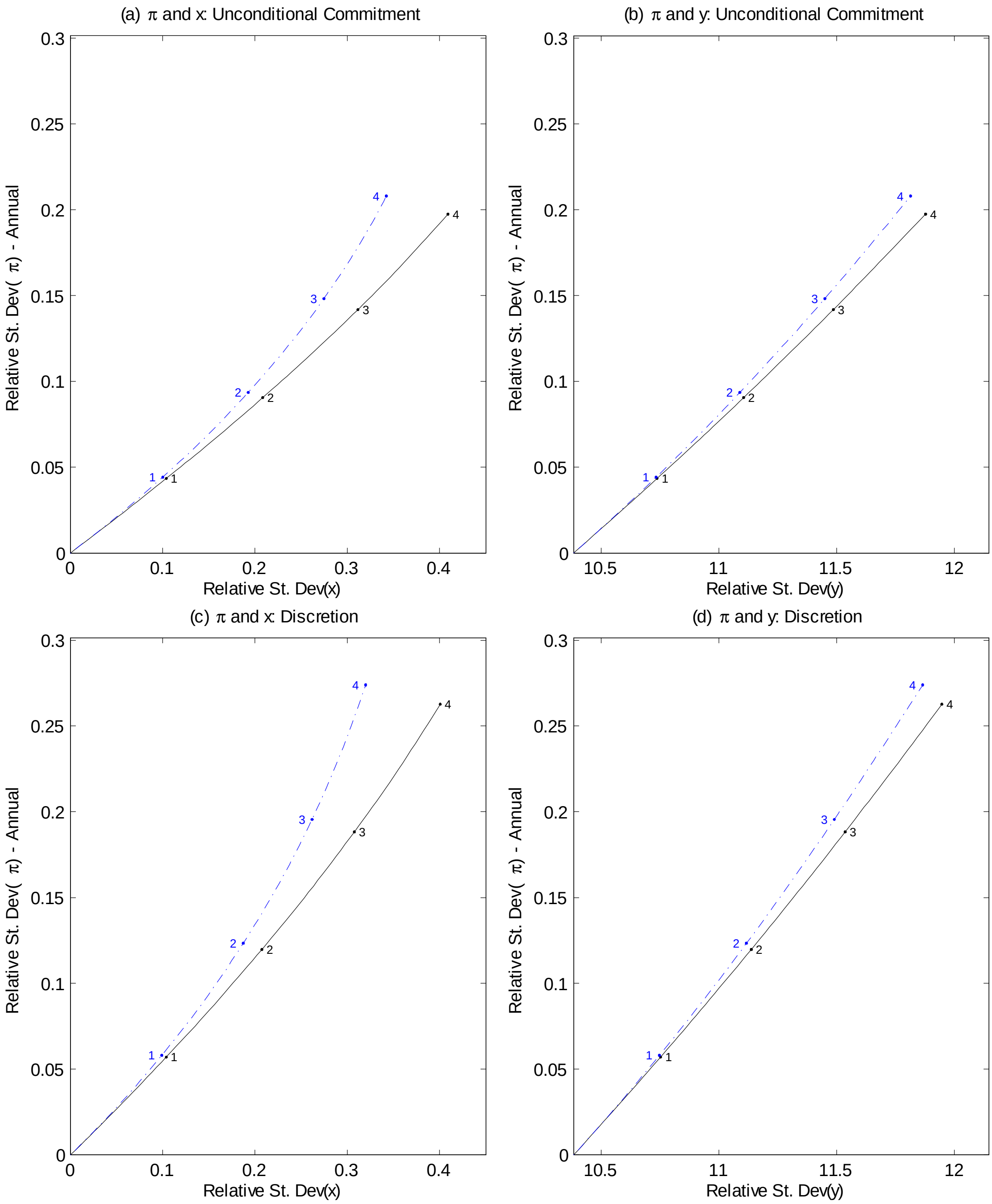

Figure 6: Relative volatilities: unconditional commitment and discretion

Note: *Optimal policy based on the TIWeB loss function (black), optimal policy based on the SWeB loss function (dashdotted blue), volatility pairs at specific levels of trend inflation (dots). Panels (a) and (b): optimal policy under commitment. Panels (c) and (d): optimal policy under discretion. Results were obtained using the alternative calibration.*

