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The Accuracy of Perturbation Methods to Solve Small Open Economy Models

Angelo M. Fasolo*

Abstract

This paper presents the evaluation of the canonical RBC models for small-open economies described in Schmitt-Grohé and Uribe (2003) when its solution is obtained by perturbation methods up to a third-order approximation. The models are evaluated in terms of accuracy of solution, ergodic moments, and local responses in extreme regions of the state vector. Results show that the gains from non-linear solutions are significant in terms of accuracy and with respect to the outcome of simulations: when compared to the linear approximation of the equilibrium conditions, non-linear solution generates very different dynamics of the stationary-inducing devices and smaller responses of consumption and output if the economy is in a state of low capital, both results highly dependent on the assumptions regarding preferences. However, significant changes in the main allocations of the economy when using different solutions appear only when GHH preferences are abandoned.

JEL Codes: C63, C68, E37, F41

Keywords: Small open economy; Stationarity; Perturbation methods; Non-linear Solutions

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1 Introduction

This paper evaluates the properties of the solutions generated by perturbation methods (first, second and third-order approximations of the equilibrium conditions) of the basic RBC models for small open economies described in Schmitt-Grohé and Uribe (2003)[21] – SGU (2003), henceforth. The solutions are evaluated in terms of accuracy, ergodic moments and local responses in extreme regions of the state variables describing the economy. The ergodic moments do not provide a complete picture of the models' dynamic, and given the use of alternative calibrations and non-linear solutions, it is worth to explore the properties of the basic RBC model for small open economies in situations of crisis, like a low level of capital stock. The analysis becomes even more important considering the evidence presented in Seoane (2011)[22], where the same models calibrated with parameters for Mexico do not provide equivalent responses as the log-linear solution in SGU (2003)[21] does.

The inclusion of high-order approximations in the analysis is justified by its increasing importance in research on asset pricing and DSGE models for small open economies (SOE's), especially the so-called Emerging Economies. As an example, Collard and Juillard (2001)[7] work with a very simple asset pricing model with a closed form representation in order to compare the properties of solutions based on perturbation methods of different orders. The authors find that the gains of solving the model with second and fourth-order approximations are significant, especially when the growth rate of dividends is very volatile or persistent over time. Recently, asset pricing models based on Epstein-Zin's (1989)[8] recursive preferences have been solved using perturbation methods, after Caldara, Fernández-Villaverde, Rubio-Ramírez and Yao (2009)[6] show the good accuracy of high-order approximation of DSGE models with these preferences. Examples of the solution of asset pricing models with recursive preferences using perturbation methods are Aldrich and Kung (2009)[3] and Malkhozov and Shamloo (2010)[23].

In terms of DSGE models, non-linear approximations have recently seen its use increased for estimation purposes, as these approximations allow the estimation of models with highly non-linear, non-Normal structure of shocks. There is evidence that non-linear solutions applied to DSGE models with Normal disturbances allow better identification of structural parameters[1]. High-order approximations are also used to correctly express the main features of a non-linear, non-Gaussian structure of a model. As an example combining all these features, Fernández-Villaverde, Guerrón-Quintana, Rubio-Ramírez and Uribe (2009)[9] solve a model with portfolio adjustment costs after adding time-varying volatility processes for the exogenous shocks. The use of a third-order perturbation is necessary, in this case, in order to capture the effect of changes in the process of volatility, as solutions based on low order approximations eliminate those effects.

In this paper, the most simple formulations of RBC models for SOE's are evaluated under different parameterization and different specifications for the utility function. The evaluation is based not only on the computation of ergodic moments and impulse response functions, like in SGU (2003)[21] and Seoane (2011)[22], but also on the accuracy of non-linear solutions and on how the model's dynamics changes once these non-linearities are incorporated in the solution. Simple models allow a clear understanding of the transmission mechanism of shocks without compromising the fitting of business cycle moments. For two main reasons, the initial simulations are based on Greenwood, Hercowitz and Huffman (1988)[12] utility function (GHH): first, it is the same utility function used in SGU (2003)[21] and Seoane (2011)[22], making the analysis here comparable with the literature; second, as GHH utility sets that labor supply is not a function of consumption level in the intratemporal Euler equation, the latter adoption of a conventional utility function allows the evaluation of the use of perturbation methods once wealth effects plays a significant role in labor supply.

Results show that the calibration proposed in SGU (2003)[21] for developed economies is invariant,

[1] See An (2008)[4] and Fernández-Villaverde and Rubio-Ramírez (2005)[10].

in terms of ergodic moments, up to second and third-order approximations, delivering equivalent results across models under GHH preferences. Ergodic moments are also invariant to reasonable increases in the volatility of the economy, given GHH preferences. However, the local analysis of the state space shows that different stationarity-inducing mechanism presents different dynamics when the model is solved using non-linear methods. More specifically, the use of high-order approximations to solve the "debt-elastic interest rate" model results in more volatile risk premium under GHH preferences. Simulations also show that, even in a model without significant real or nominal frictions, the impact of shocks over consumption and output are smaller when the economy is in an initial state of low capital. From the perspective of accuracy, the increase in volatility justifies the use of high-order approximation of the model, as the quality of the linear solution quickly deteriorates as the volatility of the economy increases.

The paper is organized as follows. The next section briefly presents the four SOE's RBC models used in SGU (2003)[21], focusing on the description of the mechanisms to make the current account stationary. Section 3 describes how to solve a model using perturbation methods and the procedures to simulate and evaluate accuracy in these models. It also describes the initial two sets of parameters used in sections 4 and 5. Section 4 shows the results of the baseline calibration, while section 5 explores the consequences of an increase in the economy's volatility, based on a new calibration for Emerging Economies. Section 6 departs from the use of GHH preferences, while trying to match the same set of moments from the calibration of section 5. Section 7 concludes.

2 The Model

The four variations of the model proposed in SGU (2003)[21] can be derived as an special case of the general formulation described below. This presentation follows exactly the same notation as the authors. In the general case, there is a large population of identical agents with the same period utility function, $U(c_t, h_t)$. These agents solve the following problem:

$$\max E_0 \sum_{t=0}^{\infty} \theta_t U(c_t, h_t)$$

$$s.t.: \quad \theta_{t+1} = \beta\left(c_t, h_t, \widetilde{c}_t, \widetilde{h}_t\right)\theta_t,\ t \geq 0,\ \theta_0 = 1$$

$$d_t = (1+r_{t-1})\, d_{t-1} - y_t + c_t + i_t + \Phi(k_{t+1} - k_t) + \tfrac{\psi_3}{2}\left(d_t - \overline{d}\right)^2$$

$$y_t = A_t F(k_t, h_t)$$

$$k_{t+1} = i_t + (1-\delta)\, k_t$$

$$r_t = r + \psi_2\left[\exp\left(\widetilde{d}_t - \overline{d}\right) - 1\right]$$

$$\log A_{t+1} = \rho \log A_t + \epsilon_{t+1}, \quad \epsilon_{t+1} \sim NIID\left(0, \sigma_\epsilon^2\right)$$

$$\lim_{j\to\infty} E_t \frac{d_{t+j}}{\prod_{s=1}^{j}(1+r_s)} \leq 0$$

In the problem, $\widetilde{c}_t$, $\widetilde{h}_t$, and $\widetilde{d}_t$ are the aggregate levels of consumption, hours worked and foreign debt, while $\overline{d}$ is the steady state level of debt. The first-order conditions of the model are taken in terms of individual consumption, c_t, hours worked, h_t, capital, k_{t+1}, foreign debt, d_t, and the discount factor, θ_t. The functional forms for $U(c,h)$, $\beta(c,h)$, $\Phi(k_{t+1} - k_t)$ and $F(k,h)$ completes the description of the model. Those are given by:

$$U(c,h) = \frac{\left[c - \omega^{-1}h^{\omega}\right]^{1-\gamma} - 1}{1-\gamma}$$

$$\beta(c,h) = \left[1 + c - \omega^{-1}h^{\omega}\right]^{-\psi_1}$$

$$F(k,h) = k^{\alpha}h^{1-\alpha}$$

$$\Phi(x) = \frac{\phi}{2} x^2$$

The equilibrium conditions common to all models, after dropping the superscript to identify aggregate values[2], are the following:

$$k_{t+1} = i_t + (1-\delta) k_t \tag{1}$$

$$y_t = A_t k_t^{\alpha} h_t^{1-\alpha} \tag{2}$$

$$\log A_{t+1} = \rho \log A_t + \epsilon_{t+1}, \quad \epsilon_{t+1} \sim NIID\left(0, \sigma_\epsilon^2\right) \tag{3}$$

$$\lim_{j \to \infty} E_t \frac{d_{t+j}}{\prod\limits_{s=1}^{j} (1 + r_s)} \leq 0 \tag{4}$$

Additionally, two identities are important regarding moments evaluated in the literature. The trade balance-output ratio and the current account-output ratio are defined as:

$$\frac{tb_t}{y_t} = \frac{y_t - c_t - i_t - \Phi(k_{t+1} - k_t)}{y_t}$$

$$\frac{ca_t}{y_t} = \frac{tb_t}{y_t} - \frac{r_{t-1} d_{t-1}}{y_t}$$

The first model considered is the "endogenous discount factor" with internalization (EDFI, henceforth), based on Mendoza (1991)[20]. In this model, agents set the discount factor, $\beta(c_t, h_t)$, when choosing the level of consumption and hours worked. Assume, additionally, that there is no cost in adjusting the portfolio and the interest rates are constant and exogenous (thus, $\psi_3 = 0$ and $r_t = r$, $\forall t$). Defining two Lagrange multipliers for the budget constraint and the functional form of the endogenous discount factor in the optimization problem, λ_t and η_t, the model is fully characterized by 9 variables: $\{c_t, h_t, k_{t+1}, i_t, y_t, d_t, A_t, \lambda_t, \eta_t\}$. The equilibrium conditions of the model are given by equations 1, 2 and 3, the transversality condition 4, plus the following set:

$$\lambda_t = \left[c_t - \omega^{-1} h_t^{\omega}\right]^{-\gamma} + \eta_t \psi_1 \left[1 + c_t - \omega^{-1} h_t^{\omega}\right]^{-\psi_1 - 1}$$

$$\lambda_t = \beta(c_t, h_t)(1+r) E_t \lambda_{t+1}$$

$$h_t^{\omega-1} = (1-\alpha) A_t k_t^{\alpha} h_t^{-\alpha}$$

$$\lambda_t \left[1 + \phi(k_{t+1} - k_t)\right] = \beta(c_t, h_t) E_t \lambda_{t+1} \left[\alpha A_{t+1} k_{t+1}^{\alpha-1} h_{t+1}^{1-\alpha} + 1 - \delta + \phi(k_{t+2} - k_{t+1})\right]$$

$$d_t = (1+r) d_{t-1} - y_t + c_t + i_t + \frac{\phi}{2}(k_{t+1} - k_t)^2$$

$$\eta_t = -U(c_{t+1}, h_{t+1}) + E_t \beta(c_{t+1}, h_{t+1}) \eta_{t+1}$$

The second model considers, again, endogenous discount factor, but this time without internalization (EDFE, henceforth) of the agents' levels of consumption and labor. When compared with the previous case, function $\beta\left(\widetilde{c}_t, \widetilde{h}_t\right)$ is set based on aggregate hours and consumption, $\left(\widetilde{c}_t, \widetilde{h}_t\right)$, making Lagrange multiplier η_t irrelevant. As a consequence, the model now is characterized by only eight variables and the new set of equilibrium conditions is given by:

$$\lambda_t = \left[c_t - \omega^{-1} h_t^{\omega}\right]^{-\gamma}$$

$$\lambda_t = \beta(c_t, h_t)(1+r) E_t \lambda_{t+1}$$

$$h_t^{\omega-1} = (1-\alpha) A_t k_t^{\alpha} h_t^{-\alpha}$$

[2]Given the assumption of a representative agent, superscripts are only adopted in the statement of the problem. Equations describing equilibrium conditions have all superscripts dropped in order to express aggregate values.

$$\lambda_t \left[1 + \phi \left(k_{t+1} - k_t\right)\right] = \beta \left(c_t, h_t\right) E_t \lambda_{t+1} \left[\alpha A_{t+1} k_{t+1}^{\alpha-1} h_{t+1}^{1-\alpha} + 1 - \delta + \phi \left(k_{t+2} - k_{t+1}\right)\right]$$
$$d_t = (1 + r) d_{t-1} - y_t + c_t + i_t + \frac{\phi}{2} \left(k_{t+1} - k_t\right)^2$$

