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A Class of Incomplete and Ambiguity Averse Preferences*

Leandro Nascimento [†] Gil Riella[‡]

Abstract

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This paper characterizes ambiguity averse preferences in the absence of the completeness axiom. We axiomatize multiple selves versions of some of the most important examples of complete and ambiguity averse preferences, and characterize when those incomplete preferences are ambiguity averse.

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1 Introduction

The subjective expected utility model Savage formulated in 1954[1] has been criticized on the basis it does not provide a good description of a decision maker's attitude towards ambiguity. It was initially suggested by Ellsberg (1961) that the decision maker does not behave as if he forms a unique subjective probability (or is surrounded by a set of priors and ignores all but one). The same critique applies to the alternative formulation of Anscombe and Aumann (1963). Here the independence axiom precludes the Ellsberg-type behavior that has been observed in experimental work.[2] A broad literature has attempted to formulate models of decision making that accommodate the Ellsberg-type behavior. A large part of this literature works within the Anscombe-Aumann framework and weakens the independence axiom.

The majority of models of decision making (under uncertainty or not) assume that preferences are complete in that every pair of alternatives is comparable. Such a postulate has been criticized as being unrealistic. For instance, in an early contribution to the study of incomplete preferences, Aumann (1962) argued that the completeness axiom is an inaccurate description of reality and also hard to accept from a normative viewpoint: "rationality" does not demand the agent to make a definite comparison of every pair of alternatives. Mandler (2005) formalizes the last point by showing that agents with incomplete preferences are not necessarily subject to money-pumps, and consequently not "irrational" in some sense.

In the context of decision making under uncertainty in the Anscombe-Aumann framework, the Knightian uncertainty model of Bewley (1986) and the recent single-prior expected multi-utility model of Ok, Ortoleva, and Riella (2008) remain the only ones which satisfy transitivity, monotonicity and allow for incompleteness of preferences.[3] Nevertheless, because both models satisfy the independence axiom, they cannot cope with the sort of criticism initially raised by Ellsberg (1961). At the same time, preferences that accommodate Ellsberg-type behavior such as the multiple priors model of Gilboa and Schmeidler (1989) and the (more general) variational preferences of Maccheroni, Marinacci, and Rustichini (2006) are complete.

Our main contribution is to identify a class of preferences that is incomplete and

[1]Savage (1972).

[2]See Camerer (1995) for a survey of the experimental work testing Ellsberg's predictions.

[3]If we do not require the agent's preferences to be monotone, then we also have the additively separable expected multi-utility model as another example of incomplete preferences under uncertainty. See Ok et al. (2008) and the references therein for the details. Faro (2008) derives a generalization of Bewley (1986) by not requiring preferences to be transitive.

at the same time can explain the Ellsberg-type of behavior. Building on behaviorally meaningful axioms on an enlarged domain of lotteries of Anscombe-Aumann acts, we construct multiple selves versions of the Gilboa and Schmeidler (1989) and Maccheroni et al. (2006) models. We also sketch a more general version of an incomplete and ambiguity averse preference relation along the lines of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2008) on the domain of Anscombe-Aumann acts.

To illustrate our representation, consider for instance the standard Gilboa and Schmeidler (1989) model. The decision maker entertains a "set of priors" M, and ranks an act f according to the single utility index

$$V_{GS}(f) = \min_{\mu \in M} \int u(f)\, d\mu.$$

In our representation the decision maker conceives a "class" $\mathcal{M}$ of possible sets of priors, and prefers the act f to g iff

$$V_{GS}^{M}(f) = \min_{\mu \in M} \int u(f)\, d\mu \geq \min_{\mu \in M} \int u(g)\, d\mu = V_{GS}^{M}(g) \text{ for all } M \in \mathcal{M}.$$

Instead of looking at a single objective function V_{GS}, his decisions are now driven by the vector $\left(V_{GS}^{M}\right)_{M \in \mathcal{M}}$ of objectives.[4] If each set M is a singleton, this is exactly the model proposed by Bewley (1986). When the class $\mathcal{M}$ is a singleton, we obtain the Gilboa-Schmeidler model. Another contribution of this paper is to show that the canonical model of Knightian uncertainty of Bewley (1986) belongs to the same class of incomplete preferences as the (complete) multiple priors and variational preferences.

This paper faces two major difficulties in axiomatizing the multiple selves version of the models mentioned above. First, we do not have an answer to what happens if one drops the completeness axiom in its entirety. Instead, we assume a weak form of completeness by requiring that the preference relation is complete on the subdomain of constant acts. That is, the Partial Completeness axiom of Bewley (1986) is assumed. Second, as we have already pointed out, in most of the paper we work with preferences defined on the domain of lotteries of acts, and not on the

[4]That collection of objectives arises from the multiplicity of sets of priors. Such multiplicity seems to be as plausible as the existence of second order beliefs. For instance, they can be interpreted as the support of a collection of second order beliefs, and the decision maker is a pessimistic agent which extracts a utility index from each of those beliefs by looking at the worst event (in this case the worst prior) in the support. As an incomplete list of recent models of second order beliefs, see Klibanoff, Marinacci, and Mukerji (2005), Nau (2006), and Seo (2007).

standard domain of Anscombe-Aumann acts. This enlarged domain is not a novel feature of this paper, and it was recently employed by Seo (2007). Our representation in such a framework induces a characterization of a class of incomplete preferences on the subdomain of Anscombe-Aumann acts whose relation to other classes of preferences in the literature is depicted in Figure 1.

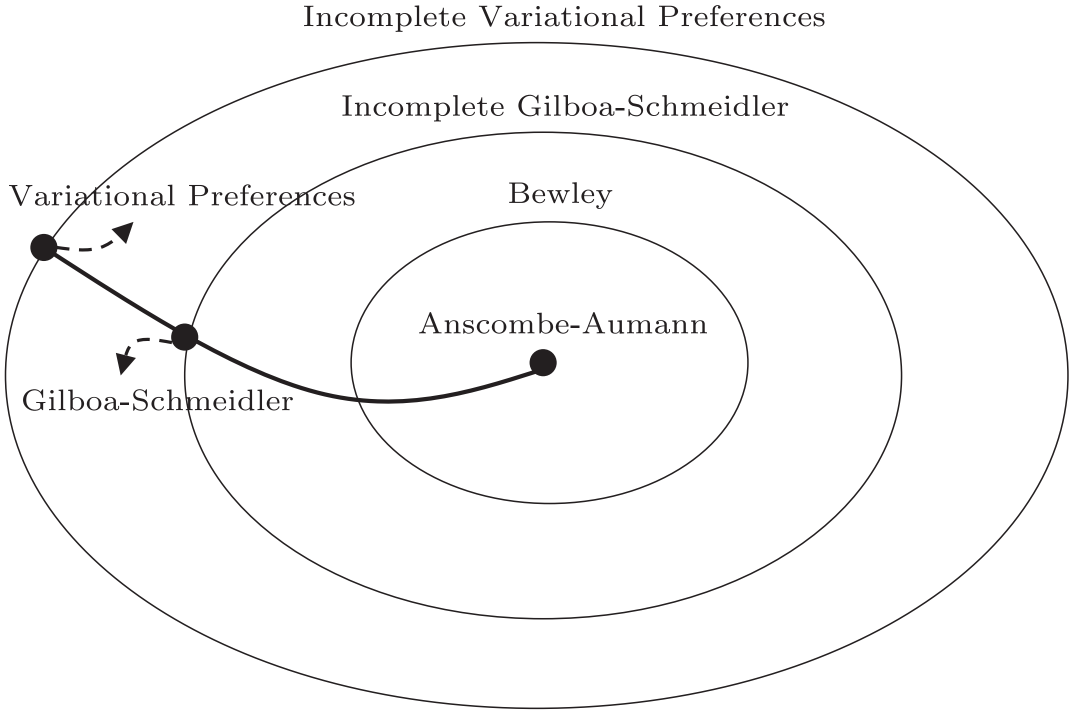

Figure 1: Preferences satisfying partial completeness and monotonicity

In spite of using the same setup of Seo (2007), who constructs a model that accommodates ambiguity aversion and does not assume reduction of compound objective lotteries, our model is not able to explain Halevy's (2007) findings of a strong empirical association between reduction of compound objective lotteries and ambiguity neutrality. We explicitly assume reduction of such lotteries in our axioms, and at the same time claim that decision makers with the preferences axiomatized in this paper are ambiguity averse provided a mild "consistency" condition among the multiple selves holds.

