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Decentralized Portfolio Management

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Abstract

Within a mean-variance model we analyze the problem of decentralized portfolio management. We find the solution for the optimal portfolio allocation for a head trader operating in *n* different markets, which is called the optimal centralized portfolio. However, as there are many traders specialized in different markets, the solution to the problem of optimal decentralized allocation should be different from the centralized case. In this paper we derive conditions for the solutions to be equivalent. We use multivariate normal returns and a negative exponential function to solve the problem analytically. We generate the equivalence of solutions by assuming that different traders face different interest rates for borrowing and lending. This interest rate is dependent on the ratio of the degrees of risk aversion of the trader and the head trader, on the excess return, and on the correlation between asset returns.

JEL: G10; G11; G23.

Keywords: risk aversion, portfolio management, Markowitz.

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1. Introduction

The optimal allocation of investment resources to different assets is one of the most well-known problems in finance and has been extensively studied. The most frequent approach to solve this problem is the mean-variance analysis, which assumes that the investor is concerned with only two parameters of the probability distribution of total returns on investment: the mean and the variance. Markowitz [1959] and Tobin [1958] were the first to put the trade-off between risk and return on a solid analytic footing. Sharpe [1964] and Lintner [1965] extended the mean-variance theory to an equilibrium theory.

Mean-variance models solve the problem of optimal portfolio allocation for a head trader operating in n different markets. The solution to this problem is called *the optimal centralized allocation*. It is interesting to notice that if investors are von Neumann-Morgenstern expected utility maximizers, then a mean-variance analysis could be justified either by assuming quadratic utility functions or that the distribution of asset returns is multivariate normal. We could solve the problem analytically for a quadratic utility function, but, as Huang and Litzenberger [1988] observed, the quadratic utility function exhibits increasing absolute risk aversion, which implies that risky assets are inferior goods. This unappealing property induced the use of a negative exponential utility function.

In practice, most investment firms operate with a head trader leading a number of traders specialized in different markets. The head trader gives general guidelines. One of these guidelines is a common benchmark rate, the interest rate that each trader faces when drawing resources from the firm to invest in risky assets. However traders solve their problem for optimal allocation in a decentralized way. The aggregation of the investment decisions by the specialized traders is called *the optimal decentralized solution*. In general, the optimal decentralized solution is different from the centralized one for a number of reasons: different risk aversion, different probability distributions of assets, etc. Therefore the mean-variance analysis is of limited use for big investment firms that have many specialized traders.

Decentralization of the investment process has not been a major theme in the finance literature for the past decades. Sharpe [1980] has been one of the first researchers to notice this gap, stating that “more research on this subject is needed”.

Sharpe [1980] explained the rationale for having multiple managers in charge of a single portfolio with two arguments, namely, specialization and diversification. Specialization would be justified by the need of superior skills for the analysis of certain types of industries and diversification would be a natural way to hedge against bad decisions taken by a single manager.

Barry and Stark [1984] analyzed the phenomenon of multiple managers using the principal-agent theory approach. They developed a model to show that risk-sharing considerations alone are sufficient to produce a decision to employ multiple agents (even in the absence of specialization and diversification).

In this paper we derive the conditions under which the centralized and decentralized solutions are equivalent. This is important because the mean-variance model could then be implemented by firms with decentralized investment decisions, as most large trading firms are.

We solve the problem with different, increasing levels of generalization. In all levels, we solve the optimal centralized and decentralized portfolio problem for n risky assets assuming that each agent has a negative exponential utility function and that rates of returns of risky assets are multivariate normally distributed, so the problem can be solved via a deterministic equivalent. The interest rate for borrowing or lending by the traders is used as control variable. We use Stevens’ [1998] characterization of the inverse of the covariance matrix to get a closed-form optimal solution.

In the basic case, we suppose each trader trades in one risky asset and one riskless asset. Allowing the interest rate for the riskless asset to be different for different traders we generate the equivalence for the centralized and decentralized solutions. Furthermore, we show that the trader interest rate depends on the ratio of the degree of risk aversion of the trader and head trader, on the excess return, and on the correlations between assets.

In the first generalization we allow each trader to trade in *m* different risky assets but with an empty intersection between subsets of risky assets managed by traders. In the second generalization we assume that traders manage the same subset of risky assets. Finally, we allow each trader to manage any subset of risky assets.