The third model assumes that there is an endogenous component on the interest rate charged on financial markets, based on the level of aggregate debt of the economy ("debt elastic interest rate", DEIR). In this version of the model, the discount factor is constant over time and there is no portfolio adjustment cost (thus, $\theta_t = \beta^t$ and $\psi_3 = 0$, $\forall t$). The only additional necessary condition is a function specifying the path for interest rates in the model. As a consequence, the equilibrium conditions of the model are given by:

$$\lambda_t = \left[c_t - \omega^{-1} h_t^{\omega}\right]^{-\gamma}$$
$$\lambda_t = \beta E_t (1 + r_{t+1}) \lambda_{t+1}$$
$$h_t^{\omega-1} = (1 - \alpha) A_t k_t^{\alpha} h_t^{-\alpha}$$
$$\lambda_t \left[1 + \phi \left(k_{t+1} - k_t\right)\right] = \beta E_t \lambda_{t+1} \left[\alpha A_{t+1} k_{t+1}^{\alpha-1} h_{t+1}^{1-\alpha} + 1 - \delta + \phi \left(k_{t+2} - k_{t+1}\right)\right]$$
$$d_t = (1 + r_t) d_{t-1} - y_t + c_t + i_t + \frac{\phi}{2} \left(k_{t+1} - k_t\right)^2$$
$$r_t = r + \psi_2 \left[\exp\left(d_{t-1} - \overline{d}\right) - 1\right]$$

Finally, the last version of the closing device assumes the presence of portfolio adjustment costs (PAC) based on the household's level of foreign debt of the economy. Interest rates and the discount factor are constant in every period (thus, $\theta_t = \beta^t$ and $r_t = r$, $\forall t$). The equilibrium conditions of the model are given by:

$$\lambda_t = \left[c_t - \omega^{-1} h_t^{\omega}\right]^{-\gamma}$$
$$\lambda_t \left(1 - \psi_3 \left(d_t - \overline{d}\right)\right) = \beta (1 + r) E_t \lambda_{t+1}$$
$$h_t^{\omega-1} = (1 - \alpha) A_t k_t^{\alpha} h_t^{-\alpha}$$
$$\lambda_t \left[1 + \phi \left(k_{t+1} - k_t\right)\right] = \beta E_t \lambda_{t+1} \left[\alpha A_{t+1} k_{t+1}^{\alpha-1} h_{t+1}^{1-\alpha} + 1 - \delta + \phi \left(k_{t+2} - k_{t+1}\right)\right]$$
$$d_t = (1 + r) d_{t-1} - y_t + c_t + i_t + \frac{\phi}{2} \left(k_{t+1} - k_t\right)^2 + \frac{\psi_3}{2} \left(d_t - \overline{d}\right)^2$$

When comparing the four variations of the basic RBC model, it is important to notice the relevant variable used in the model to induce stationarity and the asset pricing consequences of each formulation. First, in terms of variables inducing stationarity, the DEIR and PAC models use the deviations of aggregate foreign debt, a stock variable, from its steady state value to create the gap between the interest rates and the household's discount factor necessary for a mean-reversing current account. On the other hand, EDFI and EDFE models are based on flows – consumption and hours – to generate the gap. The second feature of the formulations is that EDFI and EDFE models do not alter the relative prices of capital rent with respect to foreign interest rates, which is not the case in PAC and, more explicitly, DEIR models. In models based on the endogenous discount factor, changes in the discount factor equally affect both intertemporal Euler equations, resulting in the same interest rate-return of capital ratio. The PAC and DEIR models, as they are based on changes in foreign debt, distorts only the Euler equation derived from the optimal choice of debt. Thus, changes in foreign debt also affect the relative return of investment in the economy.

The next section describes the methodology for evaluating the model's solution accuracy, which relies on the errors of the Euler equations. Using the basic RBC model for closed economies, therefore with only one intertemporal condition, Aruoba, Fernández-Villaverde and Rubio-Ramírez (2006)[5] compare the accuracy of local and global solution methods using the Euler equation errors derived from the equilibrium conditions on capital. Lim and McNelis (2008)[19] report accuracy tests based on the intertemporal conditions of both capital and foreign bonds. However, Lim and McNelis (2008)[19] use only one of the devices to induce stationarity, turning the error derived from the capital Euler equation as the only comparable measure with this work. None of these papers consider the accuracy of perturbation methods applied to the most simple RBC model for SOE's.

3 Perturbation Methods and Solution's Accuracy

This section presents an overview of perturbation methods, a description of the main statistics used to evaluate the accuracy of the solutions and details of the calibration adopted in the exercises. While solutions based on perturbation methods are widespread in the literature, and there are many references in the literature with more details than those presented here, the main objective of the section is to highlight the consequences of using non-linear polynomials to approximate the equilibrium conditions of DSGE models. In terms of calibration, this section discusses the main differences of the calibration adopted here to the one adopted in Seoane (2011)[22].

Perturbation methods became a very popular tool to solve DSGE models due to the good accuracy of the solution around the steady state, high speed of computation and the low degree of difficulty to handle large models[3]. The main idea of the method is to approximate the true policy function using Taylor series expansions around the steady state of the model. Defining d, k and A as the steady state of the state variables of the model (foreign debt, capital and productivity, respectively), σ as the vector representing the standard deviation of shocks, Y_t as a general observable (output, consumption, hours, wages, among those), the objective is to fit polynomials of the form:

$$Y\left(d_t, k_t, A_{t-1}; \sigma\right) = \sum_{i,j,m,n} \frac{1}{(i+j+m+n)!} b_{ijmn} \left(d_t - d\right)^i \left(k_t - k\right)^j \left(A_{t-1} - A\right)^m \sigma^n \epsilon_t$$

The coefficients b_{ijmn} (where $i+j+m+n=q$ for the q^{th}-order of approximation) are obtained as the cross-derivatives of the equilibrium conditions with respect to the state variables, evaluated at the model's steady state:

$$b_{ijmn} = \left.\frac{\partial^{i+j+m+n} Y\left(d, k, A; \sigma\right)}{\partial d^i \partial k^j \partial A^m \partial \sigma^n}\right|_{d,k,A;0}$$

Those derivatives can be quickly computed using numerical methods in a computer, thus not requiring very sophisticated techniques to obtain good approximations of the model. From the equation above, it is also evident why non-linear solutions breakdown the assumption of certainty equivalence, derived as a consequence of linearized models: for linear models, the expected value of the expression above does not depend on cross-terms of state variables[4]; starting from second-order solutions, the expected value depends at least on quartic terms like $(k_t - k)(k_t - k)$, $(k_t - k)(d_t - d)$, $(k_t - k)(A_{t-1} - A)$, $(k_t - k)\sigma\epsilon_t$, and so on. These terms, when integrated to compute the expected value, constitute the covariance matrix of the state vector.

Another important issue is that the vector of state deviations $\begin{bmatrix} d_t - d & k_t - k & A_{t-1} - A & \epsilon_t \end{bmatrix}$ can be defined in terms of more general functions. The most popular among transformations, log-deviations results in an approximation of the percentage deviation of the state variable from the (log of the) steady

[3] Solutions of the models were obtained using the package Dynare++. For more details on Dynare, please refer to Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto and Villemot (2011)[1]. For Dynare++, see Kamenik (2011)[16]

[4] The linear solution of the model is given by:

$$Y\left(d_t, k_t, A_{t-1}; \sigma\right) = b_{1000}\left(d_t - d\right) + b_{0100}\left(k_t - k\right) + b_{0010}\left(A_{t-1} - A\right) + b_{0001}\sigma\epsilon_t$$

state of the model. In fact, Fernández-Villaverde and Rubio-Ramírez (2006)[11] propose a general change of variable procedure based on power functions such that there is an optimal set of coefficients for the transformed variables where the quality of the solution is at least comparable in certain dimensions to the one obtained using the finite elements method – a projection method well known for the quality of its solutions.

The use of high-order perturbation methods brings an important issue when simulating the time series from the policy functions: it is very common that the solution leads to explosive paths of the series. The main reason for that, as explained in Kim, Kim, Schaumburg and Sims (2008) [17], is the use of quartic functions discussed above to approximate the polynomial of the true policy function. The solution of the equations with these quartic terms might result in the existence of a second, explosive steady state in the model. As a consequence, if the variance of the shocks is high enough, the time paths might converge to the new steady state. Kim, Kim, Schaumburg and Sims (2008) [17] suggest a "pruning" procedure, simulating the quartic terms using a linear approximation of the policy function around the deterministic steady state. Den Haan and De Wind (2010) [14] show that, despite avoiding explosive paths, the "pruning" procedure induces more distortions to the approximated policy functions. In order to minimize these distortions, they suggest the use of a linear approximation around the stochastic steady state, determined by the solution of the high-order polynomial. In this paper, the simulations follow the original proposition of Kim, Kim, Schaumburg and Sims (2008) [17], as the use of a linear approximation around the stochastic steady state did not altered significantly the results presented in the following sections.

The evaluation of a model's solution accuracy is based on the computation of Euler equation errors. Consider the Euler equations for capital and bonds in the general representation of the model, expressed in terms of its perturbation parameter and state variables:

$$\begin{aligned}
\lambda\left[1-\psi_3\left(d'-\overline{d}\right)\right] &= \beta E_t R\lambda' \\
\lambda\left[1+\phi\left(k'-k\right)\right] &= \beta E_t \lambda'\left[R_k' + 1 - \delta + \phi\left(k\left(k',d',A';\sigma\right)-k'\right)\right] \\
R_k &= \alpha A\left(k\right)^{\alpha-1}\left(h\left(k,d,A;\sigma\right)\right)^{1-\alpha} \\
\lambda &= \lambda\left(k,d,A;\sigma\right) \\
h &= h\left(k,d,A;\sigma\right) \\
R &= R\left(k,d,A;\sigma\right) \\
\beta &= \beta\left(k,d,A;\sigma\right) \\
k' &= k\left(k,d,A;\sigma\right) \\
d' &= d\left(k,d,A;\sigma\right)
\end{aligned}$$

Given a solution method i, it is possible to express the error of the approximation of the policy functions as:

$$\begin{aligned}
err_i^k\left(k,d,A;\sigma\right) &= \lambda_i\left[1+\phi\left(k_i'-k\right)\right] - \beta_i E_t \lambda_i'\left[\alpha A_i'\left(k_i'\right)^{\alpha-1}\left(h_i'\right)^{1-\alpha} + 1 - \delta + \phi\left(k_i\left(k_i',d_i',A_i';\sigma\right)-k_i'\right)\right] \\
err_i^d\left(k,d,A;\sigma\right) &= \lambda_i\left[1-\psi_3\left(d_i'-\overline{d}\right)\right] - \beta_i E_t R_i\left(k,d,A;\sigma\right)\lambda_i'
\end{aligned}$$

Following Judd (1998)[15], the previous expression is solved for consumption using the (inverse of the) functional form adopted for $\lambda_i\left(k,d,A;\sigma\right)$ and the error expressed in percentage of $c_i\left(k,d,A;\sigma\right)$[5].

[5] For the case of EDFI model, there is not a closed form solution for consumption, as the marginal utility of consumption is a function of two terms:

$$\lambda_t = U_{c,t} + \eta_t \beta_{c,t}$$

It is common practice in the literature to express this error in absolute value and transformed by the logarithm in base 10. As a consequence, a value of -3 means that, for every 1,000 dollars of consumption measured in the model, one dollar is inaccurately measured; a value of -4 implies a one-dollar error for every 10,000 dollars of consumption, and so on. The analysis of Euler error equations allows verifying the solution's accuracy outside the steady state of the model and checking error patterns with the exact location of singularity points, where the quality of the approximated solution quickly deteriorates.

From the normalized Euler equation errors, two statistics are computed. First, trying to establish an upper bound for the Euler error (see Aruoba, Fernández-Villaverde and Rubio-Ramírez, 2006 [5]), the maximum value of the normalized Euler error is found among intervals of equally spaced 300 points for each state variable (thus, the maximum value is picked among $300^3 = 27,000,000$ possible points in the state space). The extreme values for intervals for each state variable are picked based on the histogram of the simulated data. Second, in order to access the overall accuracy of the solution method, the expected value of the errors is computed around relevant intervals describing the state variables of the model. In order to compute the expected value, normalized Euler errors are integrated with respect to the distribution of the most accurate solution based on the maximum value of the Euler error.

One test statistic often used to evaluate accuracy is the den Haan-Marcet (1994)[13] procedure. The test is based on Euler equation errors of simulated data, noting that the definition of the error as a function u_t such that $u_{t+1} = f(y_t) - \phi(y_{t+1}, y_{t+2}, ...)$, where the true model is written as $f(y_t) = E_t[\phi(y_{t+1}, y_{t+2}, ...)]$. Choosing any arbitrary function $h(x_t)$, the equality must hold:

$$E_t[u_{t+1} \otimes h(x_t)] = 0$$

The empirical counterpart of the equation above, generated from a solution method i, converges to zero, as the size of the simulated series, T, goes to infinity. Thus, under the null that the equation above holds, the following test statistic converges to a χ^2 distribution with degrees of freedom equal to the size of the instrument list $h(x_t)$:

$$T\left(B_t^i\right)'\left(A_t^i\right)^{-1}\left(B_t^i\right) \longrightarrow \chi^2_{nhx}$$

$$B_t^i = \frac{1}{T}\sum_{t=1}^{T} u_{t+1}^i \otimes h\left(x_t^i\right)$$

$$A_t^i = \frac{1}{T}\sum_{t=1}^{T} \left(u_{t+1}^i\right)^2 h\left(x_t^i\right) h\left(x_t^i\right)'$$

Since the whole procedure is based on simulated data, the recommended practice suggests to calculate the test-statistic a large number of times and report the number of times the limit of the confidence interval is violated by the test. In this paper, the statistic is computed from 5000 simulations of time series with 500 observations, around ten times larger than the usual sample size (annual frequency) in US business cycle empirical analysis. It is well worth noting that den Haan-Marcet statistic tends to reject the null hypothesis for large values of T: the larger the simulated sample, the higher is the probability that the test-statistic captures the fact that policy functions drawing time series are approximations of the true model. For the sake of robustness, the test is also run from 5000 simulations of a time series with only 100 observations. The list of instruments includes a constant and lagged values of productivity and capital. The inclusion of other lags or even the inclusion of debt simultaneously with capital generated a near-singularity in the computation of matrix A_t^i. This means that the information provided by one of the stock measures of the model (capital and foreign debt) is enough to describe the behavior of the

By assumption, the Euler error is calculated based on the consumption derived from the derivative of the utility function, $U_{c,t}$.

model up to that point.