Every model is false, and ours are not immune to that. Nevertheless, we do not share the view that our models are subject to Halevy's (2007) criticisms. His experiments are a valid test of his main thesis (viz. the correlation between ambiguity neutrality and reduction of compound objective lotteries) provided his auxiliary assumptions, especially the completeness of preferences, are true. Therefore, it is not clear whether his critique applies when preferences are incomplete. For instance, the mechanism Halevy (2007) uses to elicit preferences from subjects is valid only under the completeness axiom.[5] To the best of our knowledge, there is no experimental work that explores the results of Eliaz and Ok (2006) regarding choice correspondences rationalized by an incomplete preference relation in order to correctly elicit those preferences.

1.1 Ellsberg-type behavior: example

Consider the example from Ellsberg (2001) as described by Seo (2007). There is a single urn, with 200 balls. Each ball can have one and only one of four colors: two different shades of red (RI and RII), and two different shades of black (BI and BII). One hundred balls are either RI or BI. Fifty of the remaining balls are RII, and the other fifty are BII. There are six alternative bets available to the decision maker. Bet A is such that he wins if a ball of color RI is drawn. Similarly, define the bets B, C and D on the colors BI, RII, and BII, respectively. Also define the bet AB as the bet in which the decision maker wins if a ball of color RI or BI is drawn, and the bet CD as the bet in which he wins if a ball of color RII or BII is drawn. Finally, assume the winning prize is such that the utility of winning is 1, and the utility of losing is 0.

In the original experiment, agents rank the bets according to: $C \sim D \succ A \sim B$, and $AB \sim CD$. Our model can explain the case in which $AB \sim CD$, $C \sim D \succ A, B$, and A and B are not comparable. Consider, for example, a Gilboa-Schmeidler incomplete preference relation.

The state space is $S := \{RI, BI, RII, BII\}$. The decision maker entertains two sets of priors: the first one is given by $M_1 := co\left\{\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), \left(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)\right\}$ and the second by $M_2 := co\left\{\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), \left(\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{4}\right)\right\}$.[6] That is, the decision maker is composed of two selves. One self, associated with M_1, has two extreme priors on states: a uniform prior, and one that assigns zero probability to the event a ball of color RI is drawn. The other self, associated with M_2, shares one of the extreme priors ($\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$), but is less confident about the odds of a ball of color BI: he also contemplates a prior that attaches zero probability to the event BI is drawn.

[5] The very existence of certainty equivalents to bets on Halevy's (2007) urns, which the author used to elicit preferences, hinges on the completeness assumption.

[6] The convex hull of any subset F of a vector space is denoted by $co(F)$.

The bets are ranked according to

$$U(A) = \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix}, U(B) = \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix},$$
$$U(C) = U(D) = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix},$$
$$U(AB) = U(CD) = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix},$$

where the first component of each vector denotes the utility associated with the set of priors M_1, and the second component is associated with M_2. One can check that this ranking explains the Ellsberg-type behavior mentioned above.

1.2 Outline of the paper

The paper is organized as follows. In Section 2 we introduce the basic setup. Section 3 gives a characterization of preferences represented by a multiple selves version of the maxmin expected utility model and shows its uniqueness. In Section 4 we characterize the multiple selves version of the variational preferences and prove a similar uniqueness result. Section 5 discusses when those incomplete preferences are ambiguity averse. In Section 6 we give some steps towards an axiomatization of a more general version of an incomplete and ambiguity averse preference relation. While Section 7 concludes the paper with additional remarks and open questions, the Appendix contains the proofs of our main results.

2 Setup

The set X denotes a compact metric space. Let $\Delta(X)$ be the set of Borel probability measures on X, and endow it with any metric that induces the topology of weak convergence. We denote by $\mathcal{B}(X)$ the Borel σ-algebra on X. Note that $\Delta(X)$ is a compact metric space. Let the set of states of the world be denoted by S, which we assume to be finite. The set of Anscombe-Aumann acts is $\mathcal{F} := \Delta(X)^S$, and is endowed with the product topology (hence compact).

The decision maker has preferences $\succcurlyeq$ on the set of lotteries on $\mathcal{F}$, that is, $\succcurlyeq \subseteq \Delta(\mathcal{F}) \times \Delta(\mathcal{F})$. The class of sets $\mathcal{B}(\mathcal{F})$ is the Borel σ-algebra on $\mathcal{F}$. The domain of preferences $\Delta(\mathcal{F})$ is endowed with the topology of weak convergence (hence compact). Let the binary relation $\succcurlyeq^\bullet \subseteq \Delta(X) \times \Delta(X)$ be defined as $p \succcurlyeq^\bullet q$ iff $\langle p \rangle \succcurlyeq \langle q \rangle$, where $\langle r \rangle \in \mathcal{F}$ denotes the (constant) act h,[7] where $h(s) = r \in \Delta(X)$ for all $s \in S$. That is, $\succcurlyeq^\bullet$ is the restriction of $\succcurlyeq$ to the set of all constant acts. Note that, with a slight abuse of notation, $\Delta(X) \subseteq \mathcal{F} \subseteq \Delta(\mathcal{F})$ because we can identify each $p \in \Delta(X)$ with the constant act $\langle p \rangle$, and each $f \in \mathcal{F}$ with the degenerate lottery $\delta_f \in \Delta(\mathcal{F})$.

Define two mixture operations, one on the space of Anscombe-Aumann acts, and the other on the space of lotteries of acts, as follows. Let the mixture operation $\oplus$ on $\mathcal{F}$ be such that, for all $f, g \in \mathcal{F}$, $\lambda \in [0,1]$, $(\lambda f \oplus (1-\lambda) g) \in \mathcal{F}$ is defined as $(\lambda f \oplus (1-\lambda) g)(s)(B) = \lambda f(s)(B) + (1-\lambda) g(s)(B)$ for all $s \in S$, and $B \in \mathcal{B}(X)$. That is, if we look at the inclusion $\mathcal{F} \subseteq \Delta(\mathcal{F})$, then $(\lambda f \oplus (1-\lambda) g)$ is identified with $\delta_{\lambda f + (1-\lambda) g}$. Also define the mixture operation $+$ on $\Delta(\mathcal{F})$ such that, for all $P, Q \in \Delta(\mathcal{F})$, $\lambda \in [0,1]$, $(\lambda P + (1-\lambda) Q) \in \Delta(\mathcal{F})$ is defined as $(\lambda P + (1-\lambda) Q)(B) = \lambda P(B) + (1-\lambda) Q(B)$ for all $B \in \mathcal{B}(\mathcal{F})$. Again, if we look at the inclusion $\mathcal{F} \subseteq \Delta(\mathcal{F})$, then $\lambda f + (1-\lambda) g$ is identified with $\lambda \delta_f + (1-\lambda) \delta_g$.