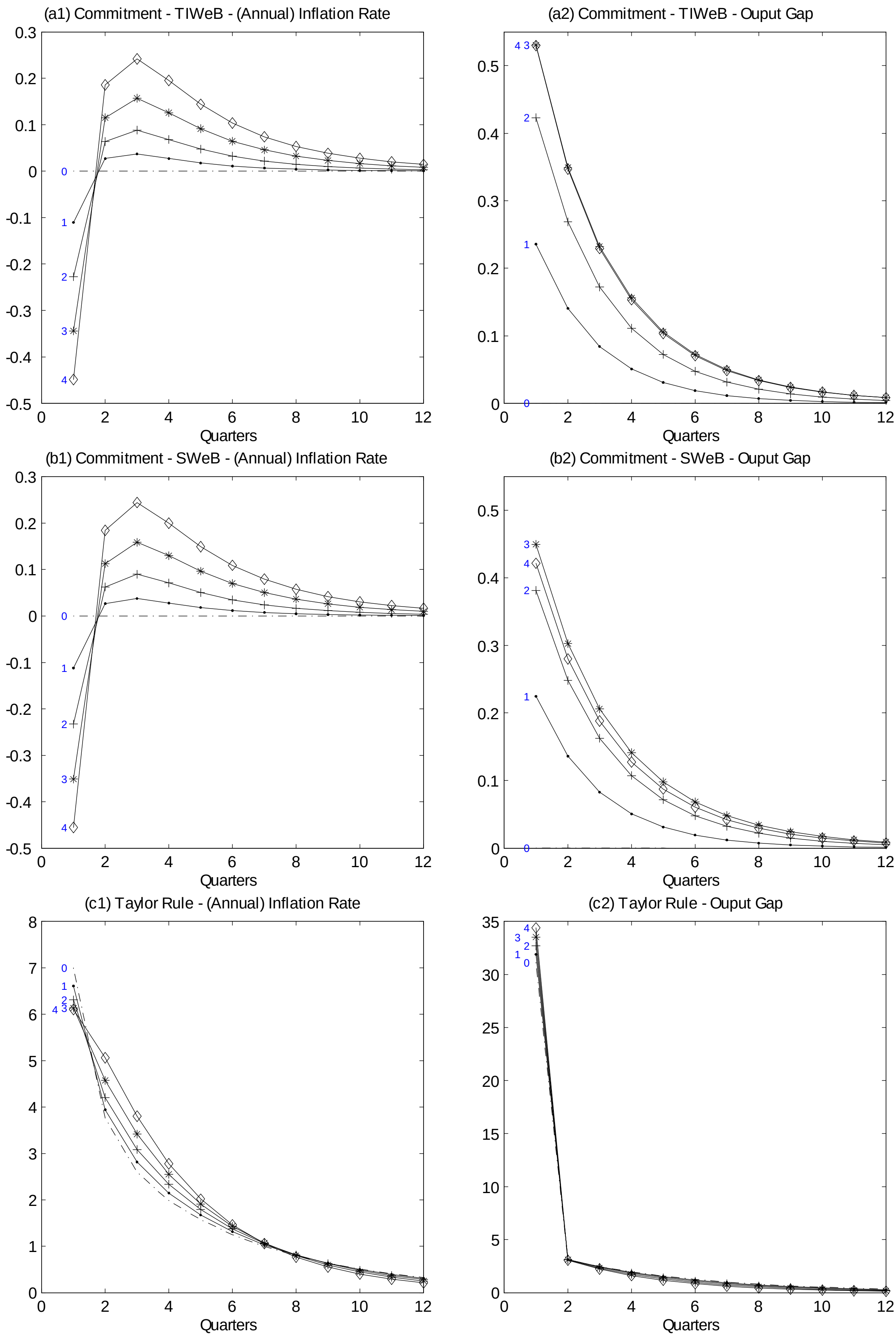

Figure 7: Relative responses to preference shock: Commitment TIWeB, Commitment SWeB, and Taylor Rule

Note: $\bar{\pi} = 0$ *(dashdotted),* $\bar{\pi} = 1$ *(dots),* $\bar{\pi} = 2$ *(crosses),* $\bar{\pi} = 3$ *(stars),* $\bar{\pi} = 4$ *(diamonds). Responses obtained using the benchmark calibration.*