In section II we find the optimal solutions for the head trader and the traders, and establish the equivalence condition for the basic case. In section III we perform comparative static. Section IV presents the generalizations. Section V summarizes the results and concludes the paper.

2. The Equivalence of Centralized and Decentralized Solutions: The Basic Case

In this section we consider the case where each trader manages one risky asset and one risk-free asset. First we calculate the head trader centralized solution, where he manages all assets, making no use of specialized traders. Then we calculate the decentralized solution, where specialized traders manage risky assets. Because each trader looks at the risk of only one asset when making his investment decision, in the aggregate the traders' solutions usually will not be compatible with the Markowitz (head trader) solution for the portfolio of all risky assets. We then calculate the conditions for the centralized and decentralized solutions to coincide.

Head Trader Centralized Solution

We consider a head trader having an initial wealth W_0 to invest in *n* risky assets and a risk-free asset f for a finite investment period. Let $\tilde{r}_j$ be the stochastic return of risky asset *j*, r_f be the return of the risk-free asset, α_j be the proportion of wealth invested in the *j-th* risky asset, and α_f be the fraction of wealth invested in the risk-free asset during the investment period. Then $\left(1+\tilde{r}_j\right)\alpha_j W_0$ is returned at the end of the investment period. The head trader wishes to maximize the expected utility of the stochastic wealth

$$\tilde{W} = W_0\left[\alpha_f\left(1+r_f\right)+\sum_{j=1}^{n}\alpha_j\left(1+\tilde{r}_j\right)\right].$$

To have a closed form solution, we assume that returns are multivariate normally distributed with mean vector $\bar{\mathbf{r}}$ and covariance matrix $\boldsymbol{\Sigma}$. We solve the optimal portfolio problem for the head trader with a negative exponential utility function, $U(W) = 1 - e^{-aW}$, where a is the strictly positive absolute risk aversion. In this case the head trader problem can be written as:

$$\max_{\alpha_f,\{\alpha_j\}_{j=1}^{n}}\left[E\left(U\left(\tilde{W}\right)\right)=\int_{-\infty}^{+\infty}U\left(\tilde{W}\right)f\left(\tilde{W}\right)d\tilde{W}\right] \tag{2.1}$$

where $f(\tilde{W})$ is a normal density function. Under the assumptions, the above problem (2.1) has a certainty equivalent given by:[1]

$$\max_{\alpha_f,\{\alpha_j\}_{j=1}^{n}}\left[\left(\alpha_f\left(1+r_f\right)+\boldsymbol{\alpha}\cdot\mathbf{r}\right)-\frac{a}{2}W_0\left(\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}\right)\right] \tag{2.2}$$

where $\boldsymbol{\alpha}' = \left(\alpha_1,\cdots,\alpha_n\right)$ is a 1xn vector of risky assets weights, $\mathbf{r}' = \left(1+\bar{r}_1 \quad . \quad . \quad . \quad 1+\bar{r}_n\right)$ is a $1\times n$ vector of expected returns of risky assets, and

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ . & . & . & . \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{bmatrix}$$

is the $n\times n$ covariance matrix for risky assets. Assuming that all initial wealth is invested, we have the following Lagrangean function for the head trader:

[1] "$\mathbf{x}'$" is the transpose of $\mathbf{x}$.

$$\ell = \left[\left(\alpha_f \left(1 + r_f \right) + \boldsymbol{\alpha} \cdot \mathbf{r} \right) - \frac{a}{2} W_0 \left(\boldsymbol{\alpha}' \boldsymbol{\Sigma} \boldsymbol{\alpha} \right) \right] - \lambda \left(\sum_{j=1}^{n} \alpha_j + \alpha_f - 1 \right). \tag{2.3}$$

The first-order conditions are:

$$\frac{\partial \ell}{\partial \alpha_f} = \left(1 + r_f \right) - \lambda = 0 \tag{2.4}$$

$$\frac{\partial \ell}{\partial \alpha_k} = \left(1 + \bar{r}_k \right) - \frac{a}{2} \left[2\alpha_k \sigma_k^2 + 2 \sum_{\substack{j=1 \\ j \neq k}}^{n} \alpha_j \sigma_{kj} \right] W_0 - \lambda = 0, \quad k = 1,2.....n \tag{2.5}$$