2.1 Remarks

The setup is the same as in Seo (2007). It ads to the standard setting a second layer of objective uncertainty through the objective mixtures of acts. Each act $f \in \mathcal{F}$ delivers an objective lottery $f(s) \in \Delta(X)$ in state s, and the decision maker is asked to make an assessment of any such act and of each possible objective lottery $P \in \Delta(\mathcal{F})$ whose prizes are Anscombe-Aumann acts.

The timing of events is the following. In the first stage, we run a spin with each outcome $f \in \mathcal{F}$ having (objective) probability $P(f)$. Next, nature selects a state $s \in S$ to be realized; this intermediate stage has subjective uncertainty. Finally, in the second stage, we run another spin, conditional on the prize f from the first stage and independently of anything else, with each outcome event $B \in \mathcal{B}(X)$ having (objective) probability $f(s)(B)$.

The introduction of an additional layer of objective uncertainty is not innocuous and will play a distinct role in the axiomatization below. In particular, the way the decision maker compares the objects $\lambda f + (1-\lambda) g$ and $\lambda f \oplus (1-\lambda) g$ determines part of the shape of his preferences. In the Anscombe-Aumann model, for instance, the decision maker is indifferent between $\lambda f + (1-\lambda) g$ and $\lambda f \oplus (1-\lambda) g$: it does

[7] Or, being more precise, the degenerate lottery that gives probability one to the constant act h.

not matter whether the randomization comes before or after the realization of the subjective state.

The indifference of the decision maker between $\lambda f + (1-\lambda) g$ and $\lambda f \oplus (1-\lambda) g$ is called "reversal of order" in the literature. In the setup of Seo (2007), ambiguity neutrality can also be characterized in terms of reduction of compound lotteries, i.e., when the decision maker is indifferent between the objects $\lambda \langle p \rangle + (1-\lambda) \langle q \rangle$ and $\lambda \langle p \rangle \oplus (1-\lambda) \langle q \rangle$. Such characterization relies on a dominance axiom that will not be assumed here. This means that, whenever we assume the weak condition that the decision maker is always indifferent between $\lambda \langle p \rangle + (1-\lambda) \langle q \rangle$ and $\lambda \langle p \rangle \oplus (1-\lambda) \langle q \rangle$, this will not imply that his preferences also satisfy reversal of order.

3 Incomplete Multiple Priors Preferences

We will use the following set of axioms to characterize preferences.

Axiom A1 (Preference Relation). *The binary relation $\succcurlyeq$ is a preorder.*

Axiom A2 (First Stage Independence). *For all $P, Q, R \in \Delta(\mathcal{F})$, $\lambda \in (0,1)$: if $P \succcurlyeq Q$, then $\lambda P + (1-\lambda) R \succcurlyeq \lambda Q + (1-\lambda) R$.*

Axiom A3 (Continuity). *If $(P^n), (Q^n) \in \Delta(\mathcal{F})^\infty$ are such that $P^n \succcurlyeq Q^n$ for all n, $P^n \to P \in \Delta(\mathcal{F})$, and $Q^n \to Q \in \Delta(\mathcal{F})$, then $P \succcurlyeq Q$.*

Axiom A4 (Partial Completeness). *The binary relation $\succcurlyeq^\bullet$ is complete.*

Axiom A5 (Monotonicity). *For all $f, g \in \mathcal{F}$: if $\langle f(s) \rangle \succcurlyeq \langle g(s) \rangle$ for all $s \in S$, then $f \succcurlyeq g$.*

Axiom A6 (C-Reduction). *For all $f \in \mathcal{F}$, $p \in \Delta(X)$, $\lambda \in (0,1)$: $\lambda f \oplus (1-\lambda) \langle p \rangle \sim \lambda f + (1-\lambda) \langle p \rangle$.*

Axiom A7 (Strong Uncertainty Aversion). *For all $f, g \in \mathcal{F}$, $\lambda \in (0,1)$: $\lambda f \oplus (1-\lambda) g \succcurlyeq \lambda f + (1-\lambda) g$.*

Axiom A8 (Nondegeneracy). $\succ \neq \emptyset$.

Axioms A1 and A4 are a weakening of the widespread "weak order" (complete preorder) assumption in the literature. By relaxing the completeness requirement, our preferences can rationalize a wide range of behavior, including whatever choice patterns were rationalized under the completeness axiom, plus, e.g., choice behavior that violates the independence of irrelevant alternatives. Axiom A4 imposes a minimum of comparability on preferences. It requires that, when facing only risk, the decision maker's preferences are complete. This Partial Completeness axiom is also present in Bewley (1986). It allows us to pin down a single utility index that represents the complete preference relation $\succcurlyeq^{\bullet}$ on the subdomain of objective lotteries (constant acts).

The First Stage Independence axiom is also present in Seo (2007). It requires the decision maker to satisfy independence when facing the objective probabilities induced by the lotteries of acts. This requirement is standard in the literature: whenever the individual faces objective uncertainty, it is common to impose independence. Our Continuity axiom A3, also called "closed-continuity", is also standard and demands that pairwise comparisons are preserved in the limit.[8]

Axiom A5 is the AA-Dominance of Seo (2007). He also uses a stronger dominance axiom to obtain a second order subjective expected utility representation, and this axiom is not assumed here. Instead, we replace his stronger dominance axiom by A6 and A7, and also relax his completeness axiom on lotteries of acts. Also note that axioms A1-A3 and A6 imply Second Stage Independence for constant acts, that is: for all $p, q, r \in \Delta(X)$, $\lambda \in (0,1)$, $\langle p \rangle \succcurlyeq \langle q \rangle$ iff $\lambda \langle p \rangle \oplus (1-\lambda) \langle r \rangle \succcurlyeq \lambda \langle q \rangle \oplus (1-\lambda) \langle r \rangle$.

From the original axioms of Gilboa and Schmeidler (1989), we only retain the Monotonicity axiom A5 and the Nondegeneracy axiom A8 in their original formats, and also part of their weak order axiom, which is weakened here to A1 and A4 after we drop the completeness requirement. The axioms A2 and A3 pertain to the domain of lotteries of acts $\Delta(\mathcal{F})$ and cannot be directly compared with the Gilboa-Schmeidler axioms.

The axioms A6 and A7 together give the shape of each utility function in the representation of $\succcurlyeq$ on $\mathcal{F}$: they are concave, positively homogeneous, and vertically invariant functions.[9] Strong Uncertainty Aversion says that the degenerate lottery of acts $\delta_{\lambda f+(1-\lambda)g}$ is preferred to $\lambda\delta_f + (1-\lambda)\delta_g$. Ultimately, the first stage mixture $\lambda\delta_f + (1-\lambda)\delta_g$ contains two sources of subjective uncertainty: one is the uncertainty

[8]Note that axioms A1-A3 imply: for all $P, Q, R \in \Delta(\mathcal{F})$, $\lambda \in (0,1)$: if $\lambda P + (1-\lambda) R \succcurlyeq \lambda Q + (1-\lambda) R$, then $P \succcurlyeq Q$. See Dubra, Maccheroni, and Ok (2004) for an account of this fact and a discussion of the Continuity axiom.