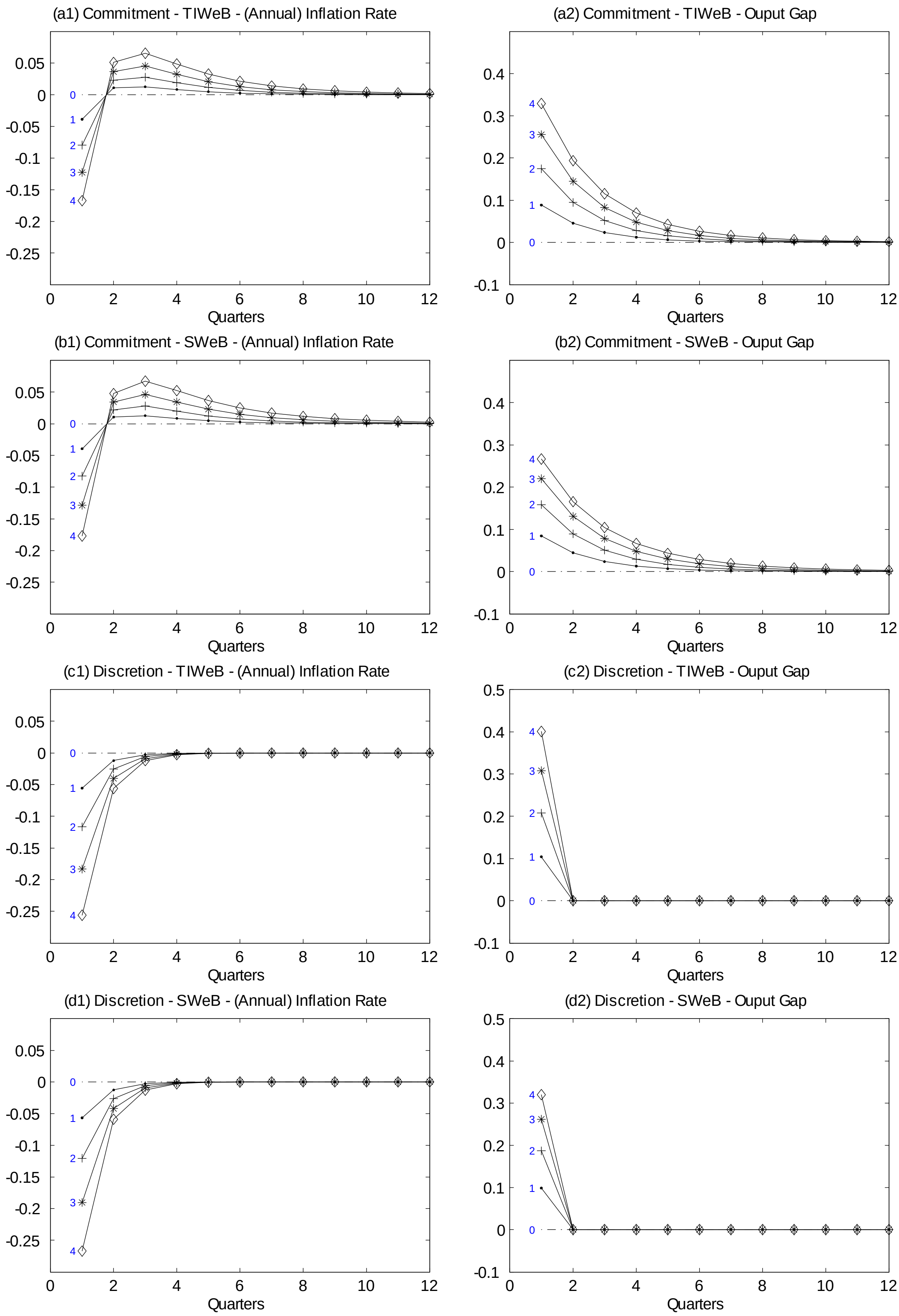

Figure 8: Relative responses to preference shock: Commitment and discretion TI-WeB, Commitment and discretion SWeB

Note: $\bar{\pi} = 0$ *(dashdotted)*, $\bar{\pi} = 1$ *(dots)*, $\bar{\pi} = 2$ *(crosses)*, $\bar{\pi} = 3$ *(stars)*, $\bar{\pi} = 4$ *(diamonds). Responses obtained using the alternative calibration.*

Figure 8 shows the relative responses of the annualized inflation rate $4\hat{\pi}_t$ and the output gap $\hat{x}_t$ to the utility shock $\hat{\epsilon}_t$ as the trend inflation rises, using the alternative calibration ($\nu = 0.50$), for the cases in which the monetary policy follows the optimal policies under unconditional commitment and discretion, based on the TIWeB and SWeB loss functions. Qualitatively, the responses using optimal policies under unconditional commitment are very similar to the ones in figure 7, and amplitudes increase as trend inflation rises. Therefore, the same comments apply. The only difference is that the amplitudes are smaller under the alternative calibration. This is due to the fact that we use a smaller value for the reciprocal of the Frisch elasticity. Using the optimal policies under discretion, the amplitudes are larger than the ones obtained using optimal policies under unconditional commitment. Furthermore, the policies do not bring the aggregate prices near where they were before the shocks. For that reason, the responses of the output gap are short lived.

7 Conclusion

The (non-linear) standard New Keynesian model is usually assumed to be consistent with the divine coincidence. When staggered price adjustment is the only nominal friction, the monetary authority does not face a trade off between stabilizing the inflation rate and the output gap in the face of preference and technology shocks. In this paper, I have shown that the divine coincidence holds only in the particular case in which the monetary authority stabilizes the inflation rate at exactly zero. A policy trade-off emerges whenever trend inflation is non-zero.

When log-linearizing the model about the equilibrium with trend inflation, an endogenous trend-inflation cost-push shock appears in the Phillips Curve. This shock, which summarizes the effects of the exogenous preference and technology shocks interacting with trend inflation, would be absent if average inflation equalled zero. When trend inflation is not zero, the presence of a shock in the Phillips Curve generates a policy trade off for the monetary authority, and the variance of this cost-push shock increases as trend inflation rises.

Deriving a second order approximation to the welfare function under trend inflation, I show that a welfare-concerned monetary authority must be more *hawkish* by putting a smaller weight on (log-deviation) output gap volatility if, as is the case for all inflation targeting central banks, average inflation is not equal to zero. Optimal policies (unconditional and timeless) for non-zero trend inflation become more inertial and history-dependent as trend inflation rises.