3.1 Calibration

Three calibrations were used for the exercises: the baseline calibration of SGU (2003)[21], a transformation of the baseline calibration, called here a "high volatility" economy, and a variation of the "high volatility" economy to accommodate the use of CRRA preferences. The baseline calibration of SGU (2003)[21] combines traditional values in the RBC literature while trying to match moments of the Canadian economy – especially the volatilities of the current account-output ratio, investment and output. It is worth noting that the calibration adopted ensures that models with different stationarity-inducing devices are perturbed around the same deterministic steady state. The authors show that, up to a first order log-approximation of the model, all four model variations deliver very similar outcomes across models, with a solid matching of statistical moments when compared to data.

There are two main reasons for the use of a "high volatility" economy. First, notice that the economy used to match empirical moments, despite being a prototypical "small open economy", is not characterized by the patterns verified in the so-called Emerging Economies. Among other stylized facts, in Emerging Economies, output, investment and consumption are more volatile, while the trade-balance is clearly countercyclical[6]. Thus, while not necessarily emphasizing the matching of empirical data, an economy with larger volatility allow us to check the robustness of results in SGU (2003)[21] about the similarities of second-order moments across different models.

Another reason for the use of a "high volatility" economy is the evaluation of the departure from the certainty equivalence in linear solutions of the model: in a calibration designed for developed economies, with low volatility of output and consumption, the role of precautionary savings and risk aversion might be so small that a first-order approximation of the equilibrium conditions delivers appropriate results in terms of dynamics. However, when uncertainty plays a large role in model dynamics, a first-order approximation might not be enough to correctly evaluate the model's dynamics, as shown in Aruoba, Fernández-Villaverde and Rubio-Ramírez (2006)[5].

Table 1 – Parameters and Moments Matched in Calibration

Moments	Mexico	Canada	Parameters	"High Volatility" Economy	SGU (2003)
$\sigma(y_t)$	2.98	2.80	β	1/1.04	1/1.04
			ω	1.455	1.455
$\sigma(c_t)$	4.26	2.50	γ	2.00	2.00
			α	0.32	0.32
$\sigma(i_t)$	13.76	9.80	δ	0.1	0.1
			ψ_1	0.11135	0.11135
$\sigma(tb_t/y_t)$	3.67	1.90	ψ_2	0.00186	7.42e-4
			ψ_3	0.00171	7.42e-4
$\rho(tb_t/y_t, y_t)$	-0.57	-0.13	ϕ	0.00617	0.028
			ρ	0.13397	0.42
$\rho(y_t, y_{t-1})$	0.26	0.61	σ_ϵ	0.0307	0.0129

Obs.: Canada moments from SGU (2003)[21]. Moments of Mexico computed from dataset of Aguiar and Gopinath (2007)[2]

The calibration used in the "high volatility" economy was constraint by the assumption that the deterministic steady state of this economy and of the baseline economy must be exactly the same. As a

[6] See, for instance, table 1 of Aguiar and Gopinath (2007)[2], with data for a panel of 13 developed SOE's and 13 Emerging Economies.

consequence, only three parameters that have no effect in the computation of the deterministic steady state were altered from the baseline calibration: the persistence and volatility of productivity, ρ and σ, and the adjustment cost of capital, ϕ, were used to match the autocorrelation of output and the volatility of consumption and investment characterizing the Mexican economy. The strategy followed SGU (2003)[21], in the sense that the moments were computed using the EDFI model solved by log-linearizing the equilibrium conditions. Parameters ψ_2 and ψ_3, characterizing DEIR and PAC models, matched the simulated volatility of the trade balance-output ratio generated in the EDFI model.

There are three main reasons for using the Mexican economy as the prototypical "high volatility" economy. First, the second moments of that economy, shown in table 1, highlight some of the main differences between developed SOE's and Emerging Economies, namely: a) consumption is more volatile than output in Mexico, while the opposite is true in Canada; b) investment is more volatile in Mexico than in Canada; c) trade balance-output ratio is slightly countercyclical in Canada, while this correlation in Mexico is much more pronounced. The second reason for using data from Mexico is availability: the dataset compiled by Aguiar and Gopinath (2007)[2], available at the authors' websites, allows comparisons between this economy and a set of Emerging and Developed Economies. Finally, the Mexican dataset was used, at a quarterly frequency, by Seoane (2011)[22] to calibrate the set of parameters for all economies described here – thus, there is base for comparison with some of the results presented here[7].

The calibration strategy adopted here differs from Seoane (2011)[22], as the author is more interested in the fitting of the basic formulations of the RBC model for Emerging Economies. Seoane (2011)[22] estimated the common parameters across models using Simulated Method of Moments (SMM) on the second-order approximation of EDFI model, with the parameters describing DEIR and PAC models calibrated to match the volatility of the current account-GDP ratio in the second-order approximation of the model. In terms of results, the calibration of the same model using quarterly data for the Mexican economy shows that statistical moments significantly differ across models. In particular, DEIR model has problems in adjusting to the countercyclical trade balance.

Finally, the economy with CRRA preferences follows the same calibration strategy described above, trying to match the dynamics of the "high volatility" economy. The CRRA economy requires different assumptions with respect to the steady state of labor supply and the trade balance-output ratio, resulting in a different set of values even for the common parameters of the economy with GHH preferences. Details of the CRRA economy, including the calibration, are presented in section 6.

4 Properties of Baseline Economy

This section explores the solution accuracy of EDFE, EDFI, DEIR and PAC models and extends the evaluation of models' global and local dynamics in extreme points of the state variables. The analysis follow three steps: first, the accuracy of each solution is evaluated using the methods presented in Aruoba, Fernández-Villaverde and Rubio-Ramírez (2006)[5] up to a third-order approximation of the equilibrium conditions; second, the computation of relevant second moments of the model replicates the exercise presented in Seoane (2011)[22]; finally, the moments and impulse response functions are computed assuming initial states of low capital in each economy.

The statistical moments are computed based on simulations of the model. Each model is simulated for 50,000 periods, with the first 1,000 observations dropped and the remaining used to compute the moments[8]. Statistical moments were computed using first, second and third-order approximations of

[7] In order to generate moments comparable to SGU (2003) [21], data from Mexico in Aguiar and Gopinath (2007)[2] was reduced to annual frequency and the logarithm of data was detrended using a quadratic polynomial. This is the procedure applied in Mendoza (1991)[20] for data from Canada, where SGU (2003) [21] picked parameters and empirical moments to compare the outcome of the model.

[8] In order to make comparisons relevant, all simulations of ergodic moments started at the deterministic steady state of

the equilibrium conditions with variables in levels. In order to generate results comparable with SGU (2003)[21] and Seoane (2011)[22], simulated values were transformed to logarithms[9]. All simulations based on non-linear approximations of the policy functions incorporated the "pruning" procedure described in Kim, Kim, Schaumburg and Sims (2008) [17], as described in section 3.

Table 2 – Den Haan-Marcet Statistic – Baseline Calibration

Order	1		2		3	
	$< 5\%$	$> 95\%$	$< 5\%$	$> 95\%$	$< 5\%$	$> 95\%$
EDFE						
Bonds	5.26	4.74	5.02	4.84	5.00	4.86
Capital	5.28	5.06	5.06	4.76	5.12	4.78
EDFI						
Bonds	5.10	4.66	5.10	4.84	5.08	4.84
Capital	5.30	4.60	5.02	4.84	5.00	4.82
DEIR						
Bonds	5.16	6.30	5.10	5.50	5.12	5.46
Capital	5.14	7.04	4.90	6.02	4.86	6.74
PAC						
Bonds	5.20	6.08	5.18	5.72	5.16	5.68
Capital	5.12	7.10	5.02	5.08	5.00	5.78

Obs.: results show percentage of simulations below (above) the 5% (95%) critical value of the chi-square distribution with 3 degrees of freedom.

Starting with the characterization of solution's accuracy, table 2 presents the den Haan-Marcet test, showing that perturbation methods do indeed a good job characterizing the dynamics of the baseline economy. Irrespective of the model or the Euler equation evaluated, the distribution of the test statistic is most of the time located inside the 90% confidence interval. The distribution of the test is slightly skewed to the left in models based on the endogenous discount factor, while the distribution is slightly skewed to the right in models based on the level of foreign debt.

Results regarding the mean and the maximum value of the Euler equation errors are presented in table 3. The maximum value of the Euler equation error is computed using intervals of 300 equally spaced points between 0.94 and 1.06 for productivity, 3.1 and 3.7 for the capital stock and -1.15 and 1.95 for foreign debt, matching the extreme values presented in simulations. The maximum value of the Euler error is found, for all formulations, at the state with the lowest levels of capital combined with the lowest productivity. Table 3 highlights the significant improvement in the quality of the approximation resulting of high-order approximations, irrespective of the model. When measured in terms of average Euler equation errors, the gains are larger moving from the first to the second-order solution: in some cases, the largest error found for a second-order approximation is smaller than the average error computed for the linear solution.

The expected value of the linear solution in the baseline economy implies that there is a $1 error for every $24,150 spent in consumption. The gains of moving to a second and third order approximation are impressive, as the expected value of the Euler error implies a $1 error for every $3,665,000 for the second-order solution and $389,050,000 for a third-order solution. Even in the worst-possible outcome,

the model and the same seed generated the random numbers characterizing the productivity shock, ϵ_t.

[9] Appendix A shows some statistics comparing the models' solutions based on levels and log-perturbations of the variables. For all models and all perturbation orders, solutions in levels were preferred to log-perturbations of the equilibrium conditions. These results are in line with those presented in Aruoba, Fernández-Villaverde and Rubio-Ramirez (2006)[5] for the baseline RBC model for closed economies.

the maximum error for a third-order approximation implies a \$1 for every \$181,950, more than 7 times smaller than the expected error of the linear solution.

Table 3 – Euler Equation Errors – Baseline Calibration

Order	1		2		3	
	Max(EE)	E(EE)	Max(EE)	E(EE)	Max(EE)	E(EE)
EDFE						
Bonds	-3.099	-4.380	-4.750	-6.619	-5.991	-9.058
Capital	-3.108	-4.407	-4.758	-6.654	-5.993	-9.280
EDFI						
Bonds	-3.162	-4.324	-3.804	-6.567	-4.334	-8.576
Capital	-3.175	-4.361	-3.809	-6.659	-4.338	-8.978
DEIR						
Bonds	-3.184	-4.341	-4.282	-6.363	-5.080	-7.806
Capital	-3.146	-4.421	-4.174	-6.322	-4.326	-7.540
PAC						
Bonds	-3.138	-4.408	-4.771	-6.659	-6.000	-8.524
Capital	-3.144	-4.421	-4.783	-6.667	-6.001	-8.954

Obs.: results measured as logarithms of base 10: a value of -4 shows an error of \$1 for every \$10,000 spent in consumption; a value of -5 represents an error of \$1 for every \$100,000; and so on.

