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Optimal Policy When the Inflation Target is not Optimal*

Sergio A. Lago Alves†

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Abstract

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I assess the optimal policy to be followed by a welfare-concerned central bank when assigned an inflation target that is not necessarily welfare-optimal. I treat the inflation target as the trend inflation and I have three main contributions: (*i*) a welfare-based loss function fully derived under trend inflation, showing how the non-optimal inflation target acts as an extra inefficiency source; (*ii*) I show that the trend inflation does affect the relative weight of the output gap: they are inversely related; (*iii*) under trend inflation, I derive time consistent optimal policies with both unconditional and timeless commitment, and I show how to translate the pursuit of the inflation target into an additional constraint in the minimization step.

Keywords: Optimal policy, trend inflation, inflation targeting
JEL Classification: E31, E52, E58

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1 Introduction

As of 2011, 27 central banks have adopted inflation targeting (IT) as their framework for monetary policy. The central government sets positive inflation targets, ranging between 2% to 5% in most countries,[1] and defines acceptance bands and horizons in which the targets are supposed to be met. The central banks are assigned the task to pursuit the targets by means of their monetary policy decisions. However, the inflation targets are not guaranteed to be welfare-optimal. Non-optimality may arise from measurement error of empirical CPIs, political economy issues (median voter preferences, lobbies, etc.) or model uncertainty, which may include a mismatch between the preferred models used by the government and the ones used by the central bank.

Currently, dynamic stochastic general equilibrium (DSGE) models not only are the preferred policy evaluation tool in central banks, but also are gauging monetary policy decisions in industrial countries.[2] While most of those models have zero (e.g. Woodford's (2003) cashless economy) or negative (e.g. Friedman's (1969) rule) inflation as the optimal level to be pursued by the monetary authority, inflation targets around the world have always been positive.[3]

Even though the targets are exogenously given, central banks have flexibility to choose what they perceive as the best policy to pursue them. Hence, I address the following question: if a welfare-concerned central bank is not seeking to bring the inflation rate back to its welfare optimum, which tool should be used to evaluate policies and pick the best one from the policy family that brings the inflation back to the target?

For this task, I treat the inflation target as a trend inflation[4] and borrow Wood-

[1] See e.g. Hammond (2011) and Roger (2010) for features of the inflation targeting frameworks and the inflation targets pursuited in each country.

[2] The Bank of Canada, Bank of England and the European Central Bank are among the central banks officially using DSGE models as their main forecasting tools. See e.g. Christoffel et al (2008), Harrison et al (2005), Murchison and Rennison (2006), Sbordone et al (2010) and Tovar (2008) for reviews on how DSGE models have been used by central banks for policy analysis and decision.

[3] Even though some authors have been imposing additional constraints to the standard DSGE model to justify higher inflation rates as the welfare-optimal, a consensus approach is yet to come. Coibion et al (2011), for instance, embed the zero lower bound for the nominal interest rates but find that the optimal level of inflation is smaller than the typical targets adopted by developed economies, even after considering different types of nominal frictions.

[4] The trend inflation, as it came to be known, is the level of the inflation rate in the steady state

ford's (2003) standard model with Calvo price setting,[5] flexible wages and a representative household, as described in Section 2. It is a simple model, but it is comprehensive enough to allow me to answer the question and obtain analytical results of easy interpretation.

Addressing the paper's question is not a simple quest in equilibria with non-zero trend inflation and staggered Calvo pricing. Even though aggregate variables have steady state levels in such equilibria,[6] there is also a steady state dispersion of relative prices. This fact is well documented in the literature of trend inflation (e.g. Ascari (2004), Ascari and Ropele (2007a, 2007b), Coibion and Gorodnichenko (2011), among others).[7]

I show that this fact is particularly critical when computing the steady sate level and the second order approximation of the aggregate labor disutility under non-zero trend inflation. Therefore I start by describing how to compute the evolution of the aggregate labor disutility considering only aggregate variables. For that purpose, I define a relevant aggregate relative price and obtain an equation describing its dynamics in a very similar way that Schmitt-Grohé and Uribe (2006) derive the aggregate relative price relevant for their model's resource constraint.

Using this result, I am able to assess the negative of the welfare function (True loss function), evaluated at steady state equilibria with different trend inflation levels. In particular, I show that the second-order derivative (curvature) of the True loss function sharply increases as the trend inflation rises above zero. With standard calibration, the curvature at the 2% annual trend inflation is about two and a half times as large as the curvature at the steady state with zero inflation. At the 4% annual trend inflation, it is ten times as large. As for the steady state level of the aggregate output, it sharply falls below the aggregate output under flexible prices,

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equilibrium.

[5] I am aware that the degree of price rigidity a la Calvo is likely to endogenously decrease as the trend inflation rises. I assume, however, that the parameter remains constant for all values of trend inflation as long as it is sufficiently small (less than 5% year, for instance).

[6] Broadly speaking, the non-stochastic steady state equilibrium is defined as the one achieved when all disturbances are fixed at their means.

[7] More reference in trend inflation is found in Amano et al (2006), Blake and Fernandez-Corugedo (2006), Cogley and Sbordone (2008), Coibion and Gorodnichenko (2011), Coibion et al (2011), Fernandez-Corugedo (2007), Kichian and Kryvtsov (2007), and Sahuc (2006). The old approach to deal with trend inflation was to embed it in full indexation rules (e.g. Yun (1996), Alves and Areosa (2005)). Analysis from micro data suggests however that there is very small or no indexation at all on individual prices (e.g. Klenow and Kryvtsov (2008) and Bils and Klenow (2004)).

which does not vary with the trend inflation. Finally, as it is expected, the True loss function has a minimum when the trend inflation is very close to zero.

Those findings are relevant because they impose limitations on any second-order approximation of the True loss function, such as Woodford's (2003) standard welfare-based loss function. The strong curvature variation implies that the Woodford's loss function is only accurate if the inflation rate oscillates sufficiently close to zero, around which the True loss function is approximated. For larger inflation rates, this approximation underestimates welfare losses. This is due to the fact that the curvature of the True loss function is relatively small around the zero trend inflation.

In this regard, I identify two main wedges between the True loss function and its approximated assessment: the *static wedge* – defined as the difference between the True and approximated loss functions, when evaluated at the steady state; and the *stochastic wedge* – defined as the extra wedge arising in the difference of the expected values of the loss functions in a stochastic equilibrium.

I also derive the trend inflation welfare-based loss function, as a second order approximation of the True loss function around the steady state with trend inflation – see Proposition 2, in Section 3.

In important aspects, it differs from the trend inflation welfare-based loss function obtained by Coibion et al. (2011), the closest paper in this regard. The authors do not completely approximate the aggregate labor disutility around the steady state with trend inflation. Instead, the main part of their approximation closely follows Woodford's (2003) approach and is done around the steady state with *flexible prices*, which is the same as the steady state with zero inflation. This approach leads to a component that directly depends on the dispersion of relative prices. The authors then depart from Woodford by approximating this term around the steady state with trend inflation. For this reason, I refer to their result as the hybrid welfare-based loss function.[8] As I conjecture, the authors must have faced the aggregation problem in the steady state with trend inflation when adapting Woodford's steps. Without an expression to describe the evolution of the aggregate disutility, they decided for the hybrid approach instead.

Since the first part of the hybrid loss function is derived around the steady state

[8] I thank Guido Ascari for suggesting this term.

with flexible prices, the endogenous weights on inflation and output gap volatilities are not the same as mine. As in any (log) second order approximation, the coefficients depend strongly on the curvature of the non-linear function. As I mentioned before, the True loss function is much flatter and its curvature is very small in the steady state with flexible prices. Therefore the weights of the hybrid loss function do not depend on the trend inflation as much as they do in the trend inflation loss function.

In particular, one of the authors' findings is that the weight of the output gap volatility does no depend on the trend inflation. My results, on the other hand, suggest that it strongly depends on the trend inflation. Using standard calibration, the relative weight of the output gap volatility decreases as the trend inflation rises. As the annual trend inflation rises from 2% to 4%, the relative weight falls from 85% of what the Woodford's loss function suggests to 70%.

Using the trend inflation loss function, I derive in Section 4 the time consistent optimal policies under both unconditional (e.g. Damjanovic et al. (2005)) and timeless (e.g. Woodford (1999 and 2003)) commitment, and show how to translate the pursuit of the inflation target into an additional constraint in the loss-minimization problem. It is known that the unconditionally optimal policy slightly dominates the timeless one (e.g. Jensen (2001), Jensen and McCallum (2002)). I show that this approach is also well-suited to cope with the pursuit of the inflation target, if defined as the unconditional expectation of the inflation rate.

An interesting feature of both trend inflation optimal policies (unconditional and timeless) is that their targeting rules are more history-dependent than the ones derived under the steady state with zero inflation, in which they only depend on the first lag of output gap and on current inflation rate (e.g. Woodford (2003) and Damjanovic et al. (2005)). Under the steady state with trend inflation, the rules also depend on the second lag of the output gap and on the first lag of the inflation rate. Finally, the inertia of the targeting rules increases as the trend inflation rises.

2 The structural model

For simplicity, I use the standard DSGE model with Calvo price-setting and flexible wages, as presented in Woodford (2003). The economy consists of a representative infinite-lived household that consumes an aggregate bundle and supplies differentiated labor to a continuum of differentiated firms indexed by $z \in (0,1)$, which produce and sell goods in a monopolistic competition environment. Firms follow a Calvo type price-setting and maximize their expected flow of profits, subjected to their own demand curves.

2.1 Households

The representative household chooses the sequence of C_t, $h_t(z)$ and B_{t+1} to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\epsilon_t^u \frac{C_t^{1-\sigma}}{1-\sigma} - \chi \int_0^1 \frac{h_t(z)^{1+\nu}}{1+\nu} dz \right),$$

subjected to the flow budget constraint $P_t C_t + E_t q_{t+1} B_{t+1} \leq B_t + \int_0^1 w_t(z) h_t(z) dz + d_t$, and a standard no-Ponzi condition. Financial markets are complete, E_t is the time-t expectations operator, C_t is the aggregate consumption bundle, P_t is the consumption price index, $h_t(z)$ is the labor supplied to firm z, ϵ_t^u is a preference shock, B_t is the state-contingent value of the portfolio of financial securities held at the beginning of period t, $w_t(z)$ is the nominal wage rate at firm z, d_t denotes nominal dividend income, and q_{t+1} is the stochastic discount factor from $(t+1)$ to t. Finally, β denotes the subject discount factor, σ^{-1} is the intertemporal elasticity of substitution and ν^{-1} is the Frisch elasticity of labor supply.