An analysis on the gains of commitment confirms that the trend-inflation optimal policy under unconditional commitment dominates all other policies under trend inflation: timeless perspective and discretion. Moreover, I find that following the optimal policies suggested by the standard welfare-based loss function that assumes a zero trend inflation rate can induce substantial welfare costs if, in fact, trend inflation deviates from zero. As a consequence, the common practice of approximating welfare around a zero-inflation steady-state may not be the best metric for policy evaluation; it tends to underestimate the welfare loss when average inflation actually exceeds zero. Empirical DSGE models in the new Keynesian tradition assume full indexation of price setting decisions to eliminate any effects of trend inflation. Given that such indexation is not observed in the micro evidence for many countries implies that the design of optimal stabilization policy is not independent of an economy's average rate of inflation.

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A Steady state levels and composite parameters

Tables 4 and 5 define the structural and composite parameters. Table 6 describes the steady state levels under trend inflation.

Table 4: Structural parameters

$\sigma \equiv$ reciprocal of intertemp elast substit	$\gamma_\pi \equiv$ coeff lag inf on index rule
$\nu \equiv$ reciprocal of the Frisch elasticity	$\varepsilon \equiv$ labor elasticity prod function
$\chi \equiv$ scale parameter on labor disutility	$\alpha \equiv$ Calvo degree of price rigidity
$\theta \equiv$ elasticity of substit between goods	$\bar{\pi} \equiv$ level of trend inflation

Table 5: Composite parameters

$\mu \equiv \frac{\theta}{\theta-1}$	$\bar{\kappa} \equiv \frac{(1-\bar{\alpha})(1-\bar{\alpha}\beta\vartheta)}{\bar{\alpha}} \frac{(\omega+\sigma)}{(1+\theta\omega)}$	$\bar{\mathcal{X}} \equiv \frac{(1-\bar{\alpha})}{(1-\bar{\alpha}\vartheta)} \frac{\bar{\kappa}}{\theta}$
$\omega \equiv \frac{1+\nu}{\varepsilon} - 1$	$\bar{\kappa}_\varpi \equiv \frac{(1-\bar{\alpha})}{(1+\theta\omega)}$	$\bar{\mathcal{V}} \equiv \frac{(\omega+\sigma)}{\bar{\mathcal{X}}} \bar{Y}^{1-\sigma}$
$\delta \equiv \frac{1}{1-\gamma_\pi}$	$\bar{\Phi}_y \equiv 1 - \frac{(1-\bar{\alpha}\beta\vartheta)(1-\bar{\alpha})}{\mu(1-\bar{\alpha}\beta)(1-\bar{\alpha}\vartheta)}$	$\mathfrak{c}_1 \equiv 1 - (\vartheta - 1)\beta \frac{\bar{\kappa}_\varpi}{\bar{\kappa}} (1-\sigma)$
$\bar{\alpha} \equiv \alpha \left(\bar{\Pi}\right)^{(\theta-1)(1-\gamma_\pi)}$	$\bar{\Phi}_\vartheta \equiv (\vartheta - 1)$	$\mathfrak{c}_2 \equiv 1 + (\vartheta - 1)\beta \frac{\bar{\kappa}_\varpi}{\bar{\kappa}} (\omega+\sigma)$
$\vartheta \equiv \left(\bar{\Pi}\right)^{(1+\theta\omega)(1-\gamma_\pi)}$	$\bar{\phi}_x \equiv \frac{1}{(\omega+\sigma)} \bar{\Phi}_y$	$\mathfrak{c}_3 \equiv \theta\bar{\kappa}\mathfrak{c}_1 + (1-\bar{\alpha}\beta\vartheta)$
$\max(\bar{\alpha}, \bar{\alpha}\vartheta) < 1$	$\bar{\phi}_\pi \equiv \frac{(1-\bar{\alpha})}{(1-\bar{\alpha}\vartheta)(1+\theta\omega)} \bar{\Phi}_\vartheta$	$\mathfrak{c}_4 \equiv \mathfrak{c}_1 - (1-\bar{\alpha}\beta\vartheta)\mathfrak{c}_2$

Table 6: Steady state levels

$\bar{I} = \beta^{-1}\left(\bar{\Pi}\right) = \frac{1}{\bar{q}}$	; $\bar{\Pi}^{ind} = \bar{\Pi}^{(\gamma_\pi+\gamma)}$	; $\bar{Y}^n = \left(\frac{\varepsilon}{\chi\mu}\right)^{\frac{1}{(\omega+\sigma)}}$	; $\bar{Y}^{ef} = \left(\frac{\varepsilon}{\chi}\right)^{\frac{1}{(\omega+\sigma)}}$
$\bar{\wp}^* = \left(\frac{1-\alpha}{1-\bar{\alpha}}\right)^{\frac{1}{\theta-1}}$	; $\frac{\bar{N}}{\bar{D}} = \left(\bar{\wp}^*\right)^{1+\theta\omega}$	; $\bar{X}^{\omega+\sigma} = \frac{1-\bar{\alpha}\beta\vartheta}{1-\bar{\alpha}\beta} \frac{\bar{N}}{\bar{D}}$	; $\bar{Y} = \bar{X}\bar{Y}^n$
$\bar{\mathcal{W}} = \frac{\bar{u}-\bar{\upsilon}}{1-\beta}$	; $\bar{u} = \frac{\bar{Y}^{1-\sigma}}{1-\sigma}$	; $\bar{\upsilon} = \frac{\chi\bar{Y}^{(1+\omega)}\bar{\mathcal{P}}^{-\theta(1+\omega)}}{1+\nu}$	; $\bar{\mathcal{P}} = \left(\frac{1-\bar{\alpha}\vartheta}{1-\alpha}\right)^{\frac{1}{\theta(1+\omega)}} \bar{\wp}^*$