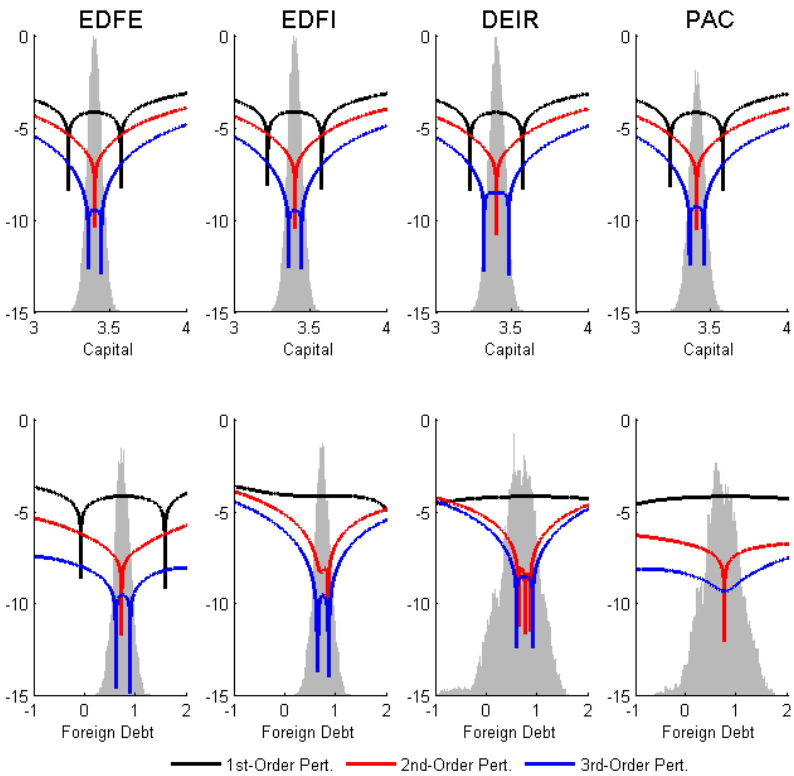

Figure 1: Euler Errors - Baseline Calibration

Figure 1 show the conditional mean of the Euler errors computed at two different dimensions of the state space: graphs in the first line of figure 1 assume that foreign debt and productivity are at the deterministic steady state value, with the horizontal axe showing the same range of capital in the histograms of figure 2; the second line of graphs follows the same procedure, with capital at the deterministic steady state and the horizontal axe showing the relevant values of foreign debt. The shaded area is the histogram of the third-order approximation, plotted here in order to capture the accuracy of the solution method in the relevant region of the state space. From the picture, it is clear that linear approximations of the equilibrium conditions are dominated by high-order solutions, even when moving far outside the deterministic steady state. Non-linear approximations errors are between two and four orders of magnitude smaller than the linear perturbation of the model, depending on the region of the state space being compared.

The picture also shows that the largest gains seems to come from a better characterization of the policy functions describing foreign debt: according to table 3, the worst fit of a third-order solution is found at the DEIR model, based on average Euler errors; in that model, the similarity between the plots of the second and third-order solutions' errors across the relevant space of foreign debt suggest it is the description of this variable that conditions the quality of the approximation.

Table 4 – Ergodic Moments – Baseline Calibration

Model	EDFI			EDFE			DEIR			PAC		
Order	1	2	3	1	2	3	1	2	3	1	2	3
Volatilities												
$\sigma(y_t)$	3.06	3.05	3.05	3.06	3.05	3.05	3.07	3.07	3.07	3.07	3.07	3.07
$\sigma(c_t)$	2.31	2.31	2.31	2.31	2.31	2.31	2.62	2.64	2.64	2.60	2.59	2.59
$\sigma(i_t)$	9.20	9.20	9.20	9.20	9.20	9.20	9.12	9.13	9.12	9.12	9.12	9.12
$\sigma(h_t)$	2.10	2.10	2.10	2.10	2.10	2.10	2.11	2.11	2.11	2.11	2.11	2.11
$\sigma\left(\frac{tb_t}{y_t}\right)$	1.54	1.54	1.54	1.54	1.54	1.54	1.75	1.77	1.77	1.73	1.72	1.72
$\sigma\left(\frac{ca_t}{y_t}\right)$	1.46	1.46	1.46	1.46	1.46	1.46	1.45	1.45	1.45	1.45	1.45	1.45
Serial correlations												
$\rho(y_t)$	0.61	0.61	0.61	0.61	0.61	0.61	0.61	0.61	0.61	0.61	0.61	0.61
$\rho(c_t)$	0.69	0.69	0.69	0.69	0.69	0.69	0.77	0.77	0.77	0.77	0.76	0.76
$\rho(i_t)$	0.08	0.08	0.08	0.08	0.08	0.08	0.07	0.07	0.07	0.07	0.07	0.07
$\rho(h_t)$	0.61	0.61	0.61	0.61	0.61	0.61	0.61	0.61	0.61	0.61	0.61	0.61
$\rho\left(\frac{tb_t}{y_t}\right)$	0.32	0.32	0.32	0.32	0.32	0.32	0.49	0.51	0.51	0.48	0.48	0.48
$\rho\left(\frac{ca_t}{y_t}\right)$	0.30	0.30	0.30	0.30	0.30	0.30	0.32	0.33	0.32	0.32	0.32	0.32
Correlation with output												
$\rho(c_t, y_t)$	0.94	0.94	0.94	0.94	0.94	0.94	0.85	0.84	0.85	0.86	0.86	0.86
$\rho(i_t, y_t)$	0.66	0.66	0.66	0.66	0.66	0.66	0.67	0.67	0.67	0.67	0.67	0.67
$\rho(h_t, y_t)$	1	1	1	1	1	1	1	1	1	1	1	1
$\rho\left(\frac{tb_t}{y_t}, y_t\right)$	-0.01	-0.01	-0.01	-0.01	-0.01	-0.01	-0.04	-0.03	-0.04	-0.04	-0.03	-0.03
$\rho\left(\frac{ca_t}{y_t}, y_t\right)$	0.02	0.02	0.02	0.02	0.02	0.02	0.04	0.05	0.04	0.05	0.05	0.05

Obs.: Parameters from SGU (2003)[21], shown in table 1. Standard-deviations in percent per year.

Moving on to the analysis of business cycle moments, table 4 shows the ergodic moments using the parameters from SGU (2003)[21]. Following Seoane (2011)[22], results in terms of unconditional moments are very similar, both across models and across perturbation order. SGU (2003)[21] had established the similarities of the ergodic moments using log-linearization of the equilibrium condition as a solution method. Indeed, when comparing results in table 4 with results from table 3 in SGU (2003)[21], even the small discrepancies across models are captured with the use of simulated data. In Seoane (2011)[22], the result in SGU (2003)[21] is expanded to second and third-order approximations of the models. As the solution based on the third-order approximation is a refinement of the stable path found in the second-order approximation, it could be argued that the ergodic distribution of the model could be affected by the solution method. However, given the calibration matching Canadian moments, the use of non-linear solutions does not alter the main results.

The similarities of the ergodic moments are also reflected in the empirical distribution of the state variables. Figure 2 shows the histograms of the distribution of capital and foreign debt. In the picture, the distributions of foreign debt for DEIR and PAC models show larger variance and a slight shift to lower levels of debt when moving to second and third-order approximations. This shift reflects the effects of relaxing the assumption of certainty equivalence imposed in the linear solution: when risk aversion plays a role in the model, households accumulate smaller amounts of debt in those two formulations, even assuming a temporary creditor position in international foreign debt markets.

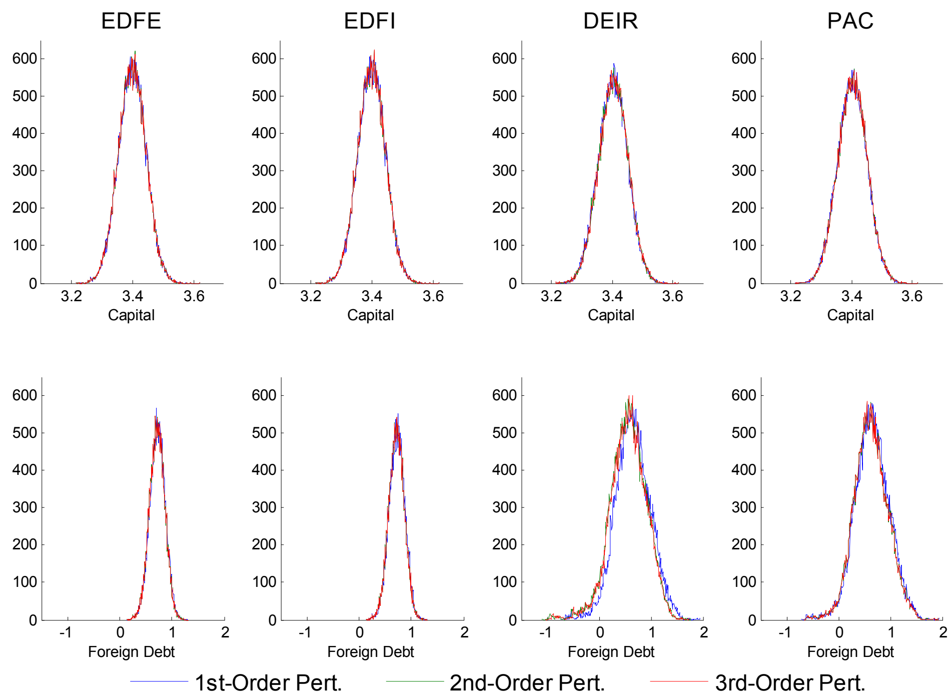

Figure 2: Histogram – Baseline Calibration

Does the improvement in solution's accuracy result in different dynamics of the baseline RBC model for SOE's? From the perspective of the ergodic moments presented in table 4, the answer is clearly negative. Seoane (2011)[22] shows that the impulse response functions of a productivity shock in the model simulated with non-linear perturbations are virtually the same as the outcome of the linear economy. Here, figure 3 shows a different computation of the impulse response functions of the model. Following the suggestion in Koop, Pesaran and Potter (1996)[18], impulse response functions generated from a positive productivity shock are calculated conditional on the initial state vector being located at the bottom 5% quantile of the distribution of capital. While the conditional expectation does not affect the IRFs of linear solutions when compared to the unconditional IRFs, non-linear solutions might result in history-dependent IRFs. Figure 3 show the cumulative IRFs for output and investment and IRF of the respective stationary-inducing mechanism in each model: the stochastic discount factor for EDFI and EDFE models, the interest rate charged in domestic markets for DEIR model and the interest rate corrected by the adjustment cost in PAC model.

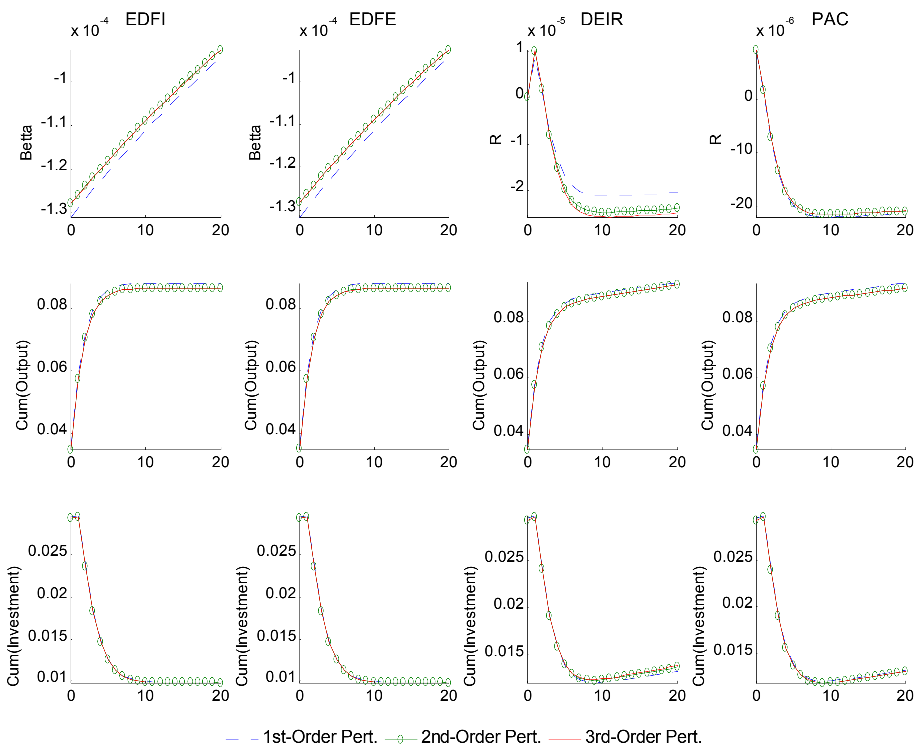

Figure 3: Impulse response functions – Productivity shock: quantile(k_0) < 5% – Baseline Calibration

In terms of output, the major effects of solving the model using non-linear solutions is on impact: the effect of a productivity shock in the region of low capital is approximately 3% smaller than the effect in the linearized economy, with the gap vanishing as time progresses. Despite the low value, the impact must be seen in a proper perspective as the policy functions of the model without any mechanism to amplify the propagation of shocks are essentially linear in the relevant regions of the state. Another interesting feature presented by figure 3 is the larger volatility of the risk premium on interest rates when using non-linear solutions to solve DEIR model. From this perspective, the cumulative effect on output offers a better intuition on the local dynamics of the model after a productivity shock. Notice how the deviations of interest rates generate in DEIR and PAC models a persistent effect, with cumulative output still growing 10 years after the initial shock. The low interest rates in domestic markets changes the intertemporal allocations from bond holdings to investment, generating the increase on output.