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=1}^{n} \alpha_j + \alpha_f - 1 = 0 \,. \tag{2.6}$$

Using (2.4) and (2.5) we get an *n*-equation system:

$$\left(\alpha_k \sigma_k^2 + \sum_{\substack{j=1 \\ j \neq k}}^{n} \alpha_j \sigma_{kj} \right) = \frac{1}{aW_0} \left(\bar{r}_k - r_f \right), \quad k = 1, 2.....n \tag{2.7}$$

or, in matrix notation:

$$\boldsymbol{\alpha} = \frac{1}{aW_0} \boldsymbol{\Sigma}^{-1} \hat{\mathbf{r}} \tag{2.8}$$

where $\hat{\mathbf{r}}$ is the $n \times 1$ vector of excess returns.

Traders' Decentralized Solution

We assume that there are *n* traders; each one specialized in a different risky asset. The *i-th* trader has initial wealth $W_{0,i}$, allocated to him by the head trader, to invest in a risky asset *i*. Let $\alpha_{i,i}$ be the fraction of his initial wealth invested in the risky asset, a_i be his coefficient of risk aversion, $r_{f,i}$ be the risk-free return the *i-th* trader pays to the firm to allocate wealth in his risky asset's investment. In other words, $r_{f,i}$ is the benchmark of the *i-th* trader. The use of different benchmarks for different traders is the instrument the head trader will use to make traders' decentralized solutions compatible with Markowitz's in the aggregate. The Lagrangean function for the *i-th* trader is:

$$\ell = \alpha_{i,i}\left(\overline{r}_i - r_{f,i}\right) - \frac{a_i}{2}\alpha_{i,i}^2\sigma_i^2 W_{0,i} + \lambda\left(1-\alpha_{i,i}\right). \tag{2.9}$$

Assuming an interior solution (the Kuhn-Tucker multiplier is zero), the first-order condition is:

$$\frac{\partial \ell}{\partial \alpha_{i,i}} = \left(\overline{r}_i - r_{f,i}\right) - a_i\alpha_{i,i}\sigma_i^2 W_{0,i} = 0 \tag{2.10}$$

and the optimal solution for the risky asset is

$$\alpha_{i,i} = \frac{1}{W_{0,i}}\frac{\left(\overline{r}_i - r_{f,i}\right)}{a_i\sigma_i^2} \tag{2.11}$$

By controlling $W_{0,i}$, the head trader can guarantee that the optimal solution is indeed an interior solution. Expression (2.11) shows that the proportion of initial wealth invested in the risky asset is decreasing in wealth, an inconvenient byproduct of the hypothesis that traders have exponential utility functions.

2.3. Equivalence of Centralized and Decentralized Solutions

The condition for the decentralized optimal choices (for the *n* traders) to be equivalent to the optimal centralized choice of the head trader is:

$$\alpha_{i,i}W_{0,i} = \alpha_i W_0, \qquad \text{for } i = 1,\ldots,n. \tag{2.12}$$

Solving for the risk-free rate for the *i-th* trader

$$r_{f,i} = \overline{r}_i - a_i\sigma_i^2\alpha_i W_0 \tag{2.13}$$

Using Stevens [1998] characterization of the inverse of the covariance matrix we derive:

$$\alpha_i = \frac{1}{W_0}\frac{1}{a\sigma_i^2\left(1-\mathrm{R}_i^2\right)}\left[\bar{r}_i - r_f - \sum_{\substack{j=1\\ j\neq i}}^{n}\beta_{ij}\left(\bar{r}_j - r_f\right)\right] \tag{2.14}$$

where R_i^2 and β_{ij} are the R-squared and the coefficients in the multiple regression of the excess return of the *i-th* asset on the excess returns of the other assets. The factor $\sigma_i^2\left(1-\mathrm{R}_i^2\right)$ is the part of the variance of the *i-th* return that cannot be explained by the regression on the other risky excess returns, which is equivalent to the estimate of the variance of the regression residual. So the denominator in this expression is the part of asset *i's* variance that cannot be diversified away times the absolute risk aversion. The numerator is proportional to the difference between asset *i's* excess return and the sum of other risky assets' excess return weighted by the respective coefficient in the multiple regression. Substituting (2.14) in (2.13) we have