B Proof of Proposition 1

PROPOSITION 1: *The true welfare function is (second-order) approximated as*

$$\mathcal{W}_t = -\frac{1}{2}\bar{\mathcal{V}} E_t \sum_{\tau=0}^{\infty} \beta^\tau \mathcal{L}_{t+\tau} + tip_t^{\mathcal{W}}$$

where

$$\mathcal{L}_t \equiv \left(\hat{\pi}_t - \hat{\pi}_t^{ind} + \bar{\phi}_\pi\right)^2 + \bar{\mathcal{X}}\left(\hat{x}_t - \bar{\phi}_x\right)^2$$

is the trend inflation welfare-based (TIWeB) loss function, $tip_t^{\mathcal{W}}$ stands for terms independent of policy at period t, $\bar{\phi}_\pi$ and $\bar{\phi}_x$ are constants that depend on the inefficiency degrees $\bar{\Phi}_\vartheta$ and $\bar{\Phi}_y$, and $\bar{\mathcal{V}}$ corrects for the aggregate reduction in the welfare when the trend inflation increases. Those coefficients are defined as follows:

$$\bar{\phi}_\pi \equiv \tfrac{(1-\bar{\alpha})}{(1-\bar{\alpha}\vartheta)(1+\theta\omega)}\bar{\Phi}_\vartheta \quad ; \bar{\phi}_x \equiv \tfrac{1}{(\omega+\sigma)}\bar{\Phi}_y \quad ; \bar{\mathcal{V}} \equiv \bar{Y}^{1-\sigma}\tfrac{(\omega+\sigma)}{\bar{\mathcal{X}}} \quad ; \bar{\mathcal{X}} \equiv \tfrac{(1-\bar{\alpha})}{(1-\bar{\alpha}\vartheta)}\tfrac{\bar{\kappa}}{\theta}$$

Proof. A) Second order approximation of $\hat{\mathcal{P}}_t$

The strategy is to make second-order approximations of the system

$$\begin{aligned} 1 &= (1-\alpha)\left(\wp_t^*\right)^{-(\theta-1)} + \alpha\left(\Pi_t/\Pi_t^{ind}\right)^{(\theta-1)} \\ \mathcal{P}_t^{-\theta(1+\omega)} &= (1-\alpha)\left(\wp_t^*\right)^{-\theta(1+\omega)} + \alpha\left(\Pi_t/\Pi_t^{ind}\right)^{\theta(1+\omega)}\mathcal{P}_{t-1}^{-\theta(1+\omega)} \end{aligned}$$

and solve for $\hat{\mathcal{P}}_t$ and $\hat{\wp}_t^*$, where $\wp_t^* \equiv p_t^*/P_t$ is the optimal resetting relative price, and $\mathcal{P}_t^{-\theta(1+\omega)} \equiv \left[\int_0^1 \left(p_t(z)/P_t\right)^{-\theta(1+\omega)} dz\right]^{-1/\theta(1+\omega)}$ is a measure of aggregate relative prices.

The (log) second-order approximation of the first equation is

$$\hat{\wp}_t^* \approx \frac{1}{2}(\theta-1)\left(\hat{\wp}_t^*\right)^2 + \frac{\bar{\alpha}}{(1-\bar{\alpha})}\left[\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) + \frac{1}{2}(\theta-1)\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right)^2\right]$$

which implies that the a first-order approximation of F_t is

$$\hat{\wp}_t^* \approx \frac{\bar{\alpha}}{(1-\bar{\alpha})}\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right)$$

I use this result to eliminate $\left(\hat{\wp}_t^*\right)^2$ from the previous second-order approximation and obtain

$$\hat{\wp}_t^* \approx \frac{\bar{\alpha}}{(1-\bar{\alpha})}\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) + \frac{1}{2}(\theta-1)\frac{\bar{\alpha}}{(1-\bar{\alpha})^2}\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right)^2$$

The (log) second-order approximation of the second equation is

$$\begin{aligned} \hat{\mathcal{P}}_t \approx\ & \frac{1}{2}\theta(1+\omega)\hat{\mathcal{P}}_t^2 + (1-\bar{\alpha}\vartheta)\hat{\wp}_t^* - \bar{\alpha}\vartheta\left[\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) - \hat{\mathcal{P}}_{t-1}\right] \\ & -\frac{1}{2}\theta(1+\omega)(1-\bar{\alpha}\vartheta)\left(\hat{\wp}_t^*\right)^2 - \frac{1}{2}\theta(1+\omega)\bar{\alpha}\vartheta\left[\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) - \hat{\mathcal{P}}_{t-1}\right]^2 \end{aligned}$$

and then the a first-order approximation of $\mathcal{P}_t$ is

$$\hat{\mathcal{P}}_t \approx (1-\bar{\alpha}\vartheta)\,\hat{\wp}_t^* - \bar{\alpha}\vartheta\left[\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) - \hat{\mathcal{P}}_{t-1}\right]$$