Table 5 summarizes the main the results of this section in terms of quantitative impacts of solving a model with non-linear solutions. First, table 5 generalizes results of figure 3, showing the difference in percentage points between impulse response functions based on linear and non-linear solutions for output, consumption and investment. The gap is described not only for the critical value of 5% of the capital stock distribution, but also for different thresholds up to the median of the distribution. Despite small, even at the median of the distribution of capital there is a small difference (between -0.62% and -0.82%) on the impact of a shock in output between linear and non-linear solutions. It is also worth noting that the absence of wealth effects in the model with GHH preferences makes the gap between linear and non-linear impulse response functions for consumption very similar to the gap for output. In terms of investment, note how DEIR model presents a significant gap between IRFs at the cumulative impact even with capital close to the median of the distribution. This gap in cumulative effect is a result of the persistent deviation of the risk premium from the steady state and the resulting change in the relative price between capital rent and bond returns. The gap is also larger in the PAC model, when compared to EDFE and EDFI models, but the effect on relative prices is magnified in DEIR model due to the functional form adopted to describe the behavior of risk premium.

Table 5 – Conditional Moments – Baseline Calibration

	EDFI	EDFE	DEIR	PAC
Gap between IRFs: Output, 3^{rd} to $1^{st}-$order: impact (cumulative)				
Quantile(K_0) < 0.05	-2.80% (-1.82%)	-2.98% (-1.93%)	-2.84% (-1.47%)	-2.84% (-1.95%)
Quantile(K_0) < 0.10	-2.42% (-1.58%)	-2.42% (-1.58%)	-2.35% (-1.18%)	-2.34% (-1.57%)
Quantile(K_0) < 0.25	-1.61% (-1.03%)	-1.61% (-1.03%)	-1.49% (-0.69%)	-1.50% (-0.93%)
Quantile(K_0) < 0.50	-0.82% (-0.50%)	-0.82% (-0.50%)	-0.62% (-0.22%)	-0.63% (-0.38%)
Gap between IRFs: Consumption, 3^{rd} to $1^{st}-$order: impact (cumulative)				
Quantile(K_0) < 0.05	-2.80% (-2.04%)	-2.99% (-2.18%)	-2.67% (-1.24%)	-2.81% (-2.04%)
Quantile(K_0) < 0.10	-2.42% (-1.77%)	-2.43% (-1.79%)	-2.20% (-0.99%)	-2.31% (-1.64%)
Quantile(K_0) < 0.25	-1.61% (-1.16%)	-1.62% (-1.17%)	-1.39% (-0.54%)	-1.48% (-0.98%)
Quantile(K_0) < 0.50	-0.82% (-0.57%)	-0.82% (-0.57%)	-0.56% (-0.13%)	-0.62% (-0.39%)
Gap between IRFs: Investment, 3^{rd} to $1^{st}-$order: impact (cumulative)				
Quantile(K_0) < 0.05	-0.49% (-0.63%)	-0.53% (-0.67%)	-0.60% (2.67%)	-0.56% (-1.17%)
Quantile(K_0) < 0.10	-0.42% (-0.54%)	-0.42% (-0.53%)	-0.50% (2.27%)	-0.46% (-0.94%)
Quantile(K_0) < 0.25	-0.28% (-0.34%)	-0.28% (-0.33%)	-0.31% (1.73%)	-0.29% (-0.56%)
Quantile(K_0) < 0.50	-0.15% (-0.15%)	-0.14% (-0.14%)	-0.14% (1.11%)	-0.12% (-0.22%)
Debt-to-GDP ratio: median – standard-deviation (SS = 0.50)				
$1^{st}-$order	0.490; 0.10	0.491; 0.10	0.434; 0.23	0.438; 0.22
$2^{nd}-$order	0.484; 0.10	0.484; 0.10	0.383; 0.24	0.418; 0.22
$3^{rd}-$order	0.484; 0.10	0.484; 0.10	0.384; 0.24	0.418; 0.22

Second, the final block of results presented in table 5 shows that the effects of precautionary savings and risk aversion, present in non-linear solutions, result in a reduction of the median debt-to-GDP ratio up to 5 percentage points, when compared to results based on the linear approximation of equilibrium. For all models, the median of the distribution of the debt-to-GDP ratio is below the steady state of the economy, but this gap for DEIR and PAC models is not only significant, but increases with the use of non-linear solutions. On the debt-to-GDP ratio, it is also worth mentioning that a second-order approximation seems to be enough to adjust the level of this variable for the nonlinearity of the model, as results on the median and standard-deviation of this ratio are very similar between second and third-order approximations.

5 The "High Volatility" Economy

As shown in section 3.1, the "high volatility" economy is characterized by a significant increase in volatility of the productivity shock, small adjustment cost of investment and low persistence of output. The main idea of the "high volatility" economy is to evaluate the quality of the solution once the overall volatility of the economy is increased to reasonable values found in the literature, at the expense of the match of additional moments of data. From this perspective, the "high volatility" economy replicates second moments of Emerging Economies in a baseline model suited for developed economies. As an example, the "high volatility" calibration shows a positive correlation between trade balance and current account with output, when the correlation for the baseline economy, fitting data for Canada, was close to zero and negative in data for Mexico. The adjustment of the model in order to fit the correlation would come at the cost of changing other structural parameters and, as a consequence, the deterministic steady state of the model. The analysis of fitting of the basic RBC model is presented in Seoane (2011)[22], where the author estimates the model to match a complete set of moments of the Mexican economy.

The den Haan-Marcet statistic for the "high volatility" economy, presented in table 6, shows that the increase in the volatility of shocks indeed seems to affect the accuracy of a model's solution. The distribution of the test statistic remains close to the boundaries of the 90% confidence interval only for models with endogenous discount factor. Indeed, for EDFE and EDFI models, the skewness to the left of the distribution of the test is still present in the linear approximation, with a better adjustment in second and third-order solutions. In both DEIR and PAC models, the boundaries of the test statistic are violated in at least one dimension, irrespective of the order of approximation used to compute the solution. Due to the bad fit of DEIR model, sometimes with worse statistics as the order of approximation increases, it is worth noting that the den Haan-Marcet test is weak, in the sense that the size of the simulation samples and the selection of instruments are determinant of its results, as discussed in section 3. From this perspective the analysis of the maximum and the average Euler errors becomes critical to access the accuracy of solutions.

Table 6 – Den Haan-Marcet Statistic – "High Volatility" Economy

Order	1		2		3	
	$< 5\%$	$> 95\%$	$< 5\%$	$> 95\%$	$< 5\%$	$> 95\%$
EDFE						
Bonds	5.44	4.58	5.08	4.94	4.96	4.94
Capital	5.84	3.30	5.14	4.88	5.24	4.80
EDFI						
Bonds	5.44	4.58	4.80	5.16	4.88	5.18
Capital	5.28	3.94	5.14	4.86	5.22	4.82
DEIR						
Bonds	3.92	7.52	3.84	8.20	3.82	8.28
Capital	6.60	2.60	4.00	9.22	3.66	10.90
PAC						
Bonds	4.00	7.70	3.68	8.46	3.66	8.40
Capital	7.06	2.12	3.84	9.06	3.78	8.84

Obs.: results show percentage of simulations below (above) the 5% (95%) critical value of the chi-square distribution with 3 degrees of freedom.

Results regarding the mean and the maximum value of the Euler equation errors are presented in table 7. Those values are computed using intervals of 300 equally spaced points between 0.85 and 1.15 for productivity, 3.10 and 3.70 for the capital stock and -1.50 and 2.00 for foreign debt. These intervals match the distributions presented in the histogram of figure 5. Table 7 highlight two important results: first, the significant improvement in the quality of the approximation resulting of high-order approximations, irrespective of the model formulation; second, the need for high-order solutions as the variance of the economy increases, shown by the comparison of the expected value of Euler errors from table 7 with results for the baseline economy. With respect to the increased accuracy with high-order solutions, notice that the maximum error with a second-order perturbation is always smaller than the average error for the linear solution. In fact, the linear solution is very poor as a whole, as the gap between the maximum and the average Euler error is very small.

Comparing the results of table 7 with results from the baseline economy in table 3, notice that the expected value and the maximum value of Euler errors are higher for the "high volatility" economy than the errors for the baseline economy. The expected value for the Euler error in linear solutions for the "high volatility" economy implies a solution error of approximately $1 for every $3,150 spent in consumption, an error much higher than the one found with the baseline calibration. It is worth noting, however, that, depending on the model, the maximum Euler error found in table 3 is similar to the value presented in table 7, meaning that in some dimensions of the states (normally, the extremes of the distribution), the quality of the approximation is very similar, irrespective of the volatility of the economy.

Table 7 – Euler Equation Errors – "High Volatility" Calibration

Order	1		2		3	
	Max(EE)	E(EE)	Max(EE)	E(EE)	Max(EE)	E(EE)
EDFE						
Bonds	-3.159	-3.495	-4.655	-6.219	-5.985	-8.469
Capital	-3.162	-3.500	-4.651	-6.202	-5.912	-8.576
EDFI						
Bonds	-3.164	-3.452	-3.531	-6.070	-4.000	-7.707
Capital	-3.169	-3.459	-3.542	-6.167	-4.008	-7.896
DEIR						
Bonds	-3.166	-3.473	-4.098	-6.129	-4.027	-6.929
Capital	-3.242	-3.505	-3.588	-5.668	-3.781	-7.049
PAC						
Bonds	-3.240	-3.507	-4.652	-5.869	-5.979	-7.889
Capital	-3.246	-3.505	-4.646	-5.772	-5.976	-7.949

Obs.: results measured as logarithms of base 10: a value of -4 shows an error of $1 for every $10,000 spent in consumption; a value of -5 represents an error of $1 for every $100,000; and so on.

As it was also the case for the baseline economy, DEIR model is the one that generates the worst fit when using high-order solutions. The improvements from first to second-order approximations are smaller in the Euler equations in the PAC and DEIR models, with larger gains showing when moving from second to third-order approximations of the policy functions. It is also worth noting that DEIR model has a very poor adjustment, when compared to other formulations, at the boundaries of the state space, as the maximum value of the error is the largest across the four models.

The picture showing the evolution of the Euler equation errors across different dimensions of the state vector provides more intuition about the loss of fitting of perturbation methods once the overall volatility of the economy is increased. Linear approximations quickly deteriorates with the increase in volatility, as the Euler error is flat all over the relevant values of the state space. The graph for linear solution confirms the intuition derived from table 7 when comparing the maximum and the average Euler error for linear approximations. While the gains from moving away of the first-order approximation of the model are evident, the gains of moving from the second to the third-order approximation are more concentrated on the tails of the distribution of the state vector. Comparing figure 4 with the plot for the baseline economy (figure 1), note that the Euler errors for third-order approximations usually break the -10 threshold in the baseline calibration[10], showing a very good solution for a large portion of the state. With the increase in volatility, the -10 limit is barely reached in some dimensions of non-linear solutions.

With respect to business cycle moments, the "high volatility" economy shows significant increases in the standard deviations of output, consumption and investment, when compared with the baseline economy. However, the main properties of the baseline economy are also present here, as the difference in ergodic moments is very small across the models. When using non-linear methods, the only difference

[10] Equivalent to $1 of error for every $10 billion spent in consumption

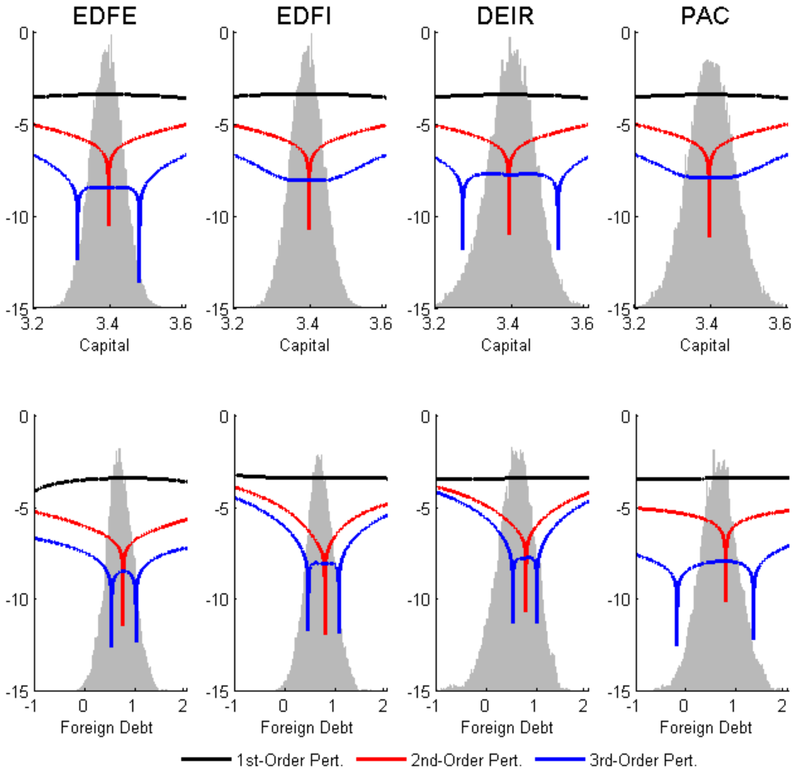

Figure 4: Euler Errors - "High-Volatility" Calibration

from the baseline economy is the slight decrease in the volatility of consumption and investment when compared to the linear solution. The decrease is more pronounced in DEIR model, with a drop of 0.5% in the volatility of consumption. Again, in order to favor the comparison with the literature, moments presented in table 8 were computed for variables in logs (except trade balance- and capital account-output ratios).