$$r_{f,i} = \bar{r}_i - \frac{1}{\left(1-\mathrm{R}_i^2\right)}\left[\bar{r}_i - r_f - \sum_{\substack{j=1\\ j\neq i}}^{n}\beta_{ij}\left(\bar{r}_j - r_f\right)\right]\frac{a_i}{a_H} \tag{2.15}$$

where a_H represents the head trader's coefficient of risk aversion. If the head trader and all traders have the same coefficient of risk aversion, the above expression simplifies to

$$r_{f,i} = \bar{r}_i - \frac{1}{\left(1-\mathrm{R}_i^2\right)}\left[\bar{r}_i - r_f - \sum_{\substack{j=1\\ j\neq i}}^{n}\beta_{ij}\left(\bar{r}_j - r_f\right)\right] \tag{2.16}$$

The head trader can use (2.15) (or (2.16)) to control traders in their optimal allocation problems. In the calculation of trader *i*'s optimal solution we have made the assumption that he maximizes the expected utility of the wealth he manages, $E\left[U\left(\tilde{W}\right)\right] = E\left[1-e^{-a_i\tilde{W}}\right]$. However, trader *i* should maximize the expected utility of his own payoff. If his payoff is a fixed proportion δ_i of the wealth he manages he should maximize the function $E\left[U\left(\tilde{W}\right)\right]=E\left[1-e^{-\eta_i\delta_i\tilde{W}}\right]$, where η_i is his true coefficient of risk aversion[2]. If we define $a_i=\delta_i\eta_i$ we can easily see that the problem does not change.

Further generalizations of the traders' payoff function could be made. An interesting case arises when traders have unlimited liability. For example, their payoff can be the excess of final wealth over a benchmark *B*, as $\tilde{W}_i=\tau_i\left(\tilde{W}-B\right)$, where the benchmark is non-stochastic and τ_i is the fraction of the excess wealth that trader *i* receives.

However, more general compensation schemes may not work within this approach. An interesting example is the introduction of options in the compensation scheme. Traders receive a fixed fee *f* and, in addition, an option based on the excess wealth that they have generated, as $\tilde{W}_i=f_i+\tau_i\max\left[0,\tilde{W}-B\right]$.

One further simplification of expression (2.15) obtains when we assume that assets returns are independent. In this case, (2.15) becomes:

$$r_{f,i}=\overline{r}_i-\left(\overline{r}_i-r_f\right)\frac{a_i}{a_H}. \tag{2.17}$$

Also, (2.14) simplifies to:

$$\alpha_i=\frac{1}{W_0}\frac{\overline{r}_i-r_f}{a\sigma_i^2} \quad . \tag{2.18}$$

In this case, if the head trader imposes $W_{0,i}=W_0$ and $r_{f,i}=r_f$, that is to say, gives all wealth to *i-th* trader and uses as benchmark rate the interest rate that the trading firm faces, then the decentralized solution is equivalent to the centralized one as (2.18) gives the same allocation as (2.11), which is the trader *i*'s solution if and only if the *i-th* trader and the head trader have the same risk aversion coefficient.

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[2] As defined in Arrow [1970] and Pratt [1964].

3. Comparative Static

Equation (2.15) gives us a sufficient condition for the decentralized and centralized problems to coincide in the sense that the dollar amount invested in each risky asset will be the same in both problems. The dollar amount invested in the riskless asset is different though. However, this is irrelevant because traders' investments in riskless assets are only notional; it is the firm paying interest to itself.

If we look at equations (2.11) and (2.14) we can say that, ceteris paribus, if the expected return of the *i-th* asset increases, both head trader and traders would invest more in this risky asset. But from (2.15), for them to increase the investment in the risky asset i by the same amount, it is necessary that the benchmark for trader i changes according to:

$$\frac{\partial r_{f,i}}{\partial \overline{r}_i} = 1 - \frac{1}{1-R_i^2}\frac{a_i}{a_H} \tag{3.1}$$

The sign of (3.1) depends on the ratio of risk aversion coefficients. The term that pre-multiplies this ratio is greater than one. If the coefficients of risk aversion are similar and the return on asset i is highly correlated with other risky assets' returns (high R_i^2), equation (3.1) implies that trader *i's* benchmark should be reduced, leading to further investment in risky asset i by trader i. This is because the head trader would reduce investments not only in the riskless asset but also in the other risky assets correlated with asset i. Since trader i does not invest in other risky assets, he has to draw relatively more money from his riskless asset. Off course, the risk-free rate of other traders that trade risky assets positively correlated with the *i-th* trader would increase to compensate the overshooting in the reduction of investment in the riskless asset made by trader i (this is illustrated by equation (3.2) below).