Consumption over all differentiated goods $c_t(z)$ is aggregated into a bundle C_t, as in Dixit and Stiglitz (1977). Aggregation and expenditure minimization relations are described by: $C_t^{\frac{\theta-1}{\theta}} = \int_0^1 c_t(z)^{\frac{\theta-1}{\theta}} dz$, $P_t^{1-\theta} = \int_0^1 p_t(z)^{1-\theta} dz$, $P_t C_t = \int_0^1 p_t(z) c_t(z) dz$ and a demand function $c_t(z) = C_t P_t^{\theta} (p_t(z))^{-\theta}$, where $\theta > 1$ is the elasticity of substitution between goods. In equilibrium,[9] the optimal real wage satisfies $\frac{w_t(z)}{P_t} = \frac{\chi h_t(z)^{\nu}}{\epsilon_t^u C_t^{-\sigma}}$ and the optimal consumption plan is described by the Euler

[9] As usual, equilibrium is defined as the equations describing the first order conditions of the representative household and firms, a transversality condition $\lim_{T\to\infty} E_T q_{t,T} B_T = 0$, where $q_{t,T} \equiv \Pi_{\tau=t+1}^{T} q_{\tau}$, and the market clearing conditions.

equation $q_t = \frac{\beta}{\Pi_t}\frac{\epsilon_t^u}{\epsilon_{t-1}^u}\left(\frac{C_t}{C_{t-1}}\right)^{-\sigma}$, where $\Pi_t = 1+\pi_t$ and $I_t = 1+i_t$ are the gross inflation and interest rates at period t, which satisfies $I_t = \frac{1}{E_t q_{t+1}}$, and i_t is the riskless one-period nominal interest rate.

The market clearing conditions are $C_t = Y_t$ and $c_t(z) = y_t(z)$, $\forall z$, where Y_t and $y_t(z)$ are the aggregate and firm z production levels.

2.2 Firms

Following Calvo (1983), firms optimally adjust their prices with probability $(1-\alpha)$. With probability α firms index their prices according to $p_t(z) = p_{t-1}(z)\Pi_t^{ind}$, in which $\Pi_t^{ind} \equiv \Pi_{t-1}^{\gamma_\pi}\bar{\Pi}^{\gamma}$ is the indexation term and $\bar{\Pi}$ is the gross inflation trend. Finally, $\gamma_\pi \in (0,1)$ and $\gamma \in (0,1)$. Firms $z \in (0,1)$ produce differentiated goods using the technology $y_t(z) = \epsilon_t^a h_t(z)^{\varepsilon}$, where ϵ_t^a is the aggregate technology shock and $\varepsilon \in (0,1)$.

When optimally readjusting at period t, firms choose the price p_t^* to maximize the expected discounted flow of each firm's nominal profits, given the demand function and the price readjusting structure. Consider the transformations $\omega \equiv \frac{1+\nu}{\varepsilon} - 1$, $\mu \equiv \frac{\theta}{\theta-1}$, $\xi_t^u \equiv \log(\epsilon_t^u)$, $\xi_t^a \equiv \log(\epsilon_t^a)$. The first order condition can be conveniently written as $(p_t^*/P_t)^{1+\theta\omega} = N_t/D_t$, where N_t and D_t can be written in recursive forms, avoiding infinite sums:

$$\begin{aligned} N_t &= (X_t)^{(\omega+\sigma)} + \alpha E_t n_{t+1} \quad , \quad n_t = q_t \mathcal{G}_t \Pi_t \left(\frac{\Pi_t}{\Pi_t^{ind}}\right)^{\theta(1+\omega)} N_t \\ D_t &= 1 + \alpha E_t d_{t+1} \quad , \quad d_t = q_t \mathcal{G}_t \Pi_t^{ind} \left(\frac{\Pi_t}{\Pi_t^{ind}}\right)^{\theta} D_t \end{aligned}$$

in which $\mathcal{G}_t \equiv Y_t/Y_{t-1}$ denotes the gross output growth rate, $X_t \equiv Y_t/Y_t^n$ is the gross output gap, and Y_t^n is the natural (flexible prices) output.[10] Finally, the Calvo pricing structure implies the following dynamics:

$$1 = (1-\alpha)\left(\frac{N_t}{D_t}\right)^{\frac{1-\theta}{1+\theta\omega}} + \alpha\left(\frac{\Pi_t^{ind}}{\Pi_t}\right)^{1-\theta} \tag{1}$$

An important feature of the staggered price structure under trend inflation is that there is a steady state dispersion of relative prices. Clarifying this point is

[10] The natural output evolves according to $(Y_t^n)^{(\omega+\sigma)} = \frac{\varepsilon}{\chi\mu}\exp\left((1+\omega)\xi_t^a + \xi_t^u\right)$.

important before presenting the main contributions of this paper.

Under trend inflation with Calvo price setting, the steady state is still dynamic at the firm level. Indeed, while structural shocks are constrained to remain at their zero means, the positive trend inflation induces a stationary dispersion of relative prices. The Calvo staggered price structure implies that there is always a fraction of firms whose prices lag behind their optimal levels.[11] As a consequence, firms adjust above the aggregate price trend when they are set to optimize. This particularity makes the individual prices and production levels to be dispersed, even under the steady state. Interestingly, the aggregate output converges to a time invariant steady state $\bar{Y}$ – the individual output dispersion is such that it cancels out when aggregating.

2.3 Welfare computation

Following, I present one of the contributions of this paper: a pair of equations describing the evolution of the aggregate disutility function $\tilde{\upsilon}_t \equiv \chi \int_0^1 \frac{h_t(z)^{1+\nu}}{1+\nu} dz$ from labor supplied to all firms, as a function of aggregate variables only. This result is important because it allows the fully-fledged derivation of the trend inflation welfare-based (TIWeB) loss function as a second order approximation of the (negative) true welfare function around the steady state with trend inflation. For contrast, Coibion et al. (2011) derives their hybrid loss function as a mixed approximation: some steps around the steady state with flexible prices (Flex StSt) and others around the steady state with trend inflation (Trend StSt). Their approach, I conjecture, was chosen due to the lack of an expression for the aggregate disutility such as the one I derive here. The hybrid approach is inadequate because it avoids dealing with the price dispersion in the Trend StSt[12] by making a first approximation around the Flex StSt, and only then approximating the remaining terms around the Trend StSt.

The following result is general and independent of any trend inflation. Using the

[11] If price indexation is full ($\gamma + \gamma_\pi = 1$) then this problem never arises. However, empirical evidence suggests that indexation is almost absent (e.g. Cogley and Sbordone (2005 and 2008), Klenow and Malin (2010), and Levin et al. (2005)).

[12] Recall that the production levels of individual firms and the corresponding individual relative prices do not have a steady state when the trend inflation is positive.

production and demand functions, I rewrite the aggregate disutility $\tilde{v}_t$ as follows

$$\begin{aligned}\tilde{v}_t = &\ \chi \int_0^1 \frac{h_t(z)^{1+\nu}}{1+\nu} dz = \frac{\chi}{1+\nu} \int_0^1 \left(y_t\left(z\right)^{\frac{1}{\varepsilon}} \exp\left(-\tfrac{1}{\varepsilon}\xi_t^a\right)\right)^{1+\nu} dz \\ = &\ \frac{\chi \exp(-(1+\omega)\xi_t^a)}{(1+\nu)} Y_t^{(1+\omega)} \int_0^1 \left(\frac{p_t(z)}{P_t}\right)^{-\theta(1+\omega)} dz\end{aligned}$$

Let $\mathcal{P}_t$ denote an aggregate relative price, relevant for the aggregate disutility:

$$\mathcal{P}_t^{-\theta(1+\omega)} \equiv \int_0^1 \left(\frac{p_t\left(z\right)}{P_t}\right)^{-\theta(1+\omega)} dz$$

Using the structure of the Calvo price setting, I am able to derive the law of motion of $\mathcal{P}_t$ in a very similar way Schmitt-Grohé and Uribe (2006) derive the aggregate relative price relevant for the resource constraint in their model. Proposition 1 describes the evolution of the aggregate disutility function $\tilde{v}_t$ and the aggregate relative price $\mathcal{P}_t$ as functions of aggregate variables only. The aggregate relative price $\mathcal{P}_t$ summarizes all information about price dispersion.

Proposition 1 *The aggregate disutility is computed as*

$$\tilde{v}_t = \frac{\chi}{1+\nu} Y_t^{(1+\omega)} \mathcal{P}_t^{-\theta(1+\omega)} \exp\left(-\left(1+\omega\right)\xi_t^a\right) \tag{2}$$

where the relevant relative price $\mathcal{P}_t$ evolves according to

$$\mathcal{P}_t^{-\theta(1+\omega)} = (1-\alpha)\left(\frac{N_t}{D_t}\right)^{-\frac{\theta(1+\omega)}{1+\theta\omega}} + \alpha\left(\frac{\Pi_t^{ind}}{\Pi_t}\right)^{-\theta(1+\omega)} \mathcal{P}_{t-1}^{-\theta(1+\omega)} \tag{3}$$

Let $\mathcal{W}_t$ denote the welfare function, computed as the discounted flow of utility evaluated at the equilibrium variables. In this case, $\mathcal{W}_t$ evolves according to a Bellman-shaped equation $\mathcal{W}_t = u_t - \tilde{v}_t + \beta E_t \mathcal{W}_{t+1}$, where $u_t = \frac{C_t^{1-\sigma}}{1-\sigma}\epsilon_t^u$.

An important issue regards the concavity of the welfare function. In the Trend StSt, $\mathcal{W}_t$ decreases fast and becomes highly concave as the inflation rate rises. For illustration, figure 1 depicts the steady state levels (see Appendix A) of the welfare function $\bar{\mathcal{W}}$, its second derivative $\frac{\partial^2 \bar{\mathcal{W}}}{\partial \bar{\pi}^2}$, the aggregate output $\bar{Y}$ and the natural output $\bar{Y}^n$ as the annual trend inflation rises from 0% to 4.5%. Since $\bar{Y}^n$ does not vary with the trend inflation, I normalize its level to 1. In this case, $\bar{Y}$ is the same as the gross output gap $\bar{X}$.