Replacing $\bar{\Phi}_\vartheta$ for $(\vartheta - 1)$ and using the last results to eliminate $\hat{\wp}_t^*$, $(\hat{\wp}_t^*)^2$ and $\hat{\mathcal{P}}_t^2$ from the previous second-order approximation, I obtain

$$\begin{aligned}\hat{\mathcal{P}}_t \;\approx\;& \bar{\alpha}\vartheta\hat{\mathcal{P}}_{t-1} - \frac{\bar{\alpha}\bar{\Phi}_\vartheta}{(1-\bar{\alpha})}\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) \\ & -\frac{1}{2}\frac{\bar{\alpha}\,(1-\bar{\alpha}\vartheta)\left[(1+\omega\theta)+\theta\,(1+\omega)\,\bar{\Phi}_\vartheta\right]}{(1-\bar{\alpha})^2}\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right)^2 \\ & +\theta\,(1+\omega)\,\bar{\alpha}\vartheta\,(1-\bar{\alpha}\vartheta)\left[\frac{1}{(1-\bar{\alpha})}\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right)\hat{\mathcal{P}}_{t-1} - \frac{1}{2}\hat{\mathcal{P}}_{t-1}^2\right]\end{aligned}$$

If $\bar{\Phi}_\vartheta$ is assumed to be a first order term (the same order of the hatted variables), then $\bar{\Phi}_\vartheta\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right)$ is a second order term and $\bar{\Phi}_\vartheta\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right)^2$ is a third order term which may be ignored in the approximation. This assumption implies that $\hat{\mathcal{P}}_t$ has no first order dynamics, other than $\bar{\alpha}\vartheta\hat{\mathcal{P}}_{t-1}$. Hence, $\hat{\mathcal{P}}_{t-1}^2$ and $\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right)\hat{\mathcal{P}}_{t-1}$ are third order variables whenever $\bar{\Phi}_\vartheta$ is a first order term. In this case the second order approximation is simplified to:

$$\hat{\mathcal{P}}_t \approx \bar{\alpha}\vartheta\hat{\mathcal{P}}_{t-1} - \frac{\bar{\alpha}\bar{\Phi}_\vartheta}{(1-\bar{\alpha})}\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) - \frac{\bar{\alpha}\,(1+\omega\theta)\,(1-\bar{\alpha}\vartheta)}{2\,(1-\bar{\alpha})^2}\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right)^2$$

Completing the squares and solving backwards to period $t=-1$, the expression is rewritten as

$$\hat{\mathcal{P}}_t \approx (\bar{\alpha}\vartheta)^{t+1}\,\hat{\mathcal{P}}_{-1} - \frac{1}{2}\frac{\bar{\alpha}\,(1-\bar{\alpha}\vartheta)}{(1-\bar{\alpha})^2}\,(1+\omega\theta)\sum_{\tau=0}^{t}(\bar{\alpha}\vartheta)^{t-\tau}\left(\hat{\pi}_\tau - \hat{\pi}_\tau^{ind} + \bar{\phi}_\pi\right)^2 + tip_t^{\mathcal{P}}$$

where $\bar{\phi}_\pi$ is a positive constant and $tip_t^{\mathcal{P}}$ denote a term independent of policy:

$$\bar{\phi}_\pi \equiv \tfrac{(1-\bar{\alpha})\bar{\Phi}_\vartheta}{(1-\bar{\alpha}\vartheta)(1+\theta\omega)} \quad ; \; tip_t^{\mathcal{P}} \equiv \tfrac{1}{2}\tfrac{\bar{\alpha}}{(1-\bar{\alpha})}\bar{\phi}_\pi\bar{\Phi}_\vartheta\tfrac{\left(1-(\bar{\alpha}\vartheta)^{t+1}\right)}{(1-\bar{\alpha}\vartheta)}$$

Note then that

$$\begin{aligned}
\sum_{t=0}^{\infty}\beta^t\hat{\mathcal{P}}_t &\approx tip_{-1}^{\beta\mathcal{P}} - \frac{1}{2}\frac{\bar{\alpha}\left(1-\bar{\alpha}\vartheta\right)}{\left(1-\bar{\alpha}\right)^2}\left(1+\omega\theta\right)\sum_{t=0}^{\infty}\beta^t\sum_{\tau=0}^{t}\left(\bar{\alpha}\vartheta\right)^{t-\tau}\left(\hat{\pi}_\tau-\hat{\pi}_\tau^{ind}+\bar{\phi}_\pi\right)^2 \\
&\approx tip_{-1}^{\beta\mathcal{P}} - \frac{1}{2}\frac{\bar{\alpha}\left(1-\bar{\alpha}\vartheta\right)}{\left(1-\bar{\alpha}\right)^2}\left(1+\omega\theta\right)\sum_{\tau=0}^{\infty}\sum_{t=\tau}^{\infty}\beta^t\left(\bar{\alpha}\vartheta\right)^{t-\tau}\left(\hat{\pi}_\tau-\hat{\pi}_\tau^{ind}+\bar{\phi}_\pi\right)^2
\end{aligned}$$