However, if the head trader is much more risk averse than trader i and risky assets are not very correlated, so that the ratio of risk averse coefficient times the inverse of $(1-R_i^2)$ is less than one, trader *i's* benchmark should increase in order to counterbalance the increase in *i-th* asset expected return. Otherwise, differences in risk aversion would generate different centralized and decentralized allocations.

If the expected return of *i-th* risky asset increases and its return is positively correlated with *j-th* risky asset we should have a substitution effect decreasing investment in the *j-th* asset. This is captured by equation (3.2) bellow:

$$\frac{\partial r_{f,j}}{\partial \overline{r_i}} = \frac{\beta_{ji}}{1-R_j^2}\frac{a_j}{a_H}. \tag{3.2}$$

When the head trader increases the risk-free rate for traders he induces a risk exposure reduction. If β_{ji} is negative then the correlation between *i-th* and *j-th* asset returns is negative. In this case, investment in the *j-th* asset increases with increase in expected return in the *i-th* asset because it can be used to hedge an increase of investment in the *i-th* asset.

From (2.13) for the *i-th* trader and (2.12) for the *k-th* trader we get

$$r_{f,i} = \overline{r_i} - a_i\sigma_i^2\frac{\alpha_i}{\alpha_k}\alpha_{k,k}W_{0i}. \tag{3.3}$$

Using (2.11) for *k-th* trader we can find

$$\frac{\partial r_{f,i}}{\partial r_{f,k}} = \frac{\alpha_i}{\alpha_k}\frac{a_i\sigma_i^2}{a_k\sigma_k^2}. \tag{3.4}$$

4. The Equivalence of Centralized and Decentralized Solutions: a Generalization

One important generalization would be to allow traders to trade in more than one market. Actually, most firms have traders specialized in a few assets. In this section we allow traders to trade in subsets of risky assets and find the equivalence condition for the optimal portfolio allocation. To simplify understanding, we make this generalization in four steps. First, we study a typical trader solution. Second, we generalize the equivalence result where traders trade in non-intersecting subsets of risky assets. Third, we study the case where traders trade in the same subset of risky assets. Finally, we allow traders to trade in any subset of risky assets.

Trader *i*'s solution

We use an amount of wealth and one benchmark for each risky asset, so if the *i-th* trader manages *m* risky assets there will be *m* risk-free rates to control his optimal decisions. This *i-th* trader has initial wealth $W_{k,i}$, allocated to him by the head trader, to invest in the *k-th* risky asset. He has a different wealth available for each asset. Let $\alpha_{k,i}$ be the fractions of initial wealth invested in the *k-th* risky asset, by the *i-th* trader. Let $r_{f_k,i}$ be the risk-free return the trader pays to the firm to use wealth in his *k-th* risky asset investment. The Lagrangean function for the *i-th* trader is:

$$\ell = \sum_{k=1}^{m} W_{k,i}\alpha_{k,i}\left(\overline{r}_k - r_{f_k,i}\right) - \frac{a}{2}\left(\sum_{k=1}^{m}\sum_{j=1}^{m} W_{k,i}W_{j,i}\alpha_{k,i}\alpha_{j,i}\sigma_{kj}\right) - \sum_{k=1}^{m}\lambda_k\left(1-\alpha_{k,i}\right).$$

Assuming interior solution for each risky asset so that all Kuhn-Tucker multipliers are zero, the first order conditions are given by:

$$\frac{\partial \ell}{\partial \alpha_{k,i}} = \left(\overline{r}_k - r_{f_k,i}\right)W_{k,i} - \frac{a}{2}\left(2\alpha_{k,i}\sigma_k^2 W_{k,i}^2 + 2\sum_{\substack{j=1\\ j\neq k}}^{m} W_{k,i}W_{j,i}\alpha_{j,i}\sigma_{jk}\right) = 0 \quad k = 1,2....m \quad (4.1)$$