My calibration is in line with accepted values for the USA.[13] Based on Cogley and Sbordone (2008): $\alpha = 0.6$, $\beta = 0.99$ (quarterly), $\theta = 10$, $\gamma_\pi = \gamma = 0$, $\varepsilon = 0.75$.[14] Based on Smets and Wouters (2007): $\sigma = 1.50$, $\nu = 1.50$.[15] In this paper, I refer to this parameter set as the benchmark calibration. As known in the literature, the effects of larger indexation are equivalent to smaller inflation trend, keeping the remaining parameters constant.[16]

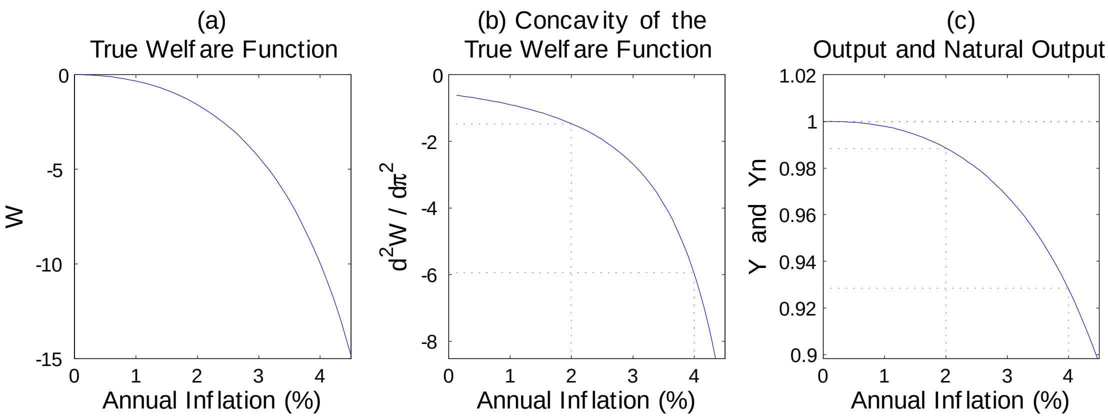

Figure 1: Steady state levels

Note on third panel: output (full line), natural output (dotted line).

As expected, the welfare function hits its maximum when the trend inflation is close to 0%. However, it is much more concave for positive values. For instance, the curvature at the 2% annual trend inflation is about two and a half times as large as the curvature at the steady state equilibrium with zero inflation (Zero StSt). At the 4% annual trend inflation, it is ten times as large. It implies that: (i) second order approximations around the Zero StSt underestimate the appropriate curvature of the welfare function when the inflation rate is actually oscillating around a positive value; and (ii) such second order approximations do not internalize the large welfare loss achieved when the trend inflation is positive. When policy is meant to keep inflation at a positive level, as in an inflation targeting framework, those findings

[13] The Calvo parameter α is also in line with micro evidence (e.g. Bils and Klenow (2004), Klenow and Kryvtsov (2008), Klenow and Malin (2010), and Nakamura and Steinsson (2008)).

[14] The authors' point estimates are $\alpha = 0.588$, $\theta = 9.8$, and $\gamma_\pi = 0$. They also calibrate $\varepsilon = 0.70$, $\beta = 0.99$.

[15] The authors actually estimate the posterior modes at $\sigma^{-1} = 1.39$, and $\nu = 1.92$.

[16] The effects ultimately depend on the term $\bar{\Pi}^{(1-\gamma_\pi-\gamma)}$.

suggest that a better policy evaluation is obtained when the approximation is done around the trend inflation.

Finally, notice that the steady state output sharply falls below the aggregate output under flexible prices as the trend inflation rises above zero. This effect clearly shows the distortion caused by high trend inflation. When the trend inflation is 2%, the output gap is open at about 1%. I recognize that the degree of price rigidity is likely to endogenously decrease as the trend inflation rises, and hence part of the inflation distortion may not be as high as the pictures suggest. However, empirical evidence for the US suggests that a Calvo parameter larger than 0.60 is consistent with annual inflation trend of 3%.[17] Therefore, my theoretical assessments may be quite reasonable in the inflation range I consider.

The bottom line is that relevant discrepancies may arise in hybrid approaches to approximate the True loss function, as the one used by Coibion et al. (2011), for they disregard the sharp increase of its curvature, and the strong fall in the level of the aggregate output, as the trend inflation rises.

2.3.1 Distortions

In order to cope with the distortion caused by the trend inflation, I expand Woodford's (2003) analysis on the efficient output under the Zero StSt. Consider a central planner who chooses the prices and the output level to maximize the welfare. The optimal solution clearly imposes every firm to produce the same efficient level Y_t^{ef}, which implies that all prices are the same, i.e. $\mathcal{P}_t = 1$. Thus the solution (see Woodford (2003)) is $\frac{\partial \tilde{v}_t^{ef}/\partial Y_t}{\partial u_t^{ef}/\partial Y_t} = 1$.[18]

Appendix A shows the steady state levels of all endogenous variables in the Trend StSt. In particular, the steady state consumption utility and the labor disutility

[17] From 1984 to 2004, the CPI annual inflation rate averaged 3.05% in the US. Using data from the same period, Smets and Wouters (2007) estimate the Calvo parameter at 0.73. Nakamura and Steinsson (2008), using micro data from 1988 to 2005, estimate the median duration between price changes at roughly 4.5 months (including sales) and 10 months (excluding sales). Those median durations τ_m are consistent with $\alpha = 0.63$ and $\alpha = 0.81$ in quarterly fequency, using $\tau_m = -\log(2)/\log(\alpha)$.

[18] This result implies that the the efficient output evolves according to $\left(Y_t^{ef}\right)^{(\omega+\sigma)} = \frac{\varepsilon}{\chi}\exp\left((1+\omega)\,\xi_t^a + \xi_t^u\right)$.

under trend inflation depends on the steady state aggregates as follows:

$$\bar{u} = \frac{\bar{Y}^{1-\sigma}}{1-\sigma} \quad , \quad \bar{\tilde{v}} = \frac{1}{1+\omega}\bar{Y}^{(1+\omega)}\left(\bar{Y}^{ef}\right)^{-(\omega+\sigma)}\bar{\mathcal{P}}^{-\theta(1+\omega)}$$

where barred variables stand for steady state levels.

Ignoring the indirect effect of $\bar{Y}$ on $\bar{\mathcal{P}}$, the steady state value of the marginal rate of substitution can be roughly approximated by the ratio of the derivatives of the steady state levels $\bar{\tilde{v}}$ and $\bar{u}$ with respect to $\bar{Y}$, i.e. $\bar{\tilde{v}}_Y/\bar{u}_Y$. Note that it is not the same as the steady state level of the ratio of the derivatives of $\tilde{v}_t$ and u_t with respect to Y_t. However, this approximation makes it easier to understand the first distortion component that exists in this model economy:

$$\frac{\bar{\tilde{v}}_Y}{\bar{u}_Y} = \frac{1}{\mu}\frac{(1-\bar{\alpha}\beta\vartheta)}{(1-\bar{\alpha}\beta)}\frac{(1-\bar{\alpha})}{(1-\bar{\alpha}\vartheta)} \tag{4}$$

where $\mu \equiv \frac{\theta}{\theta-1}$, $\omega \equiv \frac{1+\nu}{\varepsilon} - 1$, $\bar{\Pi} \equiv (1+\bar{\pi})$, $\bar{\alpha} \equiv \alpha\left(\bar{\Pi}\right)^{(\theta-1)(1-\gamma_\pi-\gamma)}$ and $\vartheta \equiv \left(\bar{\Pi}\right)^{(1+\theta\omega)(1-\gamma_\pi-\gamma)}$.

Whenever the trend inflation is positive, i.e. $\bar{\Pi} \geq 1$, note that ϑ is larger than one and the effective degree of price stickiness $\bar{\alpha}$ is greater than the Calvo degree α. Moreover, since ϑ is a positive transformation of the trend inflation, it is a convenient variable to reflect the effects of the trend inflation.

Following Woodford (2003), let $\Phi_y \equiv 1 - \bar{\tilde{v}}_Y/\bar{u}_Y$ denote the inefficiency degree of the steady state output.[19] Under the Trend StSt, the first term $\frac{1}{\mu}$ of $\frac{\bar{\tilde{v}}_Y}{\bar{u}_Y}$ is driven by the monopolistic competition distortion alone, while the second $\frac{(1-\bar{\alpha}\beta\vartheta)}{(1-\bar{\alpha}\beta)}\frac{(1-\bar{\alpha})}{(1-\bar{\alpha}\vartheta)}$ is driven by the non-zero trend inflation. Note that the second term collapses to unity when $\vartheta = 1$.

The second component of distortionary effects of the non-zero trend inflation is explained as follows. Under Calvo price setting, there is steady state dispersion of individual output and relative prices when the trend inflation is positive. In a nutshell, the indirect effect $\partial\bar{\mathcal{P}}/\partial\bar{Y}$ omitted in the previous computation of of $\bar{\tilde{v}}_Y/\bar{u}_Y$ captures this additional source of distortion. In this regard, I define $\Phi_\vartheta \equiv (\vartheta - 1)$ as an additional inefficiency parameter to measure how much of the gross inflation

[19] He only considers the Zero StSt, and hence his inefficiency degree is $\Phi_y = 1 - \frac{1}{\mu} = \frac{1}{\theta}$.

trend is above one.

As it turns out, tracking those two inefficiency parameters (Φ_y and Φ_ϑ) is sufficient for deriving the TIWeB loss function. Since efficiency requires both parameters to be zero, it is interesting to assume them to be small enough, as first order disturbance terms.[20] With such an assumption, linear terms multiplied by Φ_y and Φ_ϑ become of second-order importance and the TIWeB loss function is useful for policy analysis using the loglinearized structural equations.

2.4 The loglinearized model

The Euler equation and the pricing first order conditions can be log-linearized as the IS curve, and New Keynesian Phillips Curve (NKPC) under trend inflation. I omit the log-linearization steps, as similar ones are well documented in the literature on trend inflation.[21] For any variable $\mathfrak{U}_t$, the hatted representation $\widehat{\mathfrak{u}}_t \equiv \log\left(\mathfrak{U}_t/\overline{\mathfrak{U}}\right)$ represents its log-deviation from its Trend StSt level $\overline{\mathfrak{U}}$, which is shown in Appendix A. The model parameters and their descriptions are also shown in the same appendix.

The loglinearized IS curve is:

$$\hat{x}_t = E_t \hat{x}_{t+1} - \frac{1}{\sigma} E_t \left(\hat{\imath}_t - \hat{\pi}_{t+1} - \hat{r}_t^n\right) \tag{5}$$

where $\hat{r}_t^n \equiv E_t\left[\left(\hat{y}_{t+1}^n - \hat{y}_t^n\right) - (1+\omega)\left(\xi_{t+1} - \xi_t\right)\right]$ is the real interest rate under flexible prices, $\hat{y}_t^n = \frac{(1+\omega)\xi_t^a + \xi_t^u}{(\omega+\sigma)}$ is the natural output and $\xi_t \equiv \frac{(1-\sigma)\xi_t^a + \xi_t^u}{(\omega+\sigma)}$ is an aggregate shock.