[9]A version of A6 was used by Epstein, Marinacci, and Seo (2007) under the name of "certainty reversal of order" in the context of complete preferences over menus.

about the payoff of f, and the other about the payoff of g. Therefore, axiom A7 can be interpreted as aversion to subjective uncertainty in that the decision maker prefers (ex-ante) to face the single source of uncertainty present in $\lambda f \oplus (1-\lambda) g$ than face uncertainty on both f and g in $\lambda \delta_f + (1-\lambda) \delta_g$. Now, both $\lambda \delta_f + (1-\lambda) \delta_{\langle p \rangle}$ and $\lambda f \oplus (1-\lambda) \langle p \rangle$ have a single source of subjective uncertainty. The C-Reduction axiom says that in this case the decision maker is indifferent between those lotteries of acts.

Theorem 1. *The following are equivalent:*

(a) $\succcurlyeq$ *satisfies A1-A8.*

(b) *There exist* $u : \Delta(X) \to \mathbb{R}$ *continuous, affine, and nonconstant, and a class* $\mathcal{M}$ *of nonempty, closed and convex subsets of the* $|S|-1$*-dimensional simplex* $\Delta(S)$ *such that, for all* $P, Q \in \Delta(\mathcal{F})$,

$$\begin{gathered} P \succcurlyeq Q \\ \textit{iff} \\ \int \left[\min_{\mu \in M} \int u(f)\, d\mu \right] dP(f) \geq \int \left[\min_{\mu \in M} \int u(f)\, d\mu \right] dQ(f), \end{gathered} \tag{1}$$

for all $M \in \mathcal{M}$. *In particular, for all* $f, g \in \mathcal{F}$,

$$f \succcurlyeq g \textit{ iff } \min_{\mu \in M} \int u(f)\, d\mu \geq \min_{\mu \in M} \int u(g)\, d\mu \textit{ for all } M \in \mathcal{M}. \tag{2}$$

If we define $U_M(f) := \min_{\mu \in M} \int u(f)\, d\mu$, then (1) is the Expected Multi-Utility representation of Dubra et al. (2004) on the set of lotteries on $\mathcal{F}$ with $\{U_M : M \in \mathcal{M}\}$ being the set of utility functions on the space of prizes in their representation. The restriction of $\succcurlyeq$ to the set of Anscombe-Aumann acts admits the representation in (2). The maxmin expected utility representation of Gilboa and Schmeidler (1989) now becomes a special case of (2) when $|\mathcal{M}| = 1$. In the event each set $M \in \mathcal{M}$ is a singleton, we obtain the Knightian uncertainty model of Bewley (1986). This is easily done by strengthening A6 to the condition that, for all $f, g \in \mathcal{F}$, $\lambda \in (0,1)$: $\lambda f \oplus (1-\lambda) g \sim \lambda f + (1-\lambda) g$. By assuming in addition that $\succcurlyeq$ is complete one obtains the Anscombe and Aumann (1963) representation.

Let $\mathfrak{M}$ denote the class of all nonempty, closed and convex subsets of the $|S|-1$ dimensional simplex. The set $\mathfrak{M}$ is endowed with the Hausdorff metric d_H. A pair $(u, \mathcal{M})$ that represents $\succcurlyeq$ is unique in the sense we establish next.

Proposition 1. *Let $u, v \in C(\Delta(X))$ be affine and nonconstant, and $\mathcal{M}, \mathcal{N} \subseteq \mathfrak{M}$. The pairs $(u, \mathcal{M})$ and $(v, \mathcal{N})$ represent $\succcurlyeq$ in the sense of Theorem 1 iff u is a positive affine transformation of v, and $cl_{d_H}(co(\mathcal{M})) = cl_{d_H}(co(\mathcal{N}))$.*[10]

4 Incomplete Variational Preferences

In deriving the incomplete preferences version of Gilboa and Schmeidler (1989), we explicitly used the C-Reduction axiom to make each U vertically invariant and positively homogeneous. Incomplete variational preferences are more general and only require U to be vertically invariant. This property is satisfied if we drop A5 and A6, and replace them by the following axioms.

Axiom A5' (C-Mixture Monotonicity). *For all $f, g \in \mathcal{F}$, $p, q \in \Delta(X)$, $\lambda \in (0, 1]$: if $\lambda \langle f(s) \rangle + (1-\lambda) \langle p \rangle \succcurlyeq \lambda \langle g(s) \rangle + (1-\lambda) \langle q \rangle$ for all $s \in S$, then $\lambda f + (1-\lambda) \langle p \rangle \succcurlyeq \lambda g + (1-\lambda) \langle q \rangle$.*

Axiom A6' (Reduction of Lotteries). *For all $p, q \in \Delta(X)$, $\lambda \in (0, 1)$: $\lambda \langle p \rangle \oplus (1-\lambda) \langle q \rangle \sim \lambda \langle p \rangle + (1-\lambda) \langle q \rangle$.*

Axiom A5' is a generalization of the standard Monotonicity axiom A5. It incorporates A5 as a special case when $\lambda = 1$. Moreover, it is not difficult to show that, under the C-Reduction axiom A6, A5' is implied by A5. Note that A5 and A5' are distinct forms of monotonicity. The former is the standard Monotonicity axiom because it pertains to the domain of acts, while the latter requires some sort of monotonicity on the domain of objective mixtures (lotteries) of acts. Axiom A6' is a weakening of A6. Technically, axiom A6' is used to identify a single continuous and affine utility function representing preferences on the subdomain of constant acts.

We note in passing that axiom A5' can be replaced by the following condition:

($\frac{1}{2}$-A.5') For all $f, g \in \mathcal{F}$, $p, q \in \Delta(X)$: *if $\frac{1}{2} \langle f(s) \rangle + \frac{1}{2} \langle p \rangle \succcurlyeq \frac{1}{2} \langle g(s) \rangle + \frac{1}{2} \langle q \rangle$ for all $s \in S$, then $\frac{1}{2} f + \frac{1}{2} \langle p \rangle \succcurlyeq \frac{1}{2} g + \frac{1}{2} \langle q \rangle$.*

The condition ($\frac{1}{2}$-A.5') is a weaker version of axiom A5'. It can also be interpreted as a strengthening of the uniform continuity axiom of Cerreia-Vioglio et al. (2008)

[10]For any subset F of a metric space, $cl_d(F)$ represents its closure relative to the metric d.

provided the mixture (with equal weights) of a lottery $\langle r \rangle$ with the certainty equivalent of an act h in their framework is identified with $\frac{1}{2}h + \frac{1}{2}\langle r \rangle$. Building on an axiom along the lines of condition ($\frac{1}{2}$-A.5'), we provide in section 7 an alternative axiomatization of the variational preferences of Maccheroni et al. (2006) that does not require us to explicitly mention their weak c-independence axiom.

4.1 Remarks

We are after a multi-utility representation where each utility is a concave niveloid. The term niveloid was first introduced by Dolecki and Greco (1991, 1995). They define a niveloid as an isotone and vertically invariant functional in the space of (extended) real-valued functions. They also give an alternative characterization of a niveloid which we are about to exploit in our representation. Maccheroni et al. (2006) mention such characterization but do not exploit it as we do here. To be more concrete, let $I : \mathbb{R}^S \to \mathbb{R}$, and consider the following property:

(P) For all $\xi, \zeta \in \mathbb{R}^S$, $I(\xi) - I(\zeta) \leq \max_{s \in S} [\xi(s) - \zeta(s)]$.