The histogram of simulations in figure 5 shows a slight gap between the distributions of capital and foreign debt when using non-linear solutions of the model. This gap is more evident in DEIR and PAC models. The breakdown of the certainty equivalence principle, combined with the increase in volatility of the economy, results in larger values for accumulated capital and a foreign debt average closer to the creditor position, with the distribution shifted to the left. When compared to the histogram of the baseline economy, the low volatility of that economy resulted in changes only in the allocation of foreign debt, with mild discrepancies in the accumulation of capital.

The new calibration also results in larger volatility for capital and foreign debt in DEIR and PAC models, when compared to the simulations based on EDFE and EDFI models. This result was not evident in the baseline economy for the distribution of capital. It is also relevant that DEIR and PAC models show long tails to the left side of the distribution of foreign debt, with agents taking significant positions at the creditor side. Given the result in table 8 that the ergodic volatility of investment is lower in DEIR and PAC models, compared to values for models with endogenous discount factor, the high volatility of capital and foreign debt combined implies a lot of substitution and arbitrage movements between capital and foreign bonds in DEIR and PAC models.

Back to the analysis of impulse response functions, figure 6 shows the increase in the deviations of the stationary-inducing devices with respect to its first-order approximation counterpart, when compared to the baseline calibration. The wider gap between the simulations of linear and non-linear solutions is a consequence of the increased volatility of the overall economy, as the path for these variables is determined by the path of other endogenous variables of the model.

Table 8 – Ergodic Moments – "High Volatility" Calibration

Model	EDFI			EDFE			DEIR			PAC		
Order	1	2	3	1	2	3	1	2	3	1	2	3
Volatilities												
$\sigma(y_t)$	5.98	5.96	5.96	5.98	5.96	5.96	6.02	6.00	6.00	6.01	5.99	5.99
$\sigma(c_t)$	4.22	4.21	4.21	4.22	4.21	4.21	5.00	4.48	4.50	4.47	4.45	4.45
$\sigma(i_t)$	14.07	14.07	14.05	14.07	14.07	14.05	13.87	13.86	13.84	13.88	13.86	13.85
$\sigma(h_t)$	4.10	4.10	4.10	4.10	4.10	4.10	4.13	4.12	4.13	4.13	4.12	4.12
$\sigma\left(\frac{tb_t}{y_t}\right)$	2.32	2.32	2.33	2.32	2.32	2.32	2.30	2.32	2.32	2.29	2.29	2.30
$\sigma\left(\frac{ca_t}{y_t}\right)$	2.21	2.21	2.21	2.21	2.21	2.21	2.06	2.07	2.06	2.08	2.07	2.08
Serial correlations												
$\rho(y_t)$	0.26	0.26	0.26	0.26	0.26	0.26	0.27	0.27	0.27	0.27	0.27	0.27
$\rho(c_t)$	0.37	0.37	0.37	0.37	0.37	0.37	0.46	0.46	0.47	0.46	0.46	0.46
$\rho(i_t)$	-0.29	-0.29	-0.29	-0.29	-0.29	-0.29	-0.28	-0.28	-0.28	-0.28	-0.28	-0.28
$\rho(h_t)$	0.26	0.26	0.26	0.26	0.26	0.26	0.27	0.27	0.27	0.27	0.27	0.27
$\rho\left(\frac{tb_t}{y_t}\right)$	0.42	0.42	0.42	0.42	0.42	0.42	0.47	0.48	0.48	0.47	0.47	0.47
$\rho\left(\frac{ca_t}{y_t}\right)$	0.41	0.41	0.41	0.41	0.41	0.41	0.40	0.40	0.39	0.40	0.40	0.40
Correlation with output												
$\rho(c_t, y_t)$	0.95	0.95	0.95	0.95	0.95	0.95	0.93	0.93	0.93	0.93	0.93	0.93
$\rho(i_t, y_t)$	0.70	0.70	0.70	0.70	0.70	0.70	0.74	0.74	0.74	0.74	0.74	0.74
$\rho(h_t, y_t)$	1	1	1	1	1	1	1	1	1	1	1	1
$\rho\left(\frac{tb_t}{y_t}, y_t\right)$	0.26	0.26	0.26	0.26	0.26	0.26	0.18	0.19	0.17	0.18	0.19	0.19
$\rho\left(\frac{ca_t}{y_t}, y_t\right)$	0.33	0.33	0.33	0.33	0.33	0.33	0.31	0.32	0.31	0.32	0.32	0.32

Obs.: Parameters shown in table 1 for the "high volatility" economy. Standard-deviations in percent per year.

Table 9 shows the difference between impulse response functions of models solved by linear and non-linear approximations. The gap is measured for impulse response functions generated from alternative quantiles of the ergodic distribution of capital, just like results in table 5. As shown in figure 6, the increase in the volatility of the economy resulted in a larger gap when compared to the results for the baseline calibration. While results do not hold for DEIR model, the cumulative impact on investment, derived from the large deviations of risk premium from steady state, is 7.7% larger on non-linear solutions at the median of the distribution of capital, reaching a gap of 22% when the initial stock of capital is at the lowest threshold considered here. Thus, there is a significant amount of capital accumulation for DEIR economy not captured by linear solutions when the realization of capital in the economy is very low, with the linear solution showing households transferring wealth across periods through bonds. Also related to the increase in the overall volatility of the economy, the gap between the median of foreign debt-to-GDP ratio in linear and non-linear solutions increased at least 0.7% of the GDP (as it is the case for EDFE and EDFI models), when compared to the baseline calibration. The median debt-to-GDP ratio remains well below the steady state level (at least 2% of the GDP), but the gap between linear and non-linear solutions increased with the change in parameters of the model controlling the volatility of the economy.

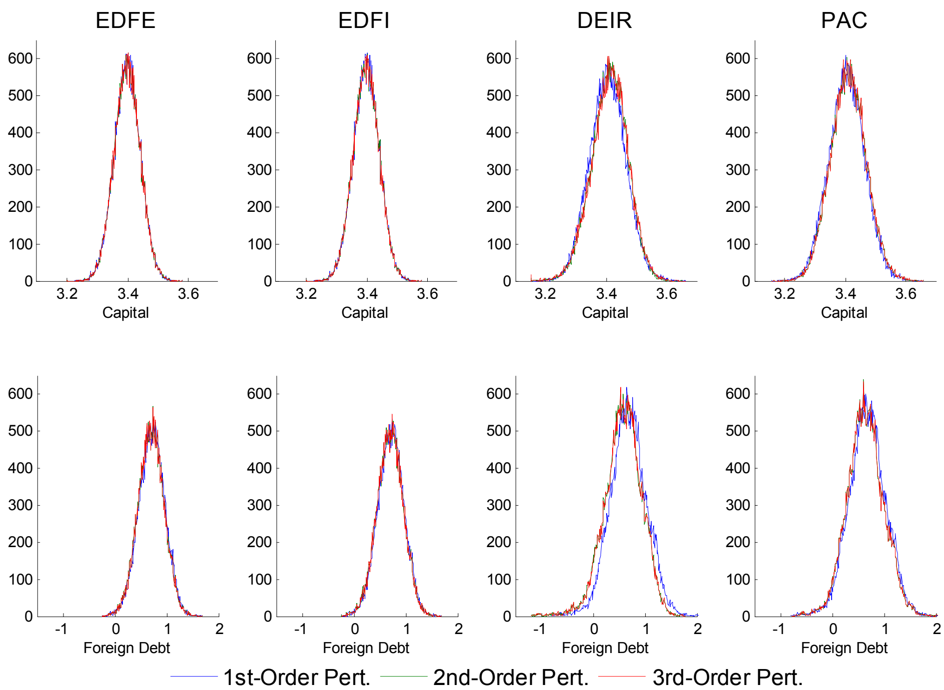

Figure 5: Histogram – "High-Volatility" Calibration

Table 9 – Conditional Moments – "High Volatility" Calibration

	EDFI	EDFE	DEIR	PAC
Gap between IRFs: Output, 3^{rd} to $1^{st}-$order: impact (cumulative)				
Quantile(K_0) < 0.05	-4.67% (-3.76%)	-4.63% (-3.71%)	-3.93% (0.20%)	-3.82% (-3.27%)
Quantile(K_0) < 0.10	-3.70% (-2.99%)	-3.74% (-2.99%)	-2.92% (0.39%)	-3.03% (-2.60%)
Quantile(K_0) < 0.25	-2.36% (-1.89%)	-2.36% (-1.89%)	-1.59% (0.52%)	-1.65% (-1.41%)
Quantile(K_0) < 0.50	-1.03% (-0.82%)	-1.03% (-0.81%)	-0.27% (0.87%)	-0.43% (-0.42%)
Gap between IRFs: Consumption, 3^{rd} to $1^{st}-$order: impact (cumulative)				
Quantile(K_0) < 0.05	-4.67% (-4.02%)	-4.64% (-4.00%)	-3.52% (1.08%)	-3.80% (-3.23%)
Quantile(K_0) < 0.10	-3.70% (-3.19%)	-3.75% (-3.22%)	-2.58% (1.12%)	-3.01% (-2.57%)
Quantile(K_0) < 0.25	-2.36% (-2.01%)	-2.36% (-2.02%)	-1.37% (1.00%)	-1.64% (-1.37%)
Quantile(K_0) < 0.50	-1.03% (-0.87%)	-1.03% (-0.86%)	-0.14% (1.17%)	-0.43% (-0.38%)
Gap between IRFs: Investment, 3^{rd} to $1^{st}-$order: impact (cumulative)				
Quantile(K_0) < 0.05	-0.88% (-0.35%)	-0.86% (-0.23%)	-0.99% (22.7%)	-1.54% (-3.14%)
Quantile(K_0) < 0.10	-0.72% (-0.18%)	-0.73% (-0.09%)	-0.70% (18.8%)	-1.25% (-2.50%)
Quantile(K_0) < 0.25	-0.50% (0.06%)	-0.49% (0.16%)	-0.37% (12.5%)	-0.76% (-1.35%)
Quantile(K_0) < 0.50	-0.32% (0.26%)	-0.31% (0.36%)	-0.05% (7.74%)	-0.36% (-0.39%)
Debt-to-GDP ratio: median – standard-deviation (SS = 0.50)				
$1^{st}-$order	0.483; 0.17	0.483; 0.17	0.454; 0.24	0.454; 0.25
$2^{nd}-$order	0.470; 0.17	0.471; 0.17	0.395; 0.25	0.425; 0.24
$3^{rd}-$order	0.471; 0.17	0.471; 0.17	0.397; 0.25	0.426; 0.24

This section shows that the increase in volatility compromises the accuracy of the models' solutions, showing the importance of solving models with high-order approximations of the equilibrium conditions. The conclusions of SGU (2003) [21] are robust to the increase in volatility in terms of ergodic moments, both in the comparison across models and across orders of perturbation. The local analysis, however,

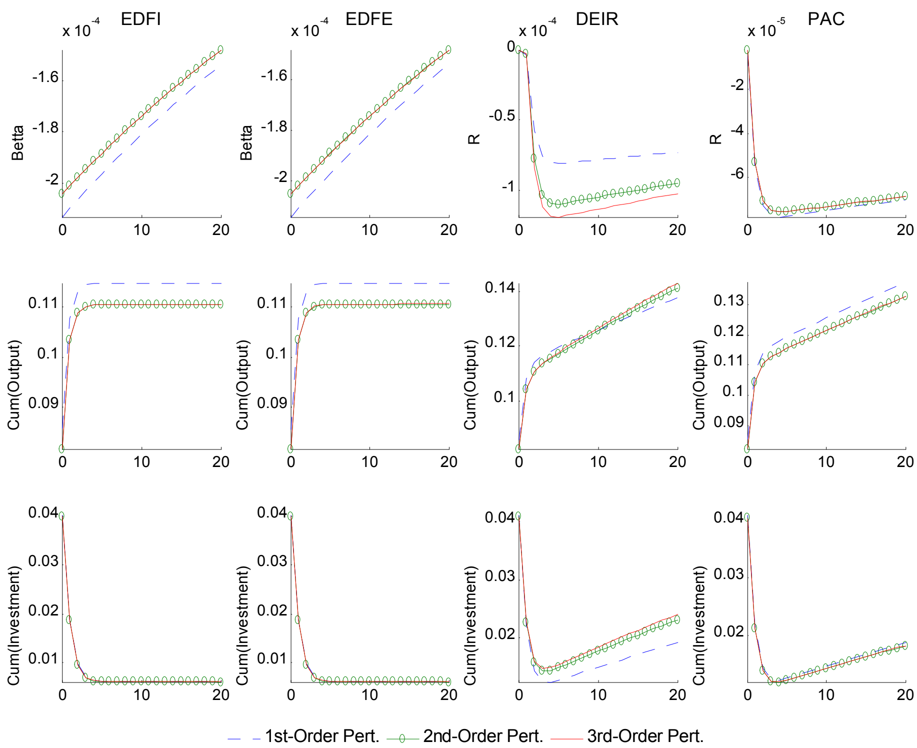

Figure 6: Impulse response functions – Productivity shock: quantile(k_0) $<$ 5% – "High-Volatility" Calibration

shows that the effect of shocks in different regions of the state space is sensitive to the overall volatility of the economy. The next section relaxes, for robustness purposes, the assumption regarding preferences, using a basic CRRA function to represent households' utility.