The NKPC under trend inflation, with indexation term $\hat{\pi}_t^{ind} = \gamma_\pi \hat{\pi}_{t-1}$, is:

$$\begin{aligned}
\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) = \quad & \beta E_t \left(\hat{\pi}_{t+1} - \hat{\pi}_{t+1}^{ind}\right) + \bar{\kappa} \hat{x}_t \\
& + \tfrac{(\vartheta-1)}{(1-\bar{\alpha}\beta\vartheta L^{-1})} \bar{\alpha} \bar{\kappa} \beta E_t \hat{x}_{t+1} + \tfrac{(\vartheta-1)}{(1-\bar{\alpha}\beta\vartheta L^{-1})} \varphi_1 \theta \beta E_t \left(\hat{\pi}_{t+1} - \hat{\pi}_{t+1}^{ind}\right) \\
& + \tfrac{(\vartheta-1)}{(1-\bar{\alpha}\beta\vartheta L^{-1})} \varphi_2 \bar{\kappa} \beta E_t \left(\hat{x}_{t+1} - \hat{x}_t\right) + \tfrac{(\vartheta-1)}{(1-\bar{\alpha}\beta\vartheta L^{-1})} \tfrac{(1-\bar{\alpha})}{(1-\alpha)} \xi_t^{cps}
\end{aligned} \tag{6}$$

where L^{-1} is the lead operator and $\xi_t^{cps} \equiv \varphi_3 \beta E_t \left(\xi_{t+1} - \xi_t\right)$ is the model consistent

[20] Woodford (2003) was the first author to model the distortion variable Φ_y as a first order disturbance term, in order to derive the SWeB loss function.

[21] See e.g. Ascari (2004), Ascari and Ropele (2007 and 2007b), Cogley and Sbordone (2008), Coibion and Gorodnichenko (2008).

cost push shock, which does not depend on the trend inflation and only affects the dynamics if the trend inflation is not zero (it is multiplied by $(\vartheta - 1)$).

Finally, the existence of the steady state with trend inflation requires $\bar{\alpha}\vartheta < 1$ and $\bar{\alpha} < 1$, which implies that the maximum level for the trend inflation to exist under the premises of the model is $\bar{\Pi} \leq \min\left(\alpha^{\frac{-1}{\theta(1+\omega)(1-(\gamma_\pi+\gamma))}}, \alpha^{\frac{-1}{(\theta-1)(1-(\gamma_\pi+\gamma))}}\right)$. Under the benchmark calibration, the annualized threshold is $\bar{\pi}^* = 6.3\%$.

Recall that ϑ equals 1 in the Zero StSt. In this case, all terms multiplied by $(\vartheta - 1)$ disappear and the standard form is obtained. I prefer this form to write the trend inflation NKPC because it requires only one equation and no infinite sums, as opposed to the way commonly presented in the literature.[22]

The term ξ_t^{cps} is model consistent because it is generated by the same structural shocks that affect both the natural and the efficient outputs. In the Zero StSt paradigm, cost push shocks are usually thought as proportional to the components of the natural output that do not affect the efficient one. Most often they are modelled as time varying tax rates on firms income or time varying markups (e.g. Clarida et al. (1999), Galí (2003), Smets and Wouters (2003, 2005, 2007), Ascari and Ropele (2007)). Nonetheless, cost push shocks play an important role in optimal policy analyses. If the only nominal rigidity is the Calvo price setting, and if the only sources of shocks are the ones considered in this paper, there is no policy trade off under the Zero StSt paradigm. As a consequence, the optimal policy is able to obtain the result of stabilizing both the inflation and the output gap. In order to obtain the trade off between both objectives, cost push shocks are more than necessary (e.g. Walsh (2003), Woodford (2003)).

3 The welfare-based loss function under trend inflation

In this section I present the trend inflation welfare-based (TIWeB) loss function, derived as the second order log-approximation of the True loss function.

[22] Since the denominators of most of the terms in the right hand side depend on the lead operator L^{-1}, there are equivalent forms with infinite sums or with the expected inflation of two periods ahead, among others.

Before showing the main result, I clarify why my approach to (second-order) approximate the aggregate disutility $\tilde{v}_t \equiv \chi \int_0^1 \frac{h_t(z)^{1+\nu}}{1+\nu} dz$ differs from Woodford's (2003). His approach consists of two parts. In the first, he approximates the integrand $h_t(z)^{1+\nu}$, which is proportional to $\left(\frac{p_t(z)}{P_t}\right)^{-\theta(1+\omega)}$, around the Zero StSt. This procedure is possible because there is no price dispersion in this steady state. Next, he integrates the approximated result and uses the Calvo structure to rewrite the variance of (log) relative prices in terms of the squared inflation rate.

Under the Trend StSt, this approach is not possible because the individual relative prices do not converge in the steady state. I conjecture that Coibion et al. (2010) must have faced this problem, and hence they decided for the hybrid approach. They start by following the first part of Woodford's steps and approximate the integrand around the Flex StSt, which is the same as the Zero StSt. The approximation around the Trend StSt is done only in the second part, in which they use the Calvo structure to cope with the variance of relative prices.

They also assume log-utility on consumption, which minimizes the effects of the hybrid approach in their final result. However, under the Trend StSt, the traditional conflict between the income and substitution effects is important in price setting. Firms discount their profit flow using the stochastic discount factor, which is linked to the household's Euler equation in equilibrium. Hence, the conflict is spread towards firms' decisions. Under the particular Zero StSt approach, this conflict is algebraically offset in the log-linearized Phillips curve. That is not the case anymore as the inflation trend becomes larger. The coefficient φ_2 and the aggregate shock ξ_t depend on the term $(1-\sigma)$, which reflects this conflict. As the following proposition states, the coefficient $\mathcal{V}_\vartheta$ of the TIWeB loss function also depends on this term.

Using proposition 1, I approximate the aggregate disutility under the Trend StSt, avoiding the problem arising from the steady state dispersion of relative prices, and derive the TIWeB loss function as follows:

Proposition 2 *The true welfare function is (second-order) approximated as*

$$\mathcal{W}_t = -\frac{1}{2}\mathcal{V}_\vartheta E_t \sum_{\tau=0}^{\infty} \beta^\tau \mathcal{L}_{t+\tau} + tip_t^{\mathcal{W}} \tag{7}$$

where

$$\mathcal{L}_t \equiv \left(\hat{\pi}_t - \hat{\pi}_t^{ind} + \phi_\pi\right)^2 + \frac{(1-\bar{\alpha})}{(1-\bar{\alpha}\vartheta)}\frac{\bar{\kappa}}{\theta}\left(\hat{x}_t - \phi_x\right)^2$$

is the trend inflation welfare-based (TIWeB) loss function, $tip_t^{\mathcal{W}}$ *stands for terms independent of policy at period* t*,* ϕ_π *and* ϕ_x *are constants that depend on the inefficiency degrees* Φ_ϑ *and* Φ_y*, and* $\mathcal{V}_\vartheta$ *corrects for the aggregate reduction in the welfare when the trend inflation increases. Those coefficients are defined as follows:*

$$\phi_\pi \equiv \tfrac{(1-\bar{\alpha})}{(1-\bar{\alpha}\vartheta)(1+\theta\omega)}\Phi_\vartheta \qquad \phi_x \equiv \tfrac{1}{(\omega+\sigma)}\Phi_y \qquad \mathcal{V}_\vartheta \equiv \bar{Y}^{1-\sigma}\tfrac{(1-\bar{\alpha}\vartheta)(\omega+\sigma)}{(1-\bar{\alpha})}\tfrac{\theta}{\bar{\kappa}}$$

The proof and details are shown in Appendix B, in which I use the assumptions that Φ_y and Φ_ϑ are first-order parameters in order to cope with linear terms in the approximation.

The first important feature is that the relative coefficient on the output gap volatility $\bar{\mathcal{X}} \equiv \frac{(1-\bar{\alpha})}{(1-\bar{\alpha}\vartheta)}\frac{\bar{\kappa}}{\theta}$ decreases as the trend inflation rises under reasonable parametrization and inflation trend levels. This result is expected due to the increasing concavity of the true welfare function in the steady state. The TIWeB loss function internalizes the big welfare loss due to inflation volatility under positive inflation trend. Hence, the monetary authority optimally places less weight into fighting volatility of the output gap, as opposed to fighting inflation volatility.

Let $\mathcal{R}$ denote the ratio of $\bar{\mathcal{X}}$ and the relative coefficient $\mathcal{X} \equiv \frac{\kappa}{\theta}$ under the Zero StSt, where κ is the coefficient of the output gap in the NKPC under the Zero StSt (defined in Appendix A).

$$\mathcal{R} = \frac{\bar{\mathcal{X}}}{\mathcal{X}} = \frac{\alpha}{(1-\alpha)(1-\alpha\beta)}\frac{(1-\bar{\alpha})^2(1-\bar{\alpha}\beta\vartheta)}{\bar{\alpha}(1-\bar{\alpha}\vartheta)} \tag{8}$$

The elasticity of $\mathcal{R}$ to the trend inflation $\bar{\Pi}$ is then computed as

$$\frac{\partial\mathcal{R}}{\partial\bar{\Pi}}\frac{\bar{\Pi}}{\mathcal{R}} = \frac{\alpha(1-\gamma_\pi-\gamma)(\theta-1)(1+\bar{\alpha})}{(1-\alpha)(1-\alpha\beta)(1-\bar{\alpha})}\left[\frac{\mu(1+\omega)\bar{\alpha}\vartheta(1-\bar{\alpha})(1-\beta)}{(1-\bar{\alpha}\vartheta)(1+\bar{\alpha})(1-\bar{\alpha}\beta\vartheta)} - 1\right] \tag{9}$$

Therefore, there is a critical level of the inflation trend $\bar{\pi}^c$ above which the elasticity is positive, i.e. in which the first term in the brackets is larger than 1. However, reasonable parametrization leads to negative elasticity when the trend inflation is not large enough. Using the benchmark calibration, and additional ones with $\alpha = 0.50$ and $\alpha = 0.65$, figure 2 plots $\mathcal{R}$ for the trend inflation ranging from 0% to 4.5%.

Note that $\mathcal{R}$ decreases fast as the inflation trend rises. This effect is slightly mitigated if the Calvo parameter α is reduced. At the 3% inflation trend, in particular, the optimal relative weight of the output gap is about 75% of the coefficient suggested by the Woodford's standard welfare-based (SWeB) loss function. The main message is that the monetary policy must increase the weight on inflation volatility when the inflation trend is larger.