Corollary 1.3 of Dolecki and Greco (1995)[11] shows that I is a niveloid (in its original sense) iff I satisfies (P). Given a multi-utility representation $\mathcal{U} \subseteq C(\mathcal{F})$ of $\succcurlyeq$ in which each U agrees with the same affine function $u \in C(\Delta(X))$ on constant acts, the following property of $\succcurlyeq$ implies that the preference on utility acts induced by each U can be represented by a niveloid:

($\mathbf{P}_{\succcurlyeq}$) For all $f, g \in \mathcal{F}$, there exists $s^* \in S$ such that $\frac{1}{2}g + \frac{1}{2}\langle f(s^*) \rangle \succcurlyeq \frac{1}{2}f + \frac{1}{2}\langle g(s^*) \rangle$.

Proposition 2. *A1, A2, A4, A5' and A6' imply ($P_{\succcurlyeq}$).*

4.2 Representation

Theorem 2. *The following are equivalent:*

(a) *$\succcurlyeq$ satisfies A1-A4, A5', A6', and A7-A8.*

(b) *There exist $u : \Delta(X) \to \mathbb{R}$ continuous, affine, and nonconstant, and a class $\mathcal{C}$ of lower semicontinuous (l.s.c.), grounded[12], and convex functions $c : \Delta(S) \to$*

[11] Also Lemma 22 of Maccheroni, Marinacci, and Rustichini (2004) and Theorem 2.2 of Dolecki and Greco (1991).

[12] That is, $\inf_{\mu \in \Delta(S)} c(\mu) = 0$.

$\overline{\mathbb{R}}_+$ *such that, for all* $P, Q \in \Delta(\mathcal{F})$,

$$P \succcurlyeq Q$$
$$iff$$
$$\int \left\{ \min_{\mu \in \Delta} \left[\int u(f) d\mu + c(\mu) \right] \right\} dP(f) \geq \int \left\{ \min_{\mu \in \Delta} \left[\int u(f) d\mu + c(\mu) \right] \right\} dQ(f),$$

for all $c \in \mathcal{C}$. *In particular, for all* $f, g \in \mathcal{F}$,

$$f \succcurlyeq g \text{ iff } \min_{\mu \in \Delta} \left[\int u(f) \, d\mu + c(\mu) \right] \geq \min_{\mu \in \Delta} \left[\int u(g) \, d\mu + c(\mu) \right] \text{ for all } c \in \mathcal{C}.$$

Moreover, given $c \in \mathcal{C}$, *there exists a unique minimal cost function* $c^* : \Delta(S) \to \mathbb{R}_+$ *such that* $U_c(f) = U_{c^*}(f)$, *for all* $f \in \mathcal{F}$, *where* $U_e(f) := \min_{\mu \in \Delta} \left[\int u(f) \, d\mu + e(\mu) \right]$, $e = c, c^*$, *and* $c^*(\mu) := \max_{f \in \mathcal{F}} \{U_c(f) - \int u(f) d\mu\}$, *for all* $\mu \in \Delta(S)$.

When each cost function c is identical to the indicator function (in the sense of convex analysis) of some closed and convex subset M of the $|S| - 1$ dimensional simplex, Theorem 2 provides a characterization of an incomplete multiple priors preference. In this case, there exists a class $\mathcal{M}$ of closed and convex subsets of $\Delta(S)$ such that $\mathcal{C} := \{\delta_M : M \in \mathcal{M}\}$, that is, for all $c \in \mathcal{C}$, $c(\mu) = \delta_M(\mu) = 0$ if $\mu \in M$, and $+\infty$ if $\mu \notin M$.

We note that each representation $(u, \mathcal{C})$ of a given preference $\succcurlyeq$ naturaly induces another representation $(u, \mathcal{C}^*)$ of $\succcurlyeq$, where $\mathcal{C}^*$ contains the minimal cost functions associated to each $c \in \mathcal{C}$. When $\mathcal{C}$ contains only minimal cost functions, or, alternatively, $\mathcal{C} = \mathcal{C}^*$, we say that $(u, \mathcal{C})$ is a representation of $\succcurlyeq$ with minimal cost functions. We can now use this concept to write a uniqueness result in the spirit of Proposition 1 for Theorem 2.

Proposition 3. *Let* $u, v \in C(\Delta(X))$ *be affine and nonconstant, and* $\mathcal{C}$ *and* $\mathcal{E}$ *be two classes of l.s.c., grounded and convex functions* $c, e : \Delta(S) \to \overline{R}_+$. *The pairs* $(u, \mathcal{C})$ *and* $(v, \mathcal{E})$ *are representations with minimal cost functions of* $\succcurlyeq$ *in the sense of Theorem 2 iff there exists* $(\alpha, \beta) \in \mathbb{R}_{++} \times \mathbb{R}$ *such that* $u = \alpha v + \beta$ *and*

$$cl_{\|\cdot\|_\infty}(co_{epi}(\mathcal{C})) = \alpha cl_{\|\cdot\|_\infty}(co_{epi}(\mathcal{E})),$$

where $co_{epi}(\mathcal{A}) := \{a : \Delta(S) \to \overline{R}_+ : epi(a) \in co(epi(\mathcal{A}))\}$, *and* $epi(\mathcal{A}) := \{epi(a) : a \in \mathcal{A}\}$, *for* $\mathcal{A} = \mathcal{C}, \mathcal{E}$.[13]

[13] We denote by $epi(a)$ the epigraph of the function a.

5 Relation to Ambiguity Aversion

Let $\succcurlyeq_1$ and $\succcurlyeq_2$ be two preference relations on $\mathcal{F}$. Following Ghirardato and Marinacci (2002), we say that $\succcurlyeq_2$ is more ambiguity averse than $\succcurlyeq_1$ if they induce the same preferences on $\Delta(X)$ and, for all $p \in \Delta(X)$, $f \in \mathcal{F}$, if $\langle p \rangle \succcurlyeq_1 f$, then $\langle p \rangle \succcurlyeq_2 f$. We take as benchmark for an ambiguity neutral preference relation the subjective expected utility model of Anscombe and Aumann (1963). Again following Ghirardato and Marinacci (2002), we say that a relation $\succcurlyeq \subseteq \mathcal{F} \times \mathcal{F}$ is ambiguity averse if it is more ambiguity averse than some Anscombe-Aumann preference $\succcurlyeq^*$.[14] In other words, $\succcurlyeq \subseteq \Delta(\mathcal{F}) \times \Delta(\mathcal{F})$ is ambiguity averse if it is more ambiguity averse than $\succcurlyeq^*$.

Here $\succcurlyeq$ stands for a preference relation on $\mathcal{F}$ such that its restriction to $\Delta(X)$ can be represented by a single utility $u : \Delta(X) \to \mathbb{R}$ that is continuous, nonconstant, and affine. The first (and less general) version of our result says that the absence of complete disagreement among the decision marker's selves about the priors on the state space is a necessary and sufficient condition for an incomplete multiple priors preference relation to be ambiguity averse.