Observing that when a trader has exponential utility the amount he invests in the risky assets is independent of wealth, we can make, without loss of generality, the simplifying assumption that $W_{k,i} = W_{j,i} \quad \forall i, j$ and $W_{0,i} \equiv W_{k,i}$. Then the first-order conditions become

$$\left(\overline{r}_k - r_{f_k,i}\right) = aW_{0,i}\left(\sum_{j=1}^{m}\alpha_{j,i}\sigma_{jk}\right), \quad k = 1,2....m \qquad (4.2)$$

Thus we have an *m* equations system to solve. Let

$$\mathbf{R} = \begin{pmatrix} \overline{r}_{1,i} - r_{f_1,i} \\ \overline{r}_{2,i} - r_{f_2,i} \\ . \\ . \\ \overline{r}_{m,i} - r_{f_m,i} \\ \end{pmatrix}.$$

The first-order conditions give

$$\mathbf{\Sigma}_{m\times m}\tilde{\boldsymbol{\alpha}}_{m\times 1} = \left(\frac{1}{aW_{0,i}}\right)\mathbf{R}_{m\times 1}.$$

Using the same approach as in section 2 we derive the proportion to be allocated in the *k-th* risky asset by the *i-th* trader.

$$\alpha_{k,i} = \frac{1}{W_{0,i}}\frac{1}{a_i\sigma_k^2\left(1-\mathrm{R}_{k,i}^2\right)}\left[\overline{r}_k - r_{f_k,i} - \sum_{\substack{j=1\\ j\neq k}}^{m}\beta_{kj}\left(\overline{r}_j - r_{f_j,i}\right)\right]. \tag{4.3}$$

Using

$$\alpha_{k,i}W_{0,i} = \alpha_k W_0 \qquad k = 1, 2......m \tag{4.4}$$

to generate the equivalence, we derive:

$$\frac{1}{a_i\sigma_k^2\left(1-\mathrm{R}_{k,i}^2\right)}\left[\overline{r}_k - r_{f_k,i} - \sum_{\substack{j=1\\ j\neq k}}^{m}\beta_{kj}\left(\overline{r}_j - r_{f_j,i}\right)\right] = \frac{1}{a\sigma_k^2\left(1-\mathrm{R}_{\mathrm{k}}^2\right)}\left[\overline{r}_k - r_f - \sum_{\substack{j=1\\ j\neq k}}^{n}\beta_{kj}\left(\overline{r}_j - r_f\right)\right]$$

where $\mathrm{R}_{k,i}^2$ and $\mathrm{R}_{\mathrm{k}}^2$ are the R-squared for the multiple regression of the excess return for the *k-th* asset on the excess returns of *m* risky markets for the *i-th* trader and the R-squared for the multiple regression of the excess return for the head trader for all the *n* risky assets, respectively. Solving this equation for the risk-free rate for the *k-th* asset we derive:

$$\left[r_{f_k,i} - \sum_{\substack{j=1 \\ j\neq k}}^{m} \beta_{kj} \left(r_{f_j,i} \right) \right] = \overline{r}_k - \frac{a_i \sigma_k^2 \left(1-\mathrm{R}_{k,i}^2\right)}{a \sigma_k^2 \left(1-\mathrm{R}_{\mathrm{k}}^2\right)} \left[\overline{r}_k - r_f - \sum_{\substack{j=1 \\ j\neq k}}^{n} \beta_{kj} \left(\overline{r}_j - r_f \right) \right] - \sum_{\substack{j=1 \\ j\neq k}}^{m} \beta_{kj} \left(\overline{r}_j \right) \qquad (4.5)$$

Thus we have m equations that must be determined simultaneously for $r_{f_k,i}$ and $r_{f_j,i}$ for $j = 1,...m;\quad j \neq k$.

The case where traders trade in non-intersecting subsets of risky assets

If there are many traders, each with a subset of the full asset basket, without intersection between these subsets, then all that the head trader has to do is to solve these systems separately for each trader. We are assuming that traders are not specialized in the same assets, but instead, all traders take decisions regarding different assets.