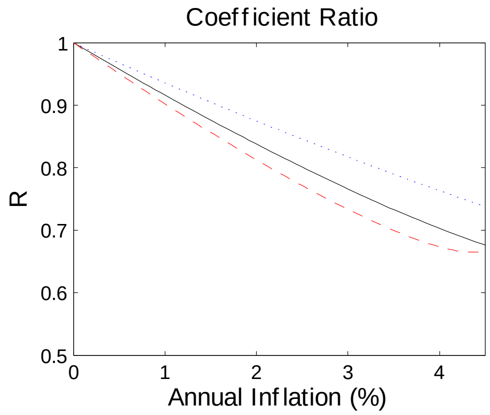

Figure 2: Coefficient Ratio (R)

Note: $\alpha = 0.50$ *(dotted blue),* $\alpha = 0.60$ *(black),* $\alpha = 0.65$ *(dashed red).*

3.1 The *static* and the *stochastic* wedges

I identify two main wedges driven between the True loss function and any approximated assessment: the *static wedge* – defined as the difference between the True and approximated loss functions, when evaluated in equilibria where the inflation remains fixed at the trend inflation, the remaining endogenous variables are fixed at the levels consistent with the trend inflation and the exogenous shocks are fixed at their means; and the *stochastic wedge* – defined as the extra wedge arising in the difference of the (unconditional) expected values of the loss functions in a stochastic equilibrium: this is a direct application of the Jensen's inequality.

Under the Zero StSt, I use plain lower case variables to represent the logdeviations. In this case, the loglinearized aggregate demand and supply curves and the

SWeB loss function are:

$$x_t = E_t x_{t+1} - \frac{1}{\sigma} E_t \left(i_t - \pi_{t+1} - r_t^n \right) \quad , \quad \left(\pi_t - \pi_t^{ind} \right) = \beta E_t \left(\pi_{t+1} - \pi_{t+1}^{ind} \right) + \kappa x_t$$
$$\mathcal{W}_t = -\frac{1}{2} \mathcal{V}_0 E_t \sum_{\tau=0}^{\infty} \beta^\tau \mathcal{L}_{0,t+\tau} + tip_t^{\mathcal{W}} \quad , \quad \mathcal{L}_{0,t} = \left(\pi_t - \pi_t^{ind} \right)^2 + \frac{\kappa}{\theta} \left(x_t - \phi_{x0} \right)^2$$

where $\pi_t^{ind} = \gamma_\pi \pi_{t-1}$, $r_t^n = \hat{r}_t^n$, $\phi_{x0} = \frac{1}{\theta(\omega+\sigma)}$ and $\mathcal{V}_0 = \theta \left(\omega + \sigma \right) \left(\frac{\varepsilon}{\chi\mu} \right)^{\frac{1-\sigma}{\omega+\sigma}} / \kappa$.

In order to compute the static wedge of the SWeB loss function, the nominal interest rate is adjusted to keep inflation fixed at the trend when satisfying the Zero StSt loglinearized equations. The SWeB static wedge is computed as $\mathcal{W} - \tilde{\mathcal{W}}$, where

$$\mathcal{W} = \bar{\mathcal{W}}_{\pi=0} - \frac{1}{2} \frac{\mathcal{V}_0}{1-\beta} \mathcal{L}_0 \quad , \quad \pi = \log \left(\bar{\Pi} \right)$$
$$\mathcal{L}_0 = \left(1 - \gamma_\pi \right)^2 \pi^2 + \frac{\kappa}{\theta} \left(x - \phi_{x0} \right)^2 \quad , \quad x = \frac{(1-\beta)(1-\gamma_\pi)}{\kappa} \pi$$

Using the benchmark calibration, panel (a) from figure 3 compares the True and the SWeB loss functions, showing the static wedges as the (red) vertical dotted lines. For simplification, the values of the loss functions at the zero inflation rate are normalized to zero. Note that both metrics agree pretty well both in level and curvature at the vicinity of the zero trend inflation (up to about $\bar{\pi} = 1\%$). After that, the curvature of the true loss function increases fast, while the SWeB curvature remains constant. Therefore the static wedges increase fast as the trend inflation rises.

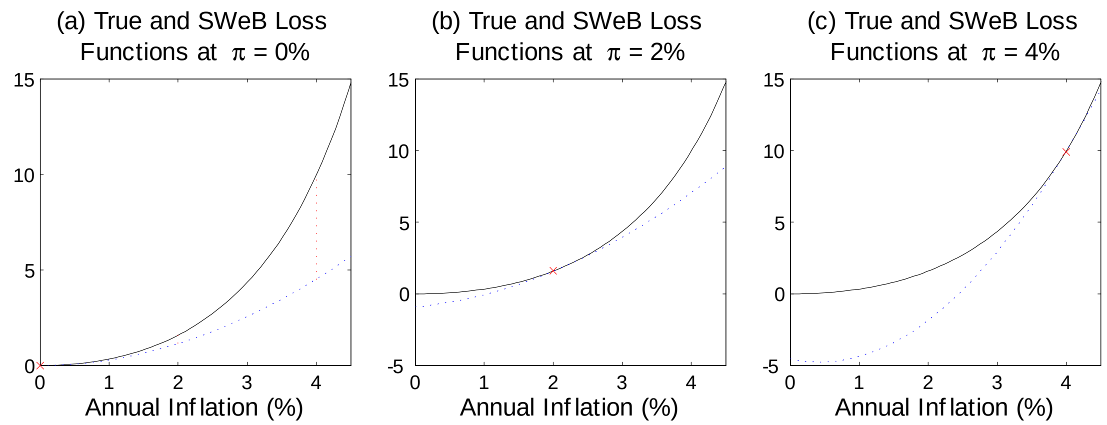

Figure 3: Loss functions

Note: True (black), SWeB and TIWeB (dotted blue), Static Wedge(dotted red).

The SWeB has a constant and small curvature relatively to the curvature of the

True loss function. The curvature ratio of both metrics $\left(\frac{\partial^2 TrueLoss}{\partial \bar{\Pi}^2} / \frac{\partial^2 SWeBLoss}{\partial \bar{\Pi}^2}\right)$ jumps from 2.6 at $\bar{\pi} = 2\%$ to 11.4 at $\bar{\pi} = 4\%$. Therefore, the stochastic wedge of the SWeB loss function is relevant. Unfortunately, there are no closed form expressions for the expected True loss function and for the stochastic wedge. Due to the model's nonlinearity, the stochastic wedge may only be obtained by simulating the model's higher order approximations. That is the reason why it is not shown in figure 3.

Regarding the TIWeB loss function, its static wedge is zero at the specific trend inflation $\bar{\pi}$ used to approximate the True loss function. Because it also has a constant curvature, the static wedge increases in the neighborhood of $\bar{\pi}$. However, the TIWeB loss wedges are smaller than the SWeB loss ones in the vicinities of each specific trend inflation $\bar{\pi}$. Panels (b) and (c) from figure 3 depict the performances of two TIWeB loss functions, approximated at $\bar{\pi}_a = 2$ and $\bar{\pi}_b = 4$.

4 Optimal policies when the trend inflation is not optimal

In order to simplify the task of having a tractable inflation targeting model, I define the objective of the inflation targeting central bank as to keep the unconditional mean of the inflation rate at the central target. Such modelling assumption is flexible enough, as it allows deviations from the central target, during different period lengths, as long as the unconditional mean is not affected. The inflation targeting objective $E\Pi_t = \bar{\Pi}$ is thus loglinearized as $E\hat{\pi}_t = 0$.

I search for time-consistent optimal policy rules under the unconditional and timeless perspectives. The unconditional approach is based on Damjanovic et al. (2005) adapted for the inflation targeting environment. The welfare-concerned central bank minimizes the unconditional expectation of the Lagrangian problem formed by the discounted sum of the TIWeB loss function, subjected to the IS, to the trend inflation NKPC curves and to $E\hat{\pi}_t = 0$.

In order to obtain the optimal policy under the timeless perspective, the inflation targeting objective must be slightly approximated. A problem arises here because the expectation operator considered in the welfare problem is conditioned on the information set at the initial period t, while the expectation operator of the targeting objective is unconditional.[23] A possible solution to this problem is to assume that the Ergodic Theorem holds: $E\hat{\pi}_t = \lim_{T\to\infty} \frac{1}{T+1}\sum_{\tau=0}^{T}\hat{\pi}_\tau$. This limiting sum can be reasonably approximated by $(1-\beta)\sum_{\tau=0}^{\infty}\beta^\tau\hat{\pi}_\tau$, provided that the subject discount factor β is close enough to unity. The inflation targeting objective is then approximated by $(1-\beta)\sum_{\tau=0}^{\infty}\beta^\tau\hat{\pi}_\tau = 0$, which implies $E_t(1-\beta)\sum_{\tau=0}^{\infty}\beta^\tau\hat{\pi}_{t+\tau} = 0$. That is what I use instead of $E\hat{\pi}_t = 0$.

Let λ_t, μ_t and Λ denote the Lagrange multipliers of the IS, the trend inflation NKPC, and the inflation target constraint. Λ is obtained by imposing the inflation targeting constraint into the equilibrium system described by the IS, the NKPC and the first order conditions. Since the model is linear, it is easy to verify that the inflation targeting constraint requires Λ to offset all constant terms in the equations. Therefore the following proposition describes both optimal policies under inflation targeting:

Proposition 3 *The unconditional and the timeless optimal policies under inflation targeting are time-consistent and represented by the following targeting rules:*

$$0 = \left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) + \frac{\digamma_4}{\theta}\left(\hat{x}_t - \beta\hat{x}_{t-1}\right) - (\vartheta - 1)\frac{\digamma_4}{\theta}\left(\digamma_7 - \beta\mathfrak{f}_{31}L\right)\frac{\beta\hat{x}_{t-1}}{(1-\beta\mathfrak{f}_{21}L)} \tag{10}$$

$$0 = \left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) + \frac{\digamma_4}{\theta}\left(\hat{x}_t - \hat{x}_{t-1}\right) - (\vartheta - 1)\frac{\digamma_4}{\theta}\left(\digamma_7 - \mathfrak{f}_{31}L\right)\frac{\hat{x}_{t-1}}{(1-\mathfrak{f}_{21}L)} \tag{11}$$

where L is the lag operator.

The coefficients, all of them positive, are defined in Appendix A. In particular, both $\digamma_4$ and ϑ converge to 1 as the trend inflation approaches zero. Therefore, the standard targeting rules (Damjanovic's unconditional and Woodford's timeless) are retrieved in the vicinities of the zero inflation steady state.

An interesting property of the time consistent optimal policies under trend inflation is that the targeting rule is more history-dependent than the one derived for the Zero StSt (see Woodford (2003)). Multiplying the first expression by $(1-\beta\mathfrak{f}_{21}L)$,

[23] Posing and solving the Lagrangian problem this way would require dealing with ratios such as $\frac{f(\varkappa_{t+j}|\mathcal{I}_t)}{f(\varkappa_{t+j})}$, where $f(\cdot)$ is the density function of the random variable $\varkappa_{t+j}$ and $\mathcal{I}_t$ is the information set at period t. The ratios never cancel out in the first order conditions.

or the latter by $(1 - \mathfrak{f}_{21}L)$, we realize that the rules depend on the second lag of the output gap and on the first lag of inflation even when there is no price indexation. Also important, the inertial intensity of both targeting rules increases as the trend inflation rises.