Proposition 4. *Let $\succcurlyeq \subseteq \mathcal{F} \times \mathcal{F}$ be an incomplete multiple priors preference relation represented by the pair $(u, \mathcal{M})$. Then $\succcurlyeq$ is ambiguity averse iff $\bigcap_{M \in \mathcal{M}} M \neq \emptyset$.*

In general, one can show that the incomplete variational preferences are also ambiguity averse, so that Proposition 4 becomes a particular case of the following.

Proposition 5. *Let $\succcurlyeq \subseteq \mathcal{F} \times \mathcal{F}$ be an incomplete variational preference relation represented by the pair $(u, \mathcal{C})$. Then $\succcurlyeq$ is ambiguity averse iff $\bigcap_{c \in \mathcal{C}} \{\mu \in \Delta(S) : c(\mu) = 0\} \neq \emptyset$.*

The nonempty intersection condition that characterizes ambiguity aversion in our framework is related to Rigotti, Shannon, and Strzalecki's (2008) notion of subjective beliefs. For a given $M \in \mathcal{M}$, the priors in M are the subjective beliefs at any constant act of one of the selves of the decision maker. For a fixed $c \in \mathcal{C}$, the set $\{\mu \in \Delta(S) : c(\mu) = 0\}$ has the same interpretation. Therefore, the incomplete preference relations above are ambiguity averse iff the selves of the decision maker share at least one subjective belief at any constant act.

[14] Alternatively, we could have taken as benchmark the single-prior expected multi-utility model of Ok et al. (2008), but since all relations considered here satisfy Partial Completeness, the requirement that the benchmark relation induce the same risk preference as the relation under study would force it to be of the Anscombe-Aumann type anyway.

6 Towards a General Case

In this section we adapt the analysis of Cerreia-Vioglio et al. (2008) to the case of incomplete preferences. We depart from the setup in the previous sections in the sense that we do not work in an environment with lotteries of acts. The reason is inherently technical. The analysis in Cerreia-Vioglio et al. (2008) is based on a duality theory for monotone quasiconcave functions. The basic advantage of working in an environment with lotteries of acts was the possibility of using the expected multi-utility theory to derive a multi-utility representation with some particular cardinal properties. Since quasiconcavity is an ordinal property, having an extra layer of objective randomization in the present section would be of little use.

Formally, we consider a binary relation $\trianglerighteq$ on the domain of AA acts $\mathcal{F}$, that is, $\trianglerighteq \subseteq \mathcal{F} \times \mathcal{F}$. Define the binary relation $\trianglerighteq^{\bullet} \subseteq \Delta(X) \times \Delta(X)$ by $p \trianglerighteq^{\bullet} q$ iff $\langle p \rangle \trianglerighteq \langle q \rangle$.The mixing operator $+$ is defined so that, for all $f, g \in \mathcal{F}$, $\lambda \in [0,1]$, $(\lambda f + (1-\lambda) g)(s)(B) = \lambda f(s)(B) + (1-\lambda) g(s)(B)$ for all $s \in S$, and $B \in \mathcal{B}(X)$. Consider the following set of axioms on $\trianglerighteq$.

Axiom B1 (Preference Relation). *The binary relation $\trianglerighteq$ is a preorder.*

Axiom B2 (Upper Semicontinuity). *For all $f \in \mathcal{F}$, the set $\{g \in \mathcal{F} : g \trianglerighteq f\}$ is closed.*

Axiom B3 (Convexity). *For all $f \in \mathcal{F}$, the set $\{g \in \mathcal{F} : g \trianglerighteq f\}$ is convex.*

Axiom B4 (Monotonicity). *For all $f \in \mathcal{F}$: if $\langle f(s) \rangle \trianglerighteq \langle g(s) \rangle$ for all $s \in S$, then $f \trianglerighteq g$.*

Axiom B5 (Partial Completeness). *The binary relation $\trianglerighteq^{\bullet}$ is complete.*

Axiom B6 (Weak Continuity). *If $(p^n), (q^n) \in \Delta(X)^{\infty}$ are such that $p^n \trianglerighteq^{\bullet} q^n$ for all n, $p^n \to p \in \Delta(X)$, and $q^n \to q \in \Delta(X)$, then $p \trianglerighteq^{\bullet} q$.*

Axiom B7 (Risk Independence). *For all $p, q, r \in \Delta(X)$, $\lambda \in (0,1)$: if $p \trianglerighteq^{\bullet} q$, then $\lambda p + (1-\lambda) r \trianglerighteq^{\bullet} \lambda q + (1-\lambda) r$.*

Axioms B5-B7 allow us to find an expected utility representation for the relation $\trianglerighteq^{\bullet}$. Axiom B4 is the same standard monotonicity property that was used in the previous sections. Convexity of preferences is necessary to guarantee that we can represent $\trianglerighteq$ by a set of quasiconcave functions. In the complete case this property is replaced by Uncertainty Aversion, but in the presence of Completeness, Monotonicity and Continuity they are equivalent.

Finally, we ask that $\trianglerighteq$ satisfy only Upper Semicontinuity. As pointed out by Evren and Ok (2007), it is fairly easy to represent an upper semicontinuous preference relation by a set of upper semicontinuous functions. However, finding a continuous multi-utility representation is a much more demanding task. Indeed, we do not know of conditions that make $\trianglerighteq$ representable by a set of continuous and quasiconcave functions. In any event, the postulates above are enough to give us a multiple selves version of the representation in Cerreia-Vioglio et al. (2008).

Theorem 3. *The following are equivalent:*

(a) $\succcurlyeq$ *satisfies B1-B7.*

(b) *There exist* $u : \Delta(X) \to \mathbb{R}$ *continuous and affine, and a collection* $\mathcal{G}$ *of upper semicontinuous functions (u.s.c.)* $G : u(\Delta(X)) \times \Delta(S) \to \mathbb{R}$ *such that:*

1. *For all* $f, g \in \mathcal{F}$,

$$f \trianglerighteq g$$
$$\textit{iff}$$
$$\inf_{\mu\in\Delta(S)} G\left(\int u(f)\, d\mu, \mu\right) \geq \inf_{\mu\in\Delta(S)} G\left(\int u(g)\, d\mu, \mu\right) \; \textit{for all } G \in \mathcal{G}.$$

2. *For all* $\mu \in \Delta(S)$, $G \in \mathcal{G}$, $G(\cdot, \mu)$ *is increasing, and there exists* $H \in \mathcal{G}$ *such that* $\inf_{\mu\in\Delta(X)} H(\cdot, \mu)$ *is strictly increasing.*

7 Discussion

7.1 Alternative axiomatization of variational preferences

The alternative axiomatization of the variational preferences of Maccheroni et al. (2006) we propose is linked to the recent generalization of Cerreia-Vioglio et al. (2008). Our goal in this alternative axiomatization is to show that one can dispense with the weak c-independence axiom of Maccheroni et al. (2006), as we do in our multiple selves version. All one needs is to replace it by independence on the subdomain of constant acts plus a stronger monotonicity axiom.

The setup is the same as in sections 2 and 3, except that the binary relation $\succcurlyeq$ is defined on the domain of Anscombe-Aumann acts $\mathcal{F}$. The restriction of $\succcurlyeq$ to the subdomain of constant acts is denoted by $\succcurlyeq^{\bullet}$. A utility index $U : \mathcal{F} \rightarrow \mathbb{R}$ that represents $\succcurlyeq$ can be constructed provided the following axioms are satisfied.

Axiom VP1 (Nondegenerate Weak Order). *The binary relation $\succcurlyeq$ is a complete preorder*, and $\succ \neq \emptyset$.