Therefore the sum is rewritten as

$$\sum_{t=0}^{\infty}\beta^t\hat{\mathcal{P}}_t \approx tip_{-1}^{\beta\mathcal{P}} - \frac{1}{2}\frac{\left(1-\bar{\alpha}\vartheta\right)\left(\omega+\sigma\right)}{\left(1-\bar{\alpha}\right)}\frac{1}{\bar{\kappa}}\sum_{\tau=0}^{\infty}\beta^\tau\left(\hat{\pi}_\tau-\hat{\pi}_\tau^{ind}+\bar{\phi}_\pi\right)^2$$

where $tip_{-1}^{\beta\mathcal{P}}$ denote a term independent of policy from period $t=0$ onward:

$$tip_{-1}^{\beta\mathcal{P}} \equiv \frac{1}{\left(1-\bar{\alpha}\beta\vartheta\right)}\left(\bar{\alpha}\vartheta\hat{\mathcal{P}}_{-1}+\frac{1}{2}\frac{\bar{\alpha}}{\left(1-\beta\right)\left(1-\bar{\alpha}\right)}\bar{\phi}_\pi\bar{\Phi}_\vartheta\right)$$

B) Second order approximations of u_t and v_t

A second order approximation of u_t is

$$u_t \approx \bar{Y}^{1-\sigma}\left[\hat{y}_t+\frac{1}{2}\left(1-\sigma\right)\hat{y}_t^2+\hat{y}_t\hat{\epsilon}_t\right]+tip_t^u$$

where $tip_t^u \equiv \bar{u}\left(1+\hat{\epsilon}_t+\frac{1}{2}\hat{\epsilon}_t^2\right)$.

Since $Y_t^{ef(\omega+\sigma)} = \left(\varepsilon/\chi\right)\epsilon_t\mathcal{A}_t^{(1+\omega)}$, the aggregate disutility $v_t = \frac{\chi}{1+\nu}\left(\frac{Y_t}{\mathcal{A}_t}\right)^{(1+\omega)}\mathcal{P}_t^{-\theta(1+\omega)}$ can be written as $v_t = \frac{1}{1+\omega}\left(\frac{Y_t}{\mathcal{P}_t^\theta}\right)^{(1+\omega)}\left(Y_t^{ef}\right)^{-(\omega+\sigma)}\epsilon_t$. Therefore, its second order approximation is

$$\begin{aligned}
v_t \;\approx\; & \bar{v}+\frac{\bar{Y}^{(1+\omega)}\left(\bar{Y}^{ef}\right)^{-(\omega+\sigma)}\bar{\mathcal{P}}^{-\theta(1+\omega)}}{1+\omega}\left[\left(1+\omega\right)\hat{y}_t-\left(\omega+\sigma\right)\hat{y}_t^{ef}-\theta\left(1+\omega\right)\hat{\mathcal{P}}_t+\hat{\epsilon}_t\right] \\
&+\frac{\bar{Y}^{(1+\omega)}\left(\bar{Y}^{ef}\right)^{-(\omega+\sigma)}\bar{\mathcal{P}}^{-\theta(1+\omega)}}{2\left(1+\omega\right)}\left[\left(1+\omega\right)\hat{y}_t-\left(\omega+\sigma\right)\hat{y}_t^{ef}-\theta\left(1+\omega\right)\hat{\mathcal{P}}_t+\hat{\epsilon}_t\right]^2
\end{aligned}$$

Note that $\bar{Y}^{(\omega+\sigma)}\left(\bar{Y}^{ef}\right)^{-(\omega+\sigma)}\bar{\mathcal{P}}^{-\theta(1+\omega)} = \left(1-\bar{\Phi}_y\right)$, where $\bar{\Phi}_y$ is assumed to be a first order disturbance term. Recall that $\hat{\mathcal{P}}_t$ is a second order variable whenever $\bar{\Phi}_\vartheta$ is assumed to be a first order disturbance term. Under such assumptions, the approximation is simplified to

$$v_t \approx \bar{Y}^{1-\sigma}\left[\hat{y}_t-\bar{\Phi}_y\hat{y}_t-\theta\hat{\mathcal{P}}_t+\frac{1}{2}\left(1+\omega\right)\hat{y}_t^2-\left(\omega+\sigma\right)\hat{y}_t\hat{y}_t^{ef}+\hat{y}_t\hat{\epsilon}_t\right]+tip_t^v$$

where $tip_t^{\upsilon} \equiv \bar{\upsilon}\left[1+\left(\hat{\epsilon}_t-(\omega+\sigma)\,\hat{y}_t^{ef}\right)+\frac{1}{2}\left(\hat{\epsilon}_t-(\omega+\sigma)\,\hat{y}_t^{ef}\right)^2\right]$.