4.1 Simulations

Since β is typically very close to 1, both optimal policy rules imply almost indistinguishable dynamics and unconditional moments. Therefore, I use only the unconditional optimal policy to simulate the model, compute volatility schedules and obtain impulse responses to shocks. I use the benchmark calibration and assume that the structural shocks are independent, serially uncorrelated and normally distributed: $\xi_t^u \overset{iid}{\sim} N\left(0, \sigma_u^2\right)$ and $\xi_t^a \overset{iid}{\sim} N\left(0, \sigma_a^2\right)$.

Figure 4 depicts the relative standard deviations of the output gap and annualized inflation, and the ratio of their unconditional standard deviations, as the annual inflation trend rises from 0 to 5%. The relative standard deviations are defined as the unconditional standard deviation of the endogenous variables divided by the unconditional standard deviation of the internally consistent cost push shock $\xi_t^{cps} \equiv \varphi_3 \beta E_t \left(\xi_{t+1} - \xi_t\right)$. Using the relative measures avoids scale problems due to the amplitude of the shocks and to the calibrated parameter values.

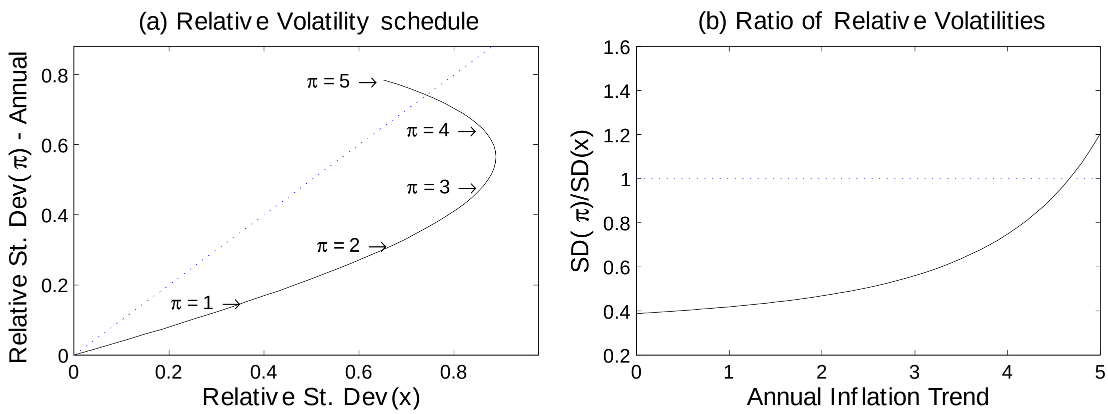

Figure 4: Relative volatilities (unconditional optimal policy)

Note: the dotted lines are the 45^o *schedule in panel (a), and the point at which both volatilities are the same in panel (b).*

The main lessons from the pictures are: (i) as in Ascari and Ropele (2007), both volatilities generally increase as the inflation trend rises; (ii) there is a threshold $\bar{\pi}_1 \in (4, 5)$ for the trend inflation above which the inflation volatility is larger than the output gap volatility; (iii) the schedule depicting the relative volatilities is hump-shaped: it initially increases with the trend inflation, but after a second threshold $\bar{\pi}_2 \in (3, 4)$ it is optimal to reduce the output gap volatility as the inflation trend rises; (iv) the ratio of the inflation volatility to the output gap volatility always increases as the trend inflation rises, irrespective of whether the inflation thresholds have been achieved or not; and (v) as the trend inflation rises, the central bank faces a policy trade off: it is not possible to completely offset the volatilities of both inflation and output gap. That is only possible when the trend inflation is zero.

The intuition behind the third lesson comes from the fact that the efficacy of the monetary policy in controlling the inflation rate is significantly reduced as the trend inflation rises: the coefficient of the output gap fades to zero as the inflation trend approaches the upper limit. Since the central bank's ability to affect the aggregate demand is not affected by the trend inflation, stabilizing the output gap eventually becomes the primary focus.

The qualitative effects of innovations in the utility ξ_t^u and in the technology ξ_t^a shocks are similar when the economy evolves around the Zero StSt: the endogenous variables respond in the same direction to positive innovations in both shocks. However, this is not the case under positive trend inflation. Consider the following reasoning: (a) the cost push shock ξ_t^{cps} only affects the dynamics if the trend inflation is not zero; and (b) ξ_t^{cps} is proportional to the expected growth of the aggregate shock $\xi_t \equiv \frac{(1-\sigma)\xi_t^a + \xi_t^u}{(\omega+\sigma)}$. Therefore, as long as the trend inflation is not zero, the direction of the direct impact of the technology shock on inflation depends on the traditional trade off between the income and substitution effects, captured in the model by the term $(1 - \sigma)$. If σ is larger than 1 then the technology shock causes the current inflation to rise, while the utility shock causes the inflation to fall. Of course, the effect of the technology shock can be either absent $(\sigma = 0)$ or negative $(\sigma < 1)$.

For that reason, I also consider the alternative cases $\sigma = 1$ and $\sigma = 0.5$ in the impulse responses exercises, depicted in figure 5.

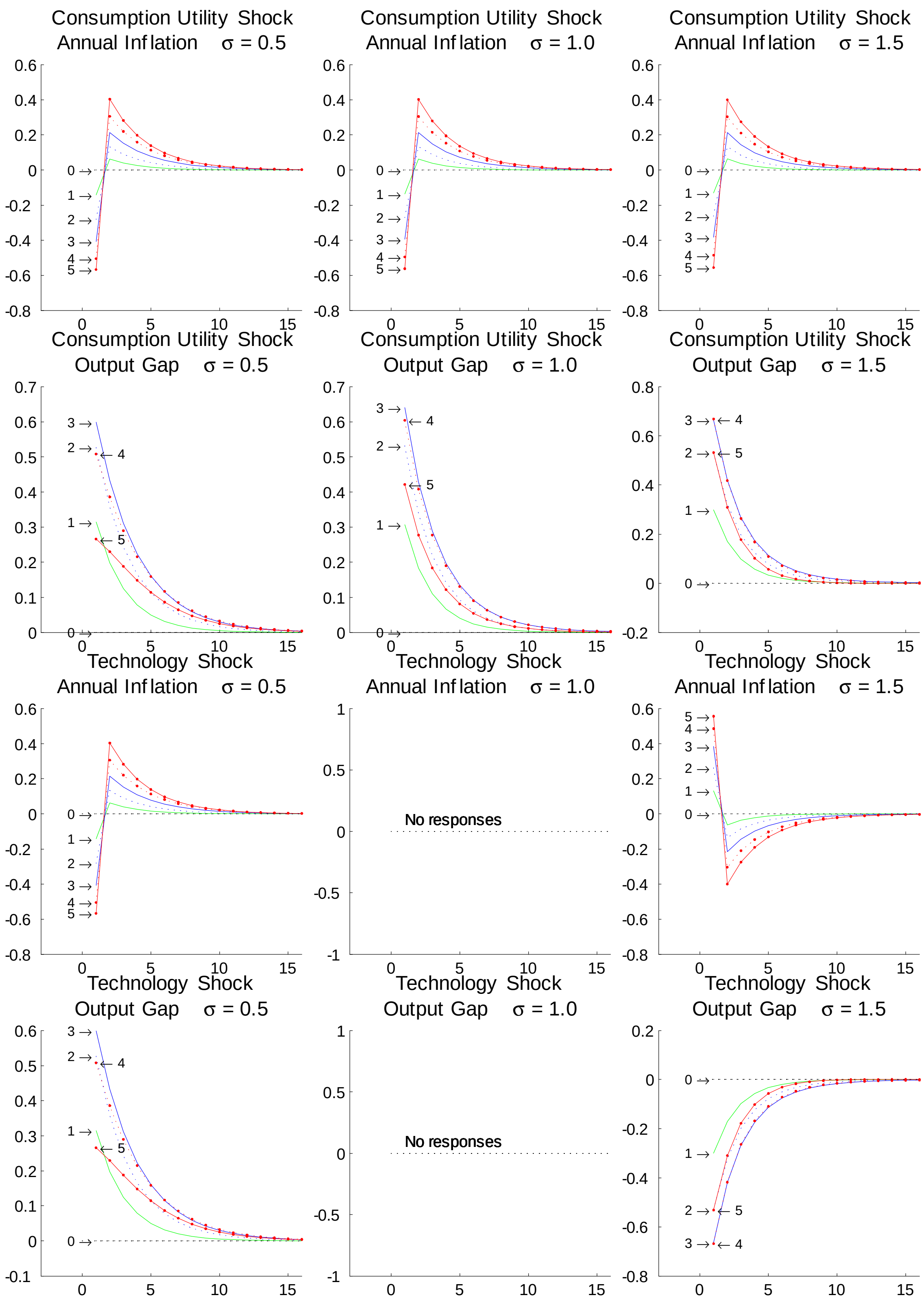

Figure 5: Relative responses to 1 St. Dev. utility and technology shocks.

Note: $\bar{\pi} = 0$ *(dotted black)*, $\bar{\pi} = 1$ *(green)*, $\bar{\pi} = 2$ *(dotted blue)*, $\bar{\pi} = 3$ *(blue)*, $\bar{\pi} = 4$ *(dotted red with circles)*, $\bar{\pi} = 5$ *(red with circles)*.

The pictures show the relative responses (divided by the standard deviation of the cost push shock ξ_t^{cps}) of the annualized inflation rate $4\hat{\pi}_t$ and the output gap $\hat{x}_t$ to one-time shocks to ξ_t^u and ξ_t^a, when the central bank follows the unconditional optimal policy. Since I only use shocks that also affect the efficient output (central planner equilibrium), there is no response when the trend inflation is zero. In this case, the monetary authority faces no policy trade off and it manages to stabilize both output gap and inflation. As expected, the relative responses to ξ_t^u and ξ_t^a are very similar when $\sigma < 1$, and the opposite when $\sigma > 1$. Also expected is that the amplitude of the inflation response increases as the trend inflation rises. As for the output gap, the responses on impact stop rising (in absolute value) as the inflation trend rises past a threshold. Beyond that point, responses on impact start to decline as the inflation trend rises. In line with result (*iii*), this fact reflects the optimal choice for enforcing output gap stabilization if the trend inflation is large enough.

The complexity and non-linearity of the coefficients make it very difficult for an analytical derivation of the thresholds.

5 Conclusion

The concavity of the welfare function increases rapidly as trend inflation rises above zero. As a consequence, the standard welfare-based loss function (Woodford (2003)) may not be the best metric for policy evaluation: it tends to underestimate the welfare loss under positive trend inflation.

Deriving a second order approximation of the welfare function under trend inflation, I show that a welfare-concerned monetary authority must be more *hawkish* by putting smaller weights on the (log-deviation) output gap volatility if she is assigned a non-zero inflation target to pursue. The time-consistent optimal policies (unconditional and timeless) for non-zero inflation targets become more inertial and history-dependent as the trend inflation rises.