Axiom VP2 (Monotonicity). *For all $f \in \mathcal{F}$: if $f(s) \succcurlyeq^{\bullet} g(s)$ for all $s \in S$, then $f \succcurlyeq g$.*

Axiom VP3 (Risk Independence). *For all $p, q, r \in \Delta(X)$, $\lambda \in (0,1)$: if $p \succcurlyeq^{\bullet} q$, then $\lambda p + (1-\lambda) r \succcurlyeq^{\bullet} \lambda q + (1-\lambda) r$.*

Axiom VP4 (Continuity). *If $(f^n), (g^n) \in \mathcal{F}^{\infty}$ are such that $f^n \succcurlyeq q^n$ for all n, $f^n \rightarrow f \in \mathcal{F}$, and $g^n \rightarrow g \in \mathcal{F}$, then $f \succcurlyeq g$.*

It is not difficult to check that axioms VP1-VP4 imply the existence of a nonconstant and affine function $u \in C(\Delta(X))$ representing $\succcurlyeq^{\bullet}$, the existence of a certainty equivalent p_f for every act f, and that the function $U : \mathcal{F} \rightarrow \mathbb{R}$ defined by $U(f) = u(p_f)$ represents $\succcurlyeq$. Assume w.l.o.g. that $u(\Delta(X)) = [-1,1]$.

Identify each $f \in \mathcal{F}$ with the vector of utils $u(f) \in [-1,1]^S$, and define the preorder $\gtrapprox$ on $[-1,1]^S$ by $u(f) = \xi_f \gtrapprox \xi_g = u(g)$ iff $f \succsim g$. Because I_U, as defined by $I_U(\xi_f) = U(f)$, represents $\gtrapprox$, this establishes the following lemma.

Lemma 1. *There exists a nonconstant, continuous and monotonic function $I_U : [-1,1]^S \rightarrow \mathbb{R}$ that represents $\gtrapprox$. Moreover, $I_U(a\mathbf{1}_S) = a$ for all $a \in [-1,1]$.*

Two additional axioms are needed. One is the standard Uncertainty Aversion axiom, and the other is a strengthening of the "uniform continuity" axiom of Cerreia-Vioglio et al. (2008).[15] We refer to our last axiom as "$\frac{1}{2}$-c-mixture monotonicity*" because of its similarity with axiom A5'.

Axiom VP5 (Uncertainty Aversion). *For all $f, g \in \mathcal{F}$, $\lambda \in (0,1)$: if $f \sim g$,* then $\lambda f + (1-\lambda) g \succcurlyeq f$.

[15] Cerreia-Vioglio et al. (2008) make use of an object (viz. the certainty equivalent of an act) that is not a primitive of the model to write that axiom. We avoid this issue here by adding one additional quantifier to our axiom VP6.

Axiom VP6 ($\frac{1}{2}$-C-Mixture Monotonicity*). *For all* $f, g \in \mathcal{F}$, $p, q \in \Delta(X)$*: if* $\frac{1}{2}f(s) + \frac{1}{2}p \succcurlyeq^{\bullet} \frac{1}{2}g(s) + \frac{1}{2}q$ *for all* $s \in S$*, then* $\frac{1}{2}r_f + \frac{1}{2}p \succcurlyeq^{\bullet} \frac{1}{2}r_g + \frac{1}{2}q$ *for any* $r_f, r_g \in \Delta(X)$ such that $f \sim r_f$ and $g \sim r_g$.

Axiom VP6 implies that I_U is a niveloid, that is, for all $\xi_f, \xi_g \in [-1,1]^S$, $I(\xi_f) - I(\xi_g) \leq \max_{s \in S}[\xi_f(s) - \xi_g(s)]$. To see this, note that all we need is to show that the following property holds:

($\mathbf{P}^*$) For all $f, g \in \mathcal{F}$, there exists $s^* \in S$ such that $\frac{1}{2}p_g + \frac{1}{2}f(s^*) \succcurlyeq^{\bullet} \frac{1}{2}p_f + \frac{1}{2}g(s^*)$.

The proof that $\mathbf{P}^*$ actually holds is an easy consequence of the representation obtained so far and axiom VP6. Lemma 20 of Maccheroni et al. (2004) guarantees that (using VP5) I_U is in fact a concave niveloid. Therefore, using the same argument as in the last paragraph of the proof of Theorem 2, we can show that $\succcurlyeq$ has a variational preference representation. The converse of the statement can be checked through standard arguments. Finally, note that one could have replaced VP6 by an axiom similar to C-Mixture Monotonicity if we replace the weight $\frac{1}{2}$ by some generic $\lambda \in (0,1]$, and assume the statement of the axiom is true for all $\lambda \in (0,1]$.

7.2 Open questions

We are mainly interested in the incomplete preference relation defined on the domain of Anscombe-Aumann acts. It is not clear, though, how to provide a direct axiomatization for preferences defined on such domain. Bewley (1986) and Ok et al. (2008) provided axiomatizations on such domain. In their cases, independence holds, and one can employ the technique of finding a set of utility functions by looking at the linear functionals which support the Aumann cone at the origin. Without the independence axiom, it is not clear how to provide a generalization of their theorems using the original domain.

A better understanding of general ambiguity averse preferences is also missing in this paper. Although we managed to sketch a representation in the format of Cerreia-Vioglio et al. (2008) in section 6, we had to work with a multi-utility representation with functions that were only upper semicontinuous. A closer multiple selves generalization of the result in Cerreia-Vioglio et al. (2008) would obtain a multi-utility representation $\mathcal{U} \subseteq C(\mathcal{F})$ and at the same time guarantee that each $U \in \mathcal{U}$ was quasiconcave and continuous. The existence of the set $\mathcal{U}$ is not a problem (e.g., Evren and Ok (2007)), but we were not able to show that each $U \in \mathcal{U}$ can be made quasiconcave and continuous at the same time.[16,17]

Finally, we conjecture that, provided we work with simple acts, our representations above (including section 6) would go through if we assume a general state space S (not necessarily finite), and that the set of consequences is a convex and compact metric space. We did not pursue such a path here because it would not add much to our understanding of incomplete and ambiguity averse preferences.

A Appendix: Proofs

A.1 Proof of Theorem 1

The proof of the direction (b)⇒(a) is standard, and thus omitted. We now prove (a)⇒(b).

Claim A.1.1. *There exists a closed and convex set $\mathcal{U} \subseteq C(\mathcal{F})$ such that, for all $P, Q \in \Delta(\mathcal{F})$, $P \succcurlyeq Q$ iff $\int_{\mathcal{F}} U dP \geq \int_{\mathcal{F}} U dQ$ for all $U \in \mathcal{U}$.*

Proof of Claim A.1.1. Because $\mathcal{F}$ is a compact metric space, $\Delta(\mathcal{F})$ is endowed with the topology of weak convergence, and $\succcurlyeq$ satisfies A1-A3, the Expected Multi-Utility Theorem of Dubra et al. (2004) applies. □

Claim A.1.2. *There exists an affine, continuous and nonconstant function $u : \Delta(X) \to \mathbb{R}$ such that, for all $p, q \in \Delta(X)$, $p \succcurlyeq^{\bullet} q$ iff $u(p) \geq u(q)$.*