6 The Model with CRRA Preferences

The final change in the model follows the robustness test from SGU (2003)[21], replacing the GHH preferences by the following functional forms for utility and the endogenous discount factor:

$$U(c,h) = \frac{\left[c^{\omega}(1-h)^{\omega}\right]^{1-\gamma} - 1}{1-\gamma}$$
$$\beta(c,h) = \left[1 + c^{\omega}(1-h)^{\omega}\right]^{-\psi_1}$$

This CRRA preferences are particularly useful, as it is the same functional form used in the basic RBC model of Aruoba, Fernández-Villaverde and Rubio-Ramírez (2006)[5]. The calibration follows again most of the strategy of SGU (2003)[21] with respect to the adjustment for the new functional forms. Thus, the model is calibrated such that hours worked corresponds to a share of 20% of total endowment time ($h = 0.2$) and the trade balance-to-GDP ratio is equal to 2 percent. This value is used to obtain parameter ψ_1 in the EDFI model and the value of foreign debt used to calibrate the remaining economies under a log-linear approximation. The main difference from the calibration adopted in SGU (2003)[21] is that, in order not to loose the effects of the increase in volatility, the new set of values for parameters ϕ, ρ and σ_ϵ matches the volatilities of output and investment and the persistence of output of the "high volatility" economy[11]. The complete set of parameters for the model with CRRA preferences are shown in table 10[12].

Table 10 – Calibration: CRRA Preferences

β	γ	α	δ	ψ_1	ω
1/1.04	2.00	0.32	0.1	0.08	0.22
ψ_2	ψ_3	ϕ	ρ	σ_ϵ	
0.000385	0.00037	0.00039	0.07925	0.028	

The analysis of den Haan-Marcet statistic, presented in table 11, show that non-linear solutions do not contribute as much as in the model with GHH preferences, since the coverage areas of the test statistic are very similar despite the increase in the approximation order. The low contribution of high-order approximations to improve the test distribution in den Haan-Marcet statistic is also a result of the baseline calibration in Aruoba, Fernández-Villaverde and Rubio-Ramírez (2006)[5], where the test statistic for the basic RBC model has almost the same distribution up to a fifth-order approximation[13]. Also as in Aruoba, Fernández-Villaverde and Rubio-Ramírez (2006)[5], the test statistic is skewed to the right of the distribution, with DEIR and PAC models with the largest degree of skewness.

Table 11 – Den Haan-Marcet Statistic – CRRA Preferences

Order	1		2		3	
	$< 5\%$	$> 95\%$	$< 5\%$	$> 95\%$	$< 5\%$	$> 95\%$
EDFE						
Bonds	4.58	7.26	4.52	7.12	4.50	7.14
Capital	4.30	7.30	4.34	7.06	4.36	7.06
EDFI						
Bonds	4.48	7.46	4.44	7.30	4.42	7.34
Capital	4.32	7.36	4.14	7.08	4.14	7.10
DEIR						
Bonds	2.88	10.28	3.02	10.03	3.00	10.01
Capital	2.96	10.24	2.74	9.98	2.76	9.93
PAC						
Bonds	3.01	10.36	2.94	10.24	2.94	10.24
Capital	2.98	10.26	2.76	10.14	2.76	10.14

Obs.: results show percentage of simulations below (above) the 5% (95%) critical value of the chi-square distribution with 3 degrees of freedom.

Table 12 shows the expected value and maximum Euler equation errors for models with CRRA preferences. The Euler equation errors were computed based on intervals of 0.58 and 0.77 for capital, -0.752 and 0.752 for foreign debt and 0.87 and 1.13 for productivity. Results still favors at least the use of second-order approximations to solve the model, based again on the comparison between the maximum Euler error of a second-order approximation and the expected value of error for a linear solution. Also similar to results presented before, the gains of moving from a second to a third-order approximation are not as significant as the gains moving from the linear to the second-order solution. Notice that the

[11] In previous exercises, calibration was based on the volatility of consumption. However, with the use of CRRA preferences, consumption becomes significantly smoother with respect to the values obtained using GHH preferences. As a consequence, the volatility of productivity needs to reach unreasonable values to replicate the volatility of consumption.

[12] With respect to the utility function, Aruoba, Fernández-Villaverde and Rubio-Ramírez (2006)[5] calibrate their model with the same value for the coefficient of risk aversion, γ, but a slightly different value for ω, as they target a share of 31% of hours worked in steady state. Parameters associated with production and investment matches empirical moments of the US economy, thus not related with calibration adopted here.

[13] See results in table 3 of Aruoba, Fernández-Villaverde and Rubio-Ramírez (2006)[5].

third-order solution presents larger errors when compared to the same solution of the model with GHH preferences.

Table 12 – Euler Equation Errors – CRRA Preferences

Order	1		2		3	
	Max(EE)	E(EE)	Max(EE)	E(EE)	Max(EE)	E(EE)
EDFE						
Bonds	-3.675	-4.294	-4.526	-6.306	-5.643	-7.374
Capital	-3.795	-3.988	-4.353	-6.180	-5.407	-7.298
EDFI						
Bonds	-3.382	-4.255	-4.335	-6.151	-5.690	-7.175
Capital	-3.461	-3.899	-4.214	-6.004	-5.484	-7.028
DEIR						
Bonds	-3.743	-4.256	-4.562	-6.026	-5.657	-7.644
Capital	-3.875	-3.993	-4.213	-6.197	-4.580	-7.059
PAC						
Bonds	-3.737	-4.297	-4.775	-6.150	-5.752	-7.340
Capital	-3.875	-3.993	-4.661	-6.050	-5.749	-7.211

Obs.: results measured as logarithms of base 10: a value of -4 shows an error of $1 for every $10,000 spent in consumption; a value of -5 represents an error of $1 for every $100,000; and so on.

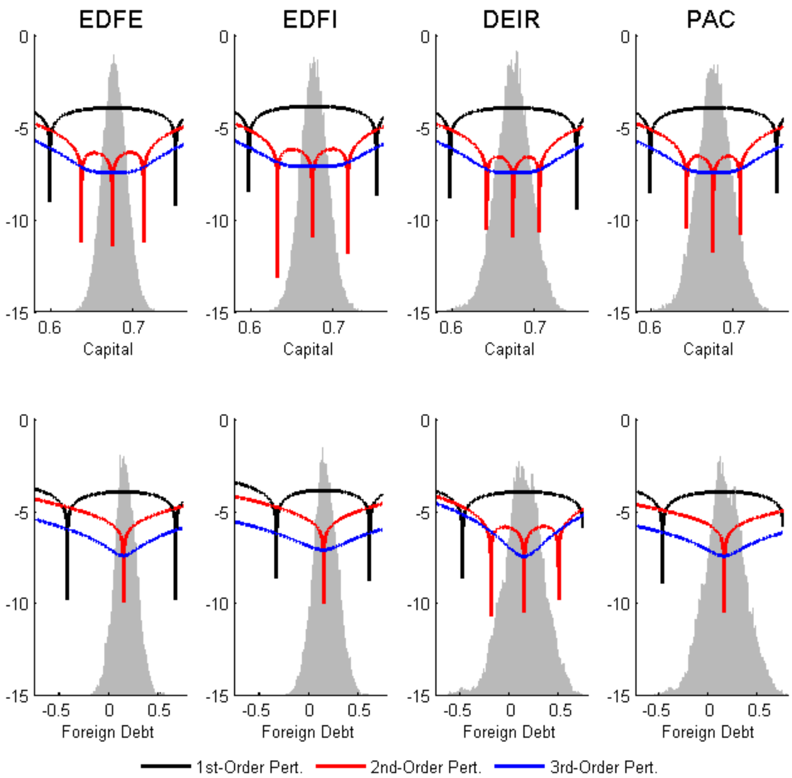

Figure 7: Euler Errors - CRRA Preferences

Figure 7 provides some intuition about the little gains derived from the use of third-order approximations of the model. While errors from a second-order perturbation of the model fluctuates a few times between negative and positive values, creating spikes in the central region of the histogram, Euler equation errors from a third-order approximation remain quite stable around the relevant state space. As a consequence, when computing the expected value of Euler equation errors, spikes in regions with high probability of occurrence receive large weights, reducing the gap between the overall quality of a second to a third-order solution.

Table 13 – Ergodic Moments – CRRA Preferences

Model	EDFI			EDFE			DEIR			PAC		
Order	1	2	3	1	2	3	1	2	3	1	2	3
Volatilities												
$\sigma(y_t)$	5.97	5.96	5.95	5.70	5.68	5.68	6.07	6.07	6.06	6.07	6.04	6.03
$\sigma(c_t)$	0.76	0.76	0.76	0.87	0.87	0.87	1.01	1.01	1.01	1.01	1.01	1.01
$\sigma(i_t)$	14.07	14.07	14.06	12.55	12.55	12.54	13.06	13.11	13.09	13.06	13.06	13.05
$\sigma(h_t)$	4.52	4.52	4.52	4.15	4.15	4.15	4.64	4.67	4.66	4.64	4.63	4.63
$\sigma\left(\frac{tb_t}{y_t}\right)$	4.75	4.76	4.79	4.38	4.39	4.41	4.70	4.75	4.77	4.70	4.72	4.75
$\sigma\left(\frac{ca_t}{y_t}\right)$	4.52	4.53	4.55	4.20	4.20	4.22	4.10	4.11	4.13	4.10	4.11	4.13
Serial correlations												
$\rho(y_t)$	0.25	0.25	0.25	0.23	0.23	0.23	0.32	0.33	0.33	0.32	0.32	0.32
$\rho(c_t)$	0.64	0.63	0.64	0.49	0.49	0.49	0.62	0.63	0.63	0.62	0.62	0.62
$\rho(i_t)$	-0.39	-0.39	-0.39	-0.39	-0.39	-0.39	-0.37	-0.37	-0.37	-0.37	-0.37	-0.37
$\rho(h_t)$	0.31	0.31	0.31	0.29	0.29	0.29	0.44	0.44	0.44	0.44	0.43	0.43
$\rho\left(\frac{tb_t}{y_t}\right)$	0.58	0.58	0.58	0.57	0.57	0.57	0.65	0.65	0.65	0.65	0.65	0.65
$\rho\left(\frac{ca_t}{y_t}\right)$	0.53	0.53	0.53	0.53	0.53	0.53	0.54	0.54	0.54	0.54	0.54	0.54
Correlation with output												
$\rho(c_t, y_t)$	0.46	0.46	0.46	0.62	0.62	0.62	0.33	0.32	0.33	0.33	0.34	0.34
$\rho(i_t, y_t)$	0.52	0.52	0.52	0.52	0.52	0.52	0.56	0.56	0.56	0.56	0.56	0.56
$\rho(h_t, y_t)$	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99
$\rho\left(\frac{tb_t}{y_t}, y_t\right)$	0.83	0.83	0.83	0.84	0.84	0.84	0.85	0.85	0.85	0.85	0.85	0.85
$\rho\left(\frac{ca_t}{y_t}, y_t\right)$	0.78	0.78	0.78	0.80	0.80	0.80	0.74	0.74	0.74	0.74	0.74	0.75

Obs.: Parameters shown in table 10. Standard-deviations in percent per year.

Simulated moments are presented in table 13. Some differences in the comparison of moments across models are evident in the economy with CRRA utility function and a high volatility. Noticeably, investment in EDFI model is more volatile than in the other models, while consumption is smoother. The current account-GDP ratio is smoother in DEIR and PAC models – both models with stationarity-inducing devices based on the level of foreign debt. Also, the correlation between consumption and output is higher in models with the endogenous discount factor. In terms of differences across solution methods, results follow those presented in the previous sections, with the ergodic moments very similar in each formulation, irrespective of the order of approximation.

One interesting feature of the model with CRRA preferences is the smaller differences in histograms of capital and foreign debt when using non-linear methods, when compared to the linear solution. In the model with GHH preferences, agents accumulate less debt and more capital as the volatility of the economy increases. In the presence of wealth effects derived from the specification of preferences, even with the model calibrated to match moments of the "high volatility" economy, the difference between linear and non-linear solutions is irrelevant. The intuition is that the consequences of abandoning the certainty equivalence principle in non-linear solutions – more capital accumulation and less foreign debt

– are cancelled by the wealth effect for the household. When comparing the histogram across models, notice again that DEIR and PAC models generate much more volatility of capital and foreign debt.