The simulations show that there exists an inflation threshold beyond which the volatility of the output gap starts to fall. This result comes from the fact that large inflation trend levels reduce the ability of the central bank to control inflation using the demand channel. As a result, optimality switches the focus of the monetary authority towards reducing the volatility of the output gap if the inflation trend is sufficiently large.

I also highlight an important feature that has not been given proper attention in the literature of trend inflation. Under the Calvo price setting, with non-zero trend inflation, a cost push shock term naturally appears in the trend inflation Phillips curve and is generated by the same shocks that affect the efficient output (solution to the central planner problem). This holds even when nominal wages are flexible. As the inflation trend rises, the central bank loses the ability to simultaneously stabilize the inflation rate and the output gap.

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A The steady state under trend inflation

Considering that the gross trend inflation rate is $\bar{\Pi}$, the steady state levels for the equilibrium variables:

Steady state levels

$$\bar{I} = \beta^{-1}\left(\bar{\Pi}\right) = \frac{1}{\bar{q}} \quad ; \; \bar{\Pi}^{ind} = \bar{\Pi}^{(\gamma_\pi + \gamma)} \quad ; \; \bar{Y}^{n} = \left(\frac{\varepsilon}{\chi\mu}\right)^{\frac{1}{(\omega+\sigma)}}$$

$$\frac{\bar{N}}{\bar{D}} = \left(\frac{1-\alpha}{1-\bar{\alpha}}\right)^{\frac{1+\theta\omega}{\theta-1}} \quad ; \; \bar{X}^{\omega+\sigma} = \frac{1-\bar{\alpha}\beta\vartheta}{1-\bar{\alpha}\beta}\frac{\bar{N}}{\bar{D}} \quad ; \; \bar{Y} = \bar{X}\bar{Y}^{n}$$

$$\bar{\mathcal{W}} = \frac{\bar{u}-\bar{\tilde{v}}}{1-\beta} \quad ; \; \bar{u} = \frac{\bar{Y}^{1-\sigma}}{1-\sigma} \quad ; \; \bar{\tilde{v}} = \frac{\chi\bar{Y}^{(1+\omega)}\bar{\mathcal{P}}^{-\theta(1+\omega)}}{1+\nu} \quad ; \; \bar{\mathcal{P}}^{\theta(1+\omega)} = \frac{1-\bar{\alpha}\vartheta}{1-\alpha}\left(\frac{\bar{N}}{\bar{D}}\right)^{\frac{\theta(1+\omega)}{1+\theta\omega}}$$

Parameters

$\sigma \equiv$ reciprocal of intertemp elast substit	$\gamma_\pi \equiv$ coeff lag inf on index rule
$\nu \equiv$ reciprocal of the Frisch elasticity	$\gamma \equiv$ coeff inf trend on index rule
$\chi \equiv$ scale parameter on labor disutility	$\varepsilon \equiv$ labor elasticity prod function
$\theta \equiv$ lasticity of substit between goods	$\alpha \equiv$ Calvo degree of price rigidity

Coefficients and restrictions

$\mu \equiv \frac{\theta}{\theta-1}$	$\Phi_y \equiv 1 - \frac{(1-\bar{\alpha}\beta\vartheta)(1-\bar{\alpha})}{\mu(1-\bar{\alpha}\beta)(1-\bar{\alpha}\vartheta)}$	$F_1 \equiv 1 - (\vartheta - 1)\beta\varphi_2$
$\omega \equiv \frac{1+\nu}{\varepsilon} - 1$	$\Phi_\vartheta \equiv (\vartheta - 1)$	$F_2 \equiv \bar{\alpha} - (\vartheta - 1)\varphi_2$
$\delta \equiv \frac{1}{1-\gamma_\pi}$	$\mathcal{V}_\vartheta \equiv \bar{Y}^{1-\sigma}\frac{(1-\bar{\alpha}\vartheta)(\omega+\sigma)}{(1-\bar{\alpha})}\frac{\theta}{\bar{\kappa}}$	$F_3 \equiv \bar{\alpha} + (1-\bar{\alpha}\beta\vartheta)\varphi_2$
$\bar{\alpha}\vartheta < 1 \qquad \bar{\alpha} < 1$	$\phi_x \equiv \frac{1}{(\omega+\sigma)}\Phi_y$	$F_4 \equiv \frac{(1-\bar{\alpha})}{(1-\bar{\alpha}\vartheta)}\frac{1}{F_1}$
$\bar{\alpha} \equiv \alpha\left(\bar{\Pi}\right)^{(\theta-1)(1-\gamma_\pi-\gamma)}$	$\phi_\pi \equiv \frac{(1-\bar{\alpha})}{(1-\bar{\alpha}\vartheta)(1+\theta\omega)}\Phi_\vartheta$	$F_5 \equiv 1 + (\vartheta - 1)\theta\varphi_1$
$\vartheta \equiv \left(\bar{\Pi}\right)^{(1+\theta\omega)(1-\gamma_\pi-\gamma)}$	$\varphi_1 \equiv \frac{(1-\bar{\alpha})(1+\omega)}{(1+\theta\omega)}$	$F_6 \equiv \bar{\alpha}\vartheta + F_5$
$\bar{\kappa} \equiv \frac{(1-\bar{\alpha})(1-\bar{\alpha}\beta\vartheta)}{\bar{\alpha}}\frac{(\omega+\sigma)}{(1+\theta\omega)}$	$\varphi_2 \equiv \frac{(1-\bar{\alpha})(1-\sigma)}{(1+\theta\omega)\bar{\kappa}}$	$F_7 \equiv \theta\varphi_1 + \mathfrak{f}_{31}$
$\kappa \equiv \frac{(1-\alpha)(1-\alpha\beta)}{\alpha}\frac{(\omega+\sigma)}{(1+\theta\omega)}$	$\varphi_3 \equiv \frac{(1-\alpha)(1+\omega)}{(1+\theta\omega)}$	$\mathfrak{f}_{21} \equiv \frac{F_2}{F_1} \qquad \mathfrak{f}_{31} \equiv \frac{F_3}{F_1}$

B Second order (log) approximation of the welfare function

B.1 Second order approximation of $\hat{\mathcal{P}}_t$

The approximation of $\mathcal{P}_t$ is easier done using the auxiliary variable $F_t \equiv N_t/D_t$. The strategy is to make second-order approximations of the system described by (1) and (3), and solve for $\hat{\mathcal{P}}_t$ and $\hat{F}_t$.

The (log) second-order approximation of equation (1) is

$$\hat{F}_t \approx \frac{1}{2}\frac{(\theta-1)}{(1+\theta\omega)}\hat{F}_t^2 + \frac{\bar{\alpha}}{(1-\bar{\alpha})}(1+\theta\omega)\left[\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) + \frac{1}{2}(\theta-1)\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right)^2\right]$$

which implies that the a first-order approximation of F_t is $\hat{F}_t \approx \frac{\bar{\alpha}(1+\theta\omega)}{(1-\bar{\alpha})}\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right)$. I use this result to eliminate $\hat{F}_t^2$ from the previous second-order approximation and obtain

$$\hat{F}_t \approx \frac{\bar{\alpha}(1+\theta\omega)}{(1-\bar{\alpha})}\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) + \frac{1}{2}\frac{\bar{\alpha}(\theta-1)(1+\theta\omega)}{(1-\bar{\alpha})^2}\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right)^2$$

The (log) second-order approximation of equation (3) is

$$\begin{aligned}\hat{\mathcal{P}}_t - \tfrac{1}{2}\theta(1+\omega)\hat{\mathcal{P}}_t^2 &\approx \frac{(1-\bar{\alpha}\vartheta)}{(1+\theta\omega)}\hat{F}_t - \bar{\alpha}\vartheta\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) + \bar{\alpha}\vartheta\hat{\mathcal{P}}_{t-1} \\ &- \frac{1}{2}\frac{\theta(1+\omega)(1-\bar{\alpha}\vartheta)}{(1+\theta\omega)^2}\hat{F}_t^2 - \tfrac{1}{2}\theta(1+\omega)\bar{\alpha}\vartheta\left[\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) - \hat{\mathcal{P}}_{t-1}\right]^2\end{aligned}$$

and then the a first-order approximation of $\mathcal{P}_t$ is $\hat{\mathcal{P}}_t \approx \frac{(1-\bar{\alpha}\vartheta)}{(1+\theta\omega)}\hat{F}_t - \bar{\alpha}\vartheta\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right)$ $+\bar{\alpha}\vartheta\hat{\mathcal{P}}_{t-1}$. Replacing Φ_ϑ for $(\vartheta - 1)$ and using the last results to eliminate $\hat{F}_t$, $\hat{F}_t^2$ and $\hat{\mathcal{P}}_t^2$ from the previous second-order approximation, I obtain

$$\begin{aligned}\hat{\mathcal{P}}_t &\approx \bar{\alpha}\vartheta\hat{\mathcal{P}}_{t-1} - \frac{\bar{\alpha}\Phi_\vartheta}{(1-\bar{\alpha})}\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) - \frac{1}{2}\frac{\bar{\alpha}(1-\bar{\alpha}\vartheta)\left[(1+\omega\theta)+\theta(1+\omega)\Phi_\vartheta\right]}{(1-\bar{\alpha})^2}\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right)^2 \\ &+ \theta(1+\omega)\bar{\alpha}\vartheta(1-\bar{\alpha}\vartheta)\left[\frac{1}{(1-\bar{\alpha})}\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right)\hat{\mathcal{P}}_{t-1} - \tfrac{1}{2}\hat{\mathcal{P}}_{t-1}^2\right]\end{aligned}$$

If Φ_ϑ is assumed to be a first order term (the same order of the hatted variables), then $\Phi_\vartheta\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right)$ is a second order term and $\Phi_\vartheta\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right)^2$ is a third order term which may be ignored in the approximation. This assumption implies that $\hat{\mathcal{P}}_t$ has no first order dynamics, other than $\bar{\alpha}\vartheta\hat{\mathcal{P}}_{t-1}$. Hence, $\hat{\mathcal{P}}_{t-1}^2$ and $\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right)\hat{\mathcal{P}}_{t-1}$ are third order variables whenever Φ_ϑ is a first order term. In this case the second order approximation is simplified to:

$$\hat{\mathcal{P}}_t \approx \bar{\alpha}\vartheta\hat{\mathcal{P}}_{t-1} - \frac{\bar{\alpha}\Phi_\vartheta}{(1-\bar{\alpha})}\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) - \frac{\bar{\alpha}(1+\theta\omega)(1-\bar{\alpha}\vartheta)}{2(1-\bar{\alpha})^2}\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right)^2$$