We can write (4.5) in matrix notation. Assuming that

$$\mathbf{b} = \begin{bmatrix} 1 & -\beta_{12} & -\beta_{13} & \cdot & -\beta_{1m} \\ -\beta_{21} & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ . & . & . & . & . \\ -\beta_{m1} & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

has full rank, we have

$$
\begin{pmatrix} r_{f_1,i} \\ r_{f_2,i} \\ \cdot \\ \cdot \\ r_{f_m,i} \end{pmatrix}_{m\times 1} = \mathbf{b}_{m\times m}^{-1} \left\{ \begin{pmatrix} \overline{r}_1 \\ \overline{r}_2 \\ \cdot \\ \cdot \\ \overline{r}_m \end{pmatrix}_{m\times 1} - \begin{pmatrix} \overline{r}_1 - r_f - \sum_{j=2}^{n} \beta_{1j}\left(\overline{r}_j - r_f\right) & 0 & \cdot & \cdot & 0 \\ 0 & \overline{r}_2 - r_f - \sum_{\substack{j=1 \\ j\neq 2}}^{n} \beta_{2j}\left(\overline{r}_j - r_f\right) & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & \overline{r}_m - r_f - \sum_{\substack{j=1 \\ j\neq m}}^{n} \beta_{mj}\left(\overline{r}_j - r_f\right) \end{pmatrix}_{m\times m} \times \begin{pmatrix} B_{1,i} \\ B_{2,i} \\ \cdot \\ \cdot \\ B_{m,i} \end{pmatrix}_{m\times 1} - \begin{pmatrix} \sum_{j=2}^{m} \beta_{1j}\left(\overline{r}_j\right) \\ \sum_{\substack{j=1 \\ j\neq 2}}^{m} \beta_{2j}\left(\overline{r}_j\right) \\ \cdot \\ \cdot \\ \sum_{\substack{j=1 \\ j\neq m}}^{m} \beta_{mj}\left(\overline{r}_j\right) \end{pmatrix}_{m\times 1} \right\}
$$

where $B_{k,i} = \dfrac{a_i \sigma_k^2 \left(1 - \mathrm{R}_{k,i}^2\right)}{a \sigma_k^2 \left(1 - \mathrm{R}_{\mathrm{k}}^2\right)}$.

If we assume that assets returns are independent then it does not matter whether the traders solve the problem in a decentralized or centralized way, the optimal solution will be the same.

The case where traders trade in the same subsets of risky assets

Another generalization would be the case were there are l traders taking decisions in m risky markets and the head trader's asset basket is composed of n risky assets, where $n > m$. We assume that all traders observe the same m risky assets and take allocation decisions in these markets based on their inferences for expected return, risk and their own degrees of risk aversion. To generate equivalence one would need:

$$\alpha_{k,1}W_{0,1} + \alpha_{k,2}W_{0,2} + ... + \alpha_{k,l}W_{0,l} = \alpha_k W_0 \qquad k = 1, 2......m \tag{4.6}$$

Let $A_{k,h} = \dfrac{1}{a_h \sigma_k^2 \left(1 - \mathrm{R}_{k,h}^2\right)}$, let $r_{f_k,i} = r_{f_k,j} \quad \forall \quad i, j$, so the riskless rate would be the same for different traders that take decisions on the same asset, let $A_k = \dfrac{1}{a\sigma_k^2 \left(1 - \mathrm{R}_{\mathrm{k}}^2\right)}$ and $z_k = \left[\bar{r}_k - r_f - \sum_{\substack{j=1 \\ j \neq k}}^{n} \beta_{kj} \left(\bar{r}_j - r_f \right) \right]$. Rearranging terms to solve for the risk-free rates, we can solve (4.6) as:

$$r_{f_k,i} - \sum_{\substack{j=1 \\ j \neq k}}^{m} \beta_{kj} r_{f_j,i} = \bar{r}_k - \sum_{\substack{j=1 \\ j \neq k}}^{m} \beta_{kj} \left(\bar{r}_j \right) - \left(\sum_{i=1}^{l} A_{k,i} \right) A_k z_k \, . \qquad i = 1, ..., l \, , \quad k = 1, ..., m \tag{4.7}$$