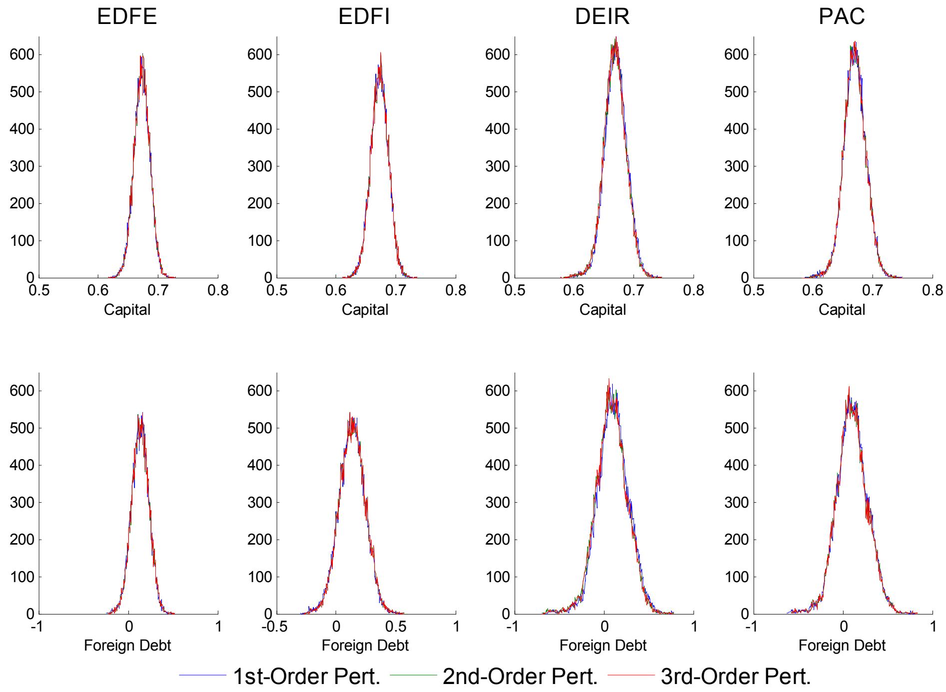

Figure 8: Histogram – CRRA Preferences

The impulse response analysis show the role of wealth effects in the model, measured by the difference of the initial impact in output when using linear and non-linear solutions in the model. While the gap in impulse response functions between linear and non-linear solutions is larger for all variables in the model with GHH preferences, the cumulative effect from the use of non-linear solutions in the simulation of investment is more pronounced for the case of CRRA preferences. As shown in table 14, even at initial capital stock close to the steady state, the cumulative gap in investment from the use of non-linear solutions is significant. Table 14 also shows that debt-to-GDP ratio is at least two times more volatile in the model with CRRA preferences, when compared to the "high volatility" economy with GHH preferences.

Another feature of the non-linear solution using CRRA preferences is the low volatility of the risk premium in DEIR model. Initial states of low capital increase the return of an additional unit of capital. Assuming an initial debtor position in foreign debt markets, households transfer wealth across time using capital, reducing the risk premium on foreign bonds, as the demand for bonds falls. The use of non-linear solutions in the model with GHH preferences exaggerates this effect, as risk aversion induces a faster convergence to the steady state level of capital through higher investment and lower consumption[14]. Under CRRA preferences, the initial state of low capital does not have the same influence, as the preference for consumption smoothness prevails in the intratemporal decision between consumption and labor of households. It is inefficient for the household to give up consumption and generate more investment for a faster convergence to the steady state of capital. As a consequence, the demand for investment is lower with CRRA preferences, and the risk premium does not fall as much as in the model with GHH preferences.

[14] In table 9, the cumulative effect of a productivity shock in investment for DEIR model is significantly higher for non-linear solutions.

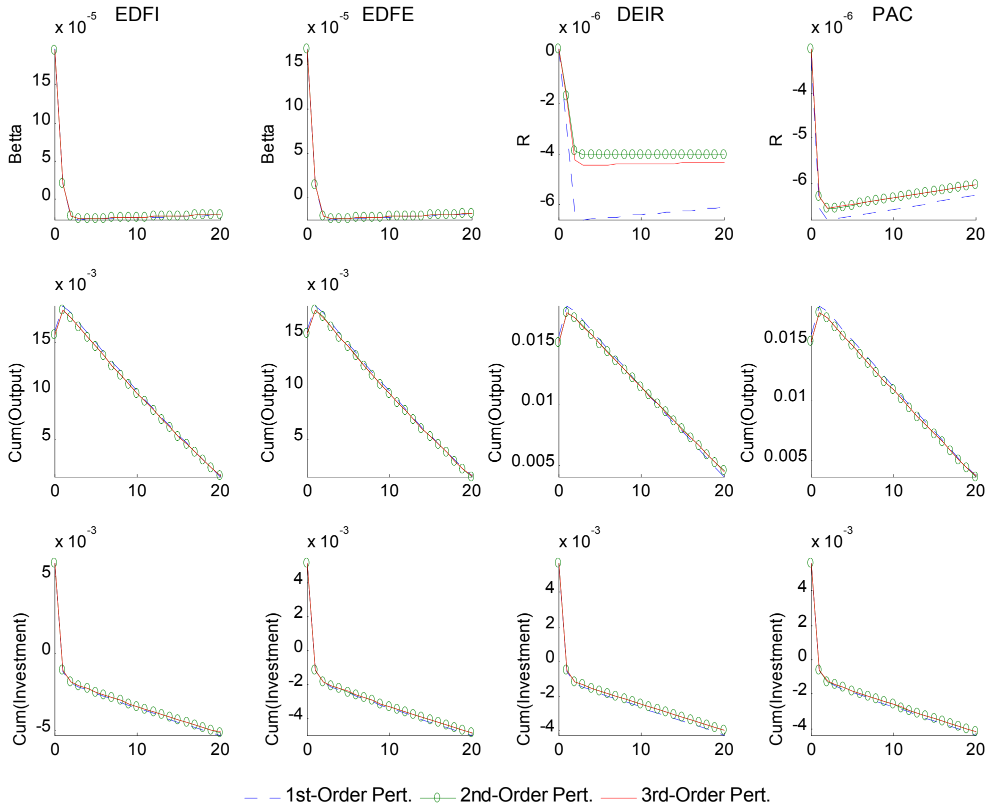

Figure 9: Impulse response functions – Productivity shock: quantile(k_0) < 5% – CRRA preferences

Table 14 – Conditional Moments – CRRA Preferences

	EDFI	EDFE	DEIR	PAC
Gap between IRFs: Output, 3^{rd} to 1^{st}−order: impact (cumulative)				
Quantile(K_0) < 0.05	-3.10% (-2.44%)	-2.72% (-2.18%)	-3.88% (-0.79%)	-3.85% (-2.97%)
Quantile(K_0) < 0.10	-2.53% (-2.03%)	-2.24% (-1.84%)	-3.19% (-0.49%)	-3.14% (-2.43%)
Quantile(K_0) < 0.25	-1.71% (-1.34%)	-1.48% (-1.21%)	-2.17% (-0.43%)	-2.14% (-1.67%)
Quantile(K_0) < 0.50	-0.89% (-0.73%)	-0.78% (-0.70%)	-1.18% (-0.34%)	-1.12% (-0.90%)
Gap between IRFs: Consumption, 3^{rd} to 1^{st}−order: impact (cumulative)				
Quantile(K_0) < 0.05	-1.42% (-1.86%)	-1.22% (-1.51%)	-2.33% (-4.60%)	-1.75% (-2.26%)
Quantile(K_0) < 0.10	-1.13% (-1.51%)	-1.03% (-1.25%)	-2.01% (-3.95%)	-1.45% (-1.84%)
Quantile(K_0) < 0.25	-0.73% (-0.98%)	-0.74% (-0.81%)	-1.39% (-2.63%)	-1.03% (-1.27%)
Quantile(K_0) < 0.50	-0.36% (-0.46%)	-0.46% (-0.44%)	-0.81% (-1.35%)	-0.62% (-0.68%)
Gap between IRFs: Investment, 3^{rd} to 1^{st}−order: impact (cumulative)				
Quantile(K_0) < 0.05	0.38% (-3.41%)	0.31% (-2.96%)	1.28% (-7.62%)	0.37% (-4.59%)
Quantile(K_0) < 0.10	0.27% (-2.82%)	0.21% (-2.47%)	1.00% (-6.43%)	0.29% (-3.75%)
Quantile(K_0) < 0.25	0.12% (-1.91%)	0.07% (-1.60%)	0.72% (-4.28%)	0.15% (-2.57%)
Quantile(K_0) < 0.50	-0.04% (-1.02%)	-0.04% (-0.88%)	0.38% (-2.11%)	0.02% (-1.36%)
Debt-to-GDP ratio: median – standard-deviation (SS = 0.50)				
1^{st}−order	0.459; 0.37	0.466; 0.33	0.362; 0.59	0.362; 0.59
2^{nd}−order	0.463; 0.37	0.472; 0.33	0.303; 0.60	0.344; 0.59
3^{rd}−order	0.463; 0.37	0.471; 0.33	0.304; 0.60	0.344; 0.59

Thus, the use of CRRA preferences generates small differences in terms of ergodic moments across models. However, the local analysis highlights the role of risk aversion on investment and on the relation between risk premium and the return of capital. The use of non-linear solutions corrects the model in dimensions where the linear solution compromises the dynamics of investment. In another property kept from the "high volatility" economy with GHH preferences, the model with CRRA preferences shows that the stationary-inducing device matters in order to characterize the initial effect of a shock. In terms of accuracy, the use of at least a second-order solution is the best recommendation, as a third-order perturbation did not alter significantly the properties of the solutions.

7 Conclusions

This paper compares solutions generated by perturbation methods in the basic RBC model for SOE's. The comparison of models is made not only in terms of ergodic moments, but also with respect to accuracy of solution and local evaluation of dynamics. Results show that the main results of SGU (2003)[21], on the similarities between EDFI, EDFE, DEIR and PAC models, are valid as long as the volatility of the economy is sufficiently low and preferences represented by GHH functions.

The use of high-order perturbation methods does not alter results based on ergodic moments. However, the use of non-linear approximations show different dynamics in constrained sets of the state space. Specifically, when the economy is located in regions of low capital, the immediate impact of a productivity shock on output generated from non-linear solutions might be 4.67% smaller than the impact computed from linear approximations, depending on the calibration and the stationary-inducing device adopted. In terms of cumulative effects, investment might be 22% higher using a non-linear solution. Again, it is worth reminding that the model presented does not have the necessary persistence of shocks, or any significant nominal or real friction capable of amplifying the impact of shocks, making policy functions essentially linear around the steady state.

In terms of accuracy of solution, the paper shows that there are significant gains obtained by departing from the linear solution, especially when the volatility of the economy is large. The loss in the quality of linear approximations increases as the variance of shocks grows, with poor approximations even around the model's deterministic steady state. This is especially relevant in for Emerging Economies, characterized by high volatility of consumption and output when compared to developed SOE's.

Finally, results point to the need of a criterion to choose between stationary-inducing devices, due to discrepancies verified in the simulations of state variables – capital and foreign debt. These models bring different dynamics for risk premium and asset pricing choices between capital and foreign debt. Another relevant topic is the use of global solution methods and the consequences for ergodic moments and local dynamics in these models, given the good quality of global approximations. Evaluating global solutions in a simple model to compare with results presented here is a useful exercise to determine the overall quality of perturbation methods for SOE's. Finally, the issue of local impulse response functions must be evaluated in models with nominal and real frictions, such that the initial impact of shocks are better described in extreme points of the state space.

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A Appendix – Euler Equation Errors for Perturbation in Logs

Table A shows the maximum and the average Euler equation errors for the model using GHH preferences and the baseline calibration. Simulated data used to generate the histogram was the same used in approximation in levels, as a third-order approximation with variables in logs generated explosive paths, irrespective of the use of "pruning" to solve for convexities in the policy function. The use of the histogram based on the variables in levels is the main reason for second-order approximations of DEIR and PAC models are worse, in terms of expected value, than the linear approximation. However, even using the histogram based on variables in logs of the second-order approximation, these models are not as accurate as when their are solved with variables in levels. Values presented in this table can be directly compared with results in table 4 in the text. Results using other calibrations and preferences display similar results, available upon request.

Table A – Euler Equation Errors – Log-Approximation of Baseline Calibration

Order	1		2		3	
	Max(EE)	E(EE)	Max(EE)	E(EE)	Max(EE)	E(EE)
EDFE						
Bonds	-2.273	-4.349	-1.583	-4.710	-1.566	-5.666
Capital	-2.309	-4.336	-1.583	-4.710	-1.566	-5.672
EDFI						
Bonds	-2.244	-4.303	-1.354	-4.527	-1.390	-5.485
Capital	-2.296	-4.284	-1.354	-4.526	-1.390	-5.492
DEIR						
Bonds	-2.279	-4.179	-1.681	-4.053	-1.622	-5.117
Capital	-2.311	-4.202	-1.686	-4.055	-1.628	-5.129
PAC						
Bonds	-2.278	-4.234	-1.699	-4.057	-1.663	-4.663
Capital	-2.311	-4.209	-1.701	-4.151	-1.666	-5.325

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