Completing the squares and solving backwards to period $t = -1$, the expression is rewritten as $\hat{\mathcal{P}}_t \approx (\bar{\alpha}\vartheta)^{(t+1)}\hat{\mathcal{P}}_{-1} - \frac{1}{2}\frac{\bar{\alpha}(1-\bar{\alpha}\vartheta)(1+\theta\omega)}{(1-\bar{\alpha})^2}\sum_{\tau=0}^{t}(\bar{\alpha}\vartheta)^{t-\tau}\left(\hat{\pi}_\tau - \hat{\pi}_\tau^{ind} + \phi_\pi\right)^2 + tip$, where $\phi_\pi \equiv \frac{(1-\bar{\alpha})\Phi_\vartheta}{(1-\bar{\alpha}\vartheta)(1+\theta\omega)}$ and tip denote a term independent of policy. Let $tip_{-1}^{\mathcal{P}}$ denote a term independent of policy from period $t = 0$ onward. Note then that

$$\begin{aligned}\sum_{t=0}^{\infty}\beta^t\hat{\mathcal{P}}_t &= tip + \sum_{t=0}^{\infty}\beta^t(\bar{\alpha}\vartheta)^{(t+1)}\hat{\mathcal{P}}_{-1} - \frac{1}{2}\frac{\bar{\alpha}(1-\bar{\alpha}\vartheta)(1+\theta\omega)}{(1-\bar{\alpha})^2}\sum_{t=0}^{\infty}\beta^t\sum_{\tau=0}^{t}(\bar{\alpha}\vartheta)^{t-\tau}\left(\hat{\pi}_\tau - \hat{\pi}_\tau^{ind} + \phi_\pi\right)^2 \\ &= tip_{-1}^{\mathcal{P}} - \frac{1}{2}\frac{\bar{\alpha}(1-\bar{\alpha}\vartheta)(1+\theta\omega)}{(1-\bar{\alpha})^2}\sum_{\tau=0}^{\infty}\sum_{t=\tau}^{\infty}\beta^t(\bar{\alpha}\vartheta)^{t-\tau}\left(\hat{\pi}_\tau - \hat{\pi}_\tau^{ind} + \phi_\pi\right)^2\end{aligned}$$

Therefore the sum is rewritten as

$$\sum_{t=0}^{\infty}\beta^t\hat{\mathcal{P}}_t \approx tip_{-1} - \frac{(1-\bar{\alpha}\vartheta)(\omega+\sigma)}{2(1-\bar{\alpha})\bar{\kappa}}\sum_{\tau=0}^{\infty}\beta^\tau\left(\hat{\pi}_\tau - \hat{\pi}_\tau^{ind} + \phi_\pi\right)^2$$

where $tip_{-1} \equiv \left[\bar{\alpha}\vartheta\hat{\mathcal{P}}_{-1} + \frac{\bar{\alpha}\Phi_\vartheta\phi_\pi}{2(1-\bar{\alpha})(1-\beta)}\right] / (1 - \bar{\alpha}\beta\vartheta)$.

B.2 Second order approximations of u_t and $\tilde{\upsilon}_t$

A second order approximation of u_t is

$$u_t \approx \bar{Y}^{1-\sigma}\left[\hat{Y}_t + \tfrac{1}{2}(1-\sigma)\hat{Y}_t^2 + \hat{Y}_t\xi_t^u\right] + tip_t^u$$

where $tip_t^u \equiv \bar{u}\left(1+\xi_t^u+\frac{1}{2}\left(\xi_t^u\right)^2\right)$. Since $\left(Y_t^{ef}\right)^{-(\omega+\sigma)} = \frac{\chi}{\varepsilon}\exp\left(-\left(1+\omega\right)\xi_t^a-\xi_t^u\right)$, the aggregate disutility $\tilde{\upsilon}_t$ can be written as $\tilde{\upsilon}_t = \frac{1}{1+\omega}\left(\frac{Y_t}{\mathcal{P}_t^\theta}\right)^{(1+\omega)}\left(Y_t^{ef}\right)^{-(\omega+\sigma)}\exp\left(\xi_t^u\right)$. Therefore, its second order approximation is

$$\begin{aligned}\tilde{\upsilon}_t \approx\ & \bar{\tilde{\upsilon}} + \tfrac{1}{1+\omega}\bar{Y}^{(1+\omega)}\left(\bar{Y}^{ef}\right)^{-(\omega+\sigma)}\bar{\mathcal{P}}^{-\theta(1+\omega)}\left[(1+\omega)\hat{Y}_t - (\omega+\sigma)\hat{Y}_t^{ef}-\theta(1+\omega)\hat{\mathcal{P}}_t+\xi_t^u\right]\\ &+ \tfrac{1}{2(1+\omega)}\bar{Y}^{(1+\omega)}\left(\bar{Y}^{ef}\right)^{-(\omega+\sigma)}\bar{\mathcal{P}}^{-\theta(1+\omega)}\left[(1+\omega)\hat{Y}_t - (\omega+\sigma)\hat{Y}_t^{ef}-\theta(1+\omega)\hat{\mathcal{P}}_t+\xi_t^u\right]^2\end{aligned}$$

Note that $\bar{Y}^{(\omega+\sigma)}\left(\bar{Y}^{ef}\right)^{-(\omega+\sigma)}\bar{\mathcal{P}}^{-\theta(1+\omega)} = \left(1-\Phi_y\right)$, where Φ_y is assumed to be a first order disturbance term. Recall that $\hat{\mathcal{P}}_t$ is a second order variable whenever Φ_ϑ is assumed to be a first order disturbance term. Under such assumptions, the approximation is simplified to

$$\tilde{\upsilon}_t \approx \bar{Y}^{1-\sigma}\left[\hat{Y}_t - \Phi_y\hat{Y}_t - \theta\hat{\mathcal{P}}_t + \tfrac{1}{2}(1+\omega)\hat{Y}_t^2 - (\omega+\sigma)\hat{Y}_t\hat{Y}_t^{ef} + \hat{Y}_t\xi_t^u\right] + tip_t^\upsilon$$

where $tip_t^\upsilon = \bar{\tilde{\upsilon}}\left[1+\left(\xi_t^u-(\omega+\sigma)\hat{Y}_t^{ef}\right)+\frac{1}{2}\left(\xi_t^u-(\omega+\sigma)\hat{Y}_t^{ef}\right)^2\right]$.

B.3 Second order approximation of the welfare function

The instantaneous utility of the representative household is $\mathcal{U}_t = u_t - \upsilon_t$, and thus is the society instantaneous utility function. Moreover, note that $\left(Y_t^{ef}\right)^{(\omega+\sigma)} = \mu\left(Y_t^n\right)^{(\omega+\sigma)}$. It implies that $\hat{Y}_t^{ef} = \hat{Y}_t^n$. Using the last results, I obtain

$$\begin{aligned}\mathcal{U}_t =\ & \bar{Y}^{1-\sigma}\left(\theta(1-\Phi_y)\hat{\mathcal{P}}_t + \Phi_y\hat{Y}_t - \tfrac{1}{2}(\omega+\sigma)\hat{Y}_t^2 + (\omega+\sigma)\hat{Y}_t\hat{Y}_t^n\right)\\ &+ \bar{Y}^{1-\sigma}\left(\Phi_y\hat{Y}_t\xi_t^u - (\omega+\sigma)\Phi_y\hat{Y}_t\hat{Y}_t^n + \tfrac{1}{2}(1+\omega)\Phi_y\hat{Y}_t^2\right)\\ &+ \bar{Y}^{1-\sigma}(1-\Phi_y)\left(\tfrac{1}{2}\theta^2(1+\omega)\hat{\mathcal{P}}_t^2 - \theta(1+\omega)\hat{Y}_t\hat{\mathcal{P}}_t + \theta(\omega+\sigma)\hat{Y}_t^n\hat{\mathcal{P}}_t - \theta\hat{\mathcal{P}}_t\xi_t^u\right) + tip_t^u - tip_t^\upsilon\end{aligned}$$

Note that there are two linear terms: $\left(1-\Phi_y\right)\hat{\mathcal{P}}_t$ and $\Phi_y\hat{Y}_t$. Therefore, this correct approximation is not useful for policy evaluation using the structural log-linearized equations. That is why I parallel Woodford's (2003) analysis and assume Φ_ϑ and Φ_y to be first order disturbance terms. Since he assesses the Zero StSt, he only makes this assumption with respect to Φ_y. Under the Trend StSt, I need to expand this assumption towards Φ_ϑ. In this case, I have already shown that $\hat{\mathcal{P}}_t$ becomes a second order variable. This implies that $\Phi_y\hat{\mathcal{P}}_t$ is a third order term. Ignoring all third-order terms and completing the squares, I simplify the second-order approximation to

$$\mathcal{U}_t = \bar{Y}^{1-\sigma}\left(\theta\hat{\mathcal{P}}_t - \tfrac{1}{2}(\omega+\sigma)(\hat{x}_t-\phi_x)^2\right) + tip_t^{\mathcal{U}} + tip_t^u - tip_t^\upsilon$$

where $\phi_x \equiv \frac{1}{(\omega+\sigma)}\Phi_y$ and $tip_t^{\mathcal{U}} \equiv \frac{1}{2(\omega+\sigma)}\bar{Y}^{1-\sigma}\left[\Phi_y^2+(\omega+\sigma)^2\left(\hat{Y}_t^n\right)^2\right]$. Using the second-order approximation on $\sum_{t=0}^{\infty}\beta^t\hat{\mathcal{P}}_t$, the social welfare $\mathcal{W}_t \equiv E_t\sum_{\tau=0}^{\infty}\beta^\tau\mathcal{U}_{t+\tau}$ is then computed as

$$\mathcal{W}_t = -\tfrac{1}{2}\mathcal{V}_\vartheta E_t\sum_{\tau=0}^{\infty}\beta^\tau\left[\left(\hat{\pi}_{t+\tau}-\hat{\pi}_{t+\tau}^{ind}+\phi_\pi\right)^2 + \tfrac{(1-\bar{\alpha})}{(1-\bar{\alpha}\vartheta)}\tfrac{\bar{\kappa}}{\theta}\left(\hat{x}_{t+\tau}-\phi_x\right)^2\right] + tip_t^{\mathcal{W}}$$

where $\mathcal{V}_\vartheta \equiv \bar{Y}^{1-\sigma} \frac{(1-\bar{\alpha}\vartheta)(\omega+\sigma)}{(1-\bar{\alpha})} \frac{\theta}{\bar{\kappa}}$ and

$$tip_t^{\mathcal{W}} \equiv \theta \bar{Y}^{1-\sigma} tip_{t-1} + E_t \sum_{\tau=0}^{\infty} \beta^\tau \left(tip_{t+\tau}^{\mathcal{U}} + tip_{t+\tau}^{u} - tip_{t+\tau}^{v} \right)$$

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