We have the following system to solve, in matrix notation:

$$\mathbf{r}_{\mathbf{f}\,m\times 1} = \mathbf{b}_{m\times m}^{-1} \times \left[\mathbf{b}_{m\times m} \times \bar{\mathbf{r}}_{m\times 1} - \mathbf{A}_{m\times 1} \right] \tag{4.8}$$

where $\mathbf{b}_{m\times m}$ is the same as defined before, $\mathbf{r}_{\mathbf{f}\,m\times 1}$ is a vector of risk-free rates for the *i-th* trader, $\bar{\mathbf{r}}_{m\times 1}$ is a vector of risky rates of returns and $\left(\sum_{i=1}^{l} A_{k,i} \right) A_k z_k$ is the *k-th* element of the vector $\mathbf{A}_{m\times 1}$.

The general case

The final generalization would be the case where each trader's portfolio is a subset of the global portfolio. Assuming that an allocation made by the head trader is a vector in $\Re^n$ we would have traders' portfolios in subspaces $\Re^l \subseteq \Re^n$, where $l \leq n$. Different traders can build portfolios in such a way that the intersection of assets between portfolios is not always an empty set. This would be the most general case. Let $\alpha_{j,i}$ be the allocation made by the *i-th* trader in *j-th* asset. Then, we have

$$\alpha_{j,i} = \alpha_{j,i}\left(\tilde{r}^i, \Sigma^i, r_{f_j,i}, r_{f_k,i}, a_i, W_{0,i}\right). \tag{4.9}$$

The allocation in the *j-th* asset depends on expected returns, covariance matrix on assets traded by the *i-th* trader, on the risk-free interest rates for each asset, on his risk aversion coefficient and on his initial endowment.

If we assume that there are l traders, and n assets, then equivalence can be achieved if the condition below is satisfied:

$$\sum_{i=1}^{l} \alpha_{j,i} W_{0,i} = \sum_{i=1}^{l} \alpha_{j,i}\left(\tilde{r}^i, \Sigma^i, r_{f_j,i}, r_{f_k,i}, a_i, W_{0,i}\right) W_{0,i} = \alpha_j W_0 \quad j = 1,2,\ldots n \tag{4.10}$$

The solution would be the same as found in the previous case with a slight difference[3]: traders trade in subsets of the global portfolio, so by construction some $\alpha_{j,i}$ are zero (for some asset j and trader i). This means that the matrixes used before would have some zero elements.

Another way to look at condition (4.10) would be that we have a system with n equations and n unknown parameters (risk-free interest rates for each asset). As this is a linear system, if the determinant of the system is different from zero (which generally is), we always have a solution for it that can be implemented numerically.

[3] As before we can assume that the benchmark on *j-th* asset is the same for all traders which trade on that asset.

To do that we need to estimate the risk aversion coefficient for each trader, which is not an especially difficult task. Then the implementation of the methodology of this paper in a real world situation could be easily done by allowing traders to trade with some predetermined amount of wealth and benchmark for each asset he trades.

5. Summary

In this paper conditions were derived for the solutions to the centralized and decentralized problems to be equivalent. This is important because the mean-variance model can be implemented by firms with decentralized decisions, as most large trading firms are. This equivalence is established for the case when traders have no better information than the head trader. The next step in our research agenda is to study the case when traders are better informed than the head trader.

We show that if the head trader uses a different interest rate for each trader to borrow or lend as control variable, he can achieve the same allocation with the decentralized portfolio decision problem as in the centralized portfolio decision problem. Furthermore, we found that this interest rate depends on the ratio of the degree of risk aversion of the trader and head trader, on excess returns, and on correlation of assets.

We allow traders to trade in m markets and found that in this framework it is possible to achieve the same allocation with the decentralized portfolio decision problem as in the centralized one as we use the interest rate to borrow or lend as a control variable. In this case the head trader must solve m simultaneous equations to find these interest rates simultaneously. The same problem is solved for the case where all traders trade at the same subset of the global assets. Finally, a general solution is shown, where traders are allowed to trade in any subset of risky assets.

Further generalizations would be dropping the CARA utility function assumption and using an intertemporal framework. Besides, allowing for short sales would be an interesting issue to examine. Sometimes traders may want to stay short in the market and the principal (head trader) could supply them with collateral (instead of money).

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