C) Second order approximation of the welfare function

The instantaneous utility of the representative household is $\mathcal{U}_t = u_t - \upsilon_t$, and thus is the society instantaneous utility function. Moreover, note that $Y_t^{ef} = \mu^{1/(\omega+\sigma)} Y_t^n$. It implies that $\hat{y}_t^{ef} = \hat{y}_t^n$. Using the last results, I obtain

$$\begin{aligned}
\mathcal{U}_t &= \bar{Y}^{1-\sigma}\left(\theta\left(1-\bar{\Phi}_y\right)\hat{\mathcal{P}}_t+\bar{\Phi}_y\hat{y}_t-\frac{1}{2}(\omega+\sigma)\,\hat{y}_t^2+(\omega+\sigma)\,\hat{y}_t\hat{y}_t^n\right)\\
&\quad+\bar{Y}^{1-\sigma}\left(\bar{\Phi}_y\hat{y}_t\hat{\epsilon}_t-(\omega+\sigma)\,\bar{\Phi}_y\hat{y}_t\hat{y}_t^n+\frac{1}{2}(1+\omega)\,\bar{\Phi}_y\hat{y}_t^2\right)\\
&\quad+\bar{Y}^{1-\sigma}\left(1-\bar{\Phi}_y\right)\left(\frac{1}{2}\theta^2(1+\omega)\,\hat{\mathcal{P}}_t^2-\theta(1+\omega)\,\hat{y}_t\hat{\mathcal{P}}_t+\theta(\omega+\sigma)\,\hat{y}_t^n\hat{\mathcal{P}}_t-\theta\hat{\mathcal{P}}_t\hat{\epsilon}_t\right)\\
&\quad+tip_t^u-tip_t^{\upsilon}
\end{aligned}$$

Note that there are two linear terms: $\left(1-\bar{\Phi}_y\right)\hat{\mathcal{P}}_t$ and $\bar{\Phi}_y\hat{y}_t$. Therefore, this correct approximation is not useful for policy evaluation using the structural log-linearized equations. That is the reason I parallel the analysis in Woodford (2003, chap. 6) and assume $\bar{\Phi}_\vartheta$ and $\bar{\Phi}_y$ to be first order disturbance terms. Since he assesses the Zero StSt, he only makes this assumption with respect to $\bar{\Phi}_y$. Under the Trend StSt, I need to expand this assumption towards $\bar{\Phi}_\vartheta$. In this case, I have already shown that $\hat{\mathcal{P}}_t$ becomes a second order variable. This implies that $\bar{\Phi}_y\hat{\mathcal{P}}_t$ is a third order term. Ignoring all third-order terms and completing the squares, I simplify the second-order approximation to

$$\mathcal{U}_t = \bar{Y}^{1-\sigma}\left(\theta\hat{\mathcal{P}}_t-\frac{1}{2}(\omega+\sigma)\left(\hat{x}_t-\bar{\phi}_x\right)^2\right)+tip_t^{\mathcal{U}}+tip_t^u-tip_t^{\upsilon}$$

where $\bar{\phi}_x \equiv \frac{1}{(\omega+\sigma)}\bar{\Phi}_y$ and $tip_t^{\mathcal{U}} \equiv \frac{1}{2(\omega+\sigma)}\bar{Y}^{1-\sigma}\left[\bar{\Phi}_y^2+(\omega+\sigma)^2\left(\hat{y}_t^n\right)^2\right]$. Using the second-order approximation on $\sum_{t=0}^{\infty}\beta^t\hat{\mathcal{P}}_t$, the social welfare $\mathcal{W}_t \equiv E_t\sum_{\tau=0}^{\infty}\beta^\tau\mathcal{U}_{t+\tau}$ is then computed as

$$\mathcal{W}_t \equiv -\frac{1}{2}\bar{\mathcal{V}}E_t\sum_{\tau=0}^{\infty}\beta^\tau\left[\left(\hat{\pi}_{t+\tau}-\hat{\pi}_{t+\tau}^{ind}+\bar{\phi}_\pi\right)^2+\frac{(1-\bar{\alpha})}{(1-\bar{\alpha}\vartheta)}\frac{\bar{\kappa}}{\theta}\left(\hat{x}_{t+\tau}-\bar{\phi}_x\right)^2\right]+tip_t^{\mathcal{W}}$$

where $\bar{\mathcal{V}} \equiv \bar{Y}^{1-\sigma}\frac{(1-\bar{\alpha}\vartheta)(\omega+\sigma)}{(1-\bar{\alpha})}\frac{\theta}{\bar{\kappa}}$ and $tip_t^{\mathcal{W}} \equiv \theta\bar{Y}^{1-\sigma}tip_{t-1}^{\beta\mathcal{P}}+E_t\sum_{\tau=0}^{\infty}\beta^\tau\left(tip_{t+\tau}^{\mathcal{U}}+tip_{t+\tau}^u-tip_{t+\tau}^{\upsilon}\right)$.

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