Proof of Claim A.1.2. The binary relation $\succcurlyeq^{\bullet}$ is a preorder on $\Delta(X)$. One can verify A3 implies that $\succcurlyeq^{\bullet}$ is closed-continuous. Moreover, it is complete by A4. Now use A2, A3, and A6 to obtain that, for all $p, q, r \in \Delta(X)$, $\lambda \in (0,1)$, $p \succcurlyeq^{\bullet} q$ iff $\langle p \rangle \succcurlyeq \langle q \rangle$ iff $\lambda \langle p \rangle \oplus (1-\lambda) \langle r \rangle \sim \lambda \langle p \rangle + (1-\lambda) \langle r \rangle \succcurlyeq \lambda \langle q \rangle + (1-\lambda) \langle r \rangle \sim \lambda \langle p \rangle \oplus (1-\lambda) \langle r \rangle$ iff $\lambda p + (1-\lambda) r \succcurlyeq^{\bullet} \lambda q + (1-\lambda) r$. Therefore, $\succcurlyeq^{\bullet}$ satisfies all the assumptions of the Expected Utility Theorem, and it can be represented by an affine and nonconstant function $u \in C(\Delta(X))$. Moreover, using A5 and A8 one can show u is nonconstant.[18] □

[16] In particular, if the set $\mathcal{U}$ were compact and the function $e : \mathcal{F} \to C(\mathcal{U})$ as defined by $e(f)(u) = u(f)$ were K-quasiconcave in the sense of Benoist, Borwein, and Popovici (2002), one could have applied their theorem 3.1. We were not successful in establishing those two properties.

[17] A general problem is that convex incomplete preferences may admit multi-utility representations with some functions that fail to be quasiconcave.

[18] For any compact subset F of a normed vector space, $C(F)$ stands for the set of continuous functions on F, and is endowed with the sup norm.

The set $\mathcal{U}$ may contain constant functions. They are not essential to the representation and can be discarded at this point. Therefore, assume w.l.o.g. that $\mathcal{U}$ contains only nonconstant functions. By axiom A8, $\mathcal{U} \neq \emptyset$.

We can employ standard arguments to prove the existence of $\overline{x}, \underline{x} \in X$ such that $\langle \delta_{\overline{x}} \rangle \succ^{\bullet} \langle \delta_{\underline{x}} \rangle$, and $\langle \delta_{\overline{x}} \rangle \succcurlyeq^{\bullet} \langle p \rangle \succcurlyeq^{\bullet} \langle \delta_{\underline{x}} \rangle$ for all $p \in \Delta(X)$. Moreover, because of C-Reduction, Continuity, Partial Completeness and Independence over lotteries, it can also be shown that, for all $p \in \Delta(X)$, there exists $\lambda_p \in [0,1]$ such that $\langle p \rangle \sim \lambda_p \langle \delta_{\overline{x}} \rangle \oplus (1 - \lambda_p) \langle \delta_{\underline{x}} \rangle$. The implication $\langle p \rangle \succ \langle q \rangle \Rightarrow \lambda_p > \lambda_q$ is also true.

Fix any $U \in \mathcal{U}$, and use Monotonicity to show that $U(\langle \delta_{\overline{x}} \rangle) > U(\langle \delta_{\underline{x}} \rangle)$. As a consequence, whenever $\langle p \rangle \succ \langle q \rangle$, it is false that $U(\langle p \rangle) = U(\langle q \rangle)$. If this equality were true, then using axiom A6 and Independence on the subdomain of constant acts we obtain

$$\begin{aligned} U(\langle p \rangle) &= \lambda_p \left(U(\langle \delta_{\overline{x}} \rangle) - U(\langle \delta_{\underline{x}} \rangle) \right) + U(\langle \delta_{\underline{x}} \rangle) \\ &= \lambda_q \left(U(\langle \delta_{\overline{x}} \rangle) - U(\langle \delta_{\underline{x}} \rangle) \right) + U(\langle \delta_{\underline{x}} \rangle) = U(\langle q \rangle), \end{aligned}$$

implying $(\lambda_p - \lambda_q)\left(U(\langle \delta_{\overline{x}} \rangle) - U(\langle \delta_{\underline{x}} \rangle) \right) = 0$. Because of $U(\langle \delta_{\overline{x}} \rangle) > U(\langle \delta_{\underline{x}} \rangle)$, we have $\lambda_p = \lambda_q$, a contradiction. Conclusion: for any fixed $U \in \mathcal{U}$, $U|_{\Delta(X)}$ is affine and represents $\succcurlyeq^{\bullet}$.

Claim A.1.3. *Each $U \in \mathcal{U}$ can be normalized so that $U|_{\Delta(X)} = u$.*

Proof of Claim A.1.3. Fix any $U \in \mathcal{U}$. Because $\succcurlyeq^{\bullet}$ is complete, for all $p, q \in \Delta(X)$, $p \succcurlyeq^{\bullet} q$ iff $U(\langle p \rangle) \geq U(\langle q \rangle)$. Therefore, $U|_{\Delta(X)}$ and u are both affine representations of $\succcurlyeq^{\bullet}$. By cardinal uniqueness, we know there exists $(\alpha_U, \beta_U) \in \mathbb{R}_{++} \times \mathbb{R}$ such that $U|_{\Delta(X)} = \alpha_U u + \beta_U$. □

Because $\Delta(X)$ is weak* compact and u is continuous, there exist $\overline{p}, \underline{p} \in \Delta(X)$ such that $u(\overline{p}) \geq u(p) \geq u(\underline{p})$ for all $p \in \Delta(X)$. By A5 and A8 it must be that $\overline{p} \neq \underline{p}$. W.l.o.g. normalize u so that $u(\overline{p}) = 1$ and $u(\underline{p}) = -1$. Then $u(\Delta(X)) = [-1, 1]$ (use Second Stage Independence for constant acts). Given $p \in \Delta(X)$, it follows from our normalization of U in the previous step that $U(\langle p \rangle) = u(p)$. Therefore, for all $f \in \mathcal{F}$, let $\xi_f := u \circ f \in [-1,1]^S$. Let the functional $I_U : [-1,1]^S \to \mathbb{R}$ be defined by $I_U(\xi_f) = U(f)$, for all $\xi_f \in [-1,1]^S$ (A5 guarantees that I_U is well-defined).

Claim A.1.4. *I_U is positively homogeneous.*

Proof of Claim A.1.4. Take any $U \in \mathcal{U}$. Let $p_0 \in \Delta(X)$ be such that $u(p_0) = 0$. Let $\xi_f \in [-1,1]^S$, $\lambda \in (0,1)$. Axiom A6 implies $\lambda f \oplus (1 - \lambda) \langle p_0 \rangle \sim \lambda f + (1 - \lambda) \langle p_0 \rangle$, and hence $I_U(\lambda\xi_f) = U(\lambda f + (1-\lambda)\langle p_0\rangle) = \lambda U(f) + (1-\lambda) U(\langle p_0\rangle) = \lambda I_U(\xi_f)$. If $\lambda > 1$ and $\lambda\xi_f \in [-1,1]^S$, then $I_U(\xi_f) = I_U\left(\frac{1}{\lambda}(\lambda\xi_f)\right)$ iff $\lambda I_U(\xi_f) = I_U(\lambda\xi_f)$ (because $\frac{1}{\lambda} < 1$). □

Using an argument similar to Gilboa and Schmeidler (1989), we extend I_U to $\mathbb{R}^S$ (call this extension I_U^*): for all $\xi \in \mathbb{R}^S$, let $I_U^*(\xi) = \frac{1}{\lambda} I_U^*(\lambda\xi)$, for all $\lambda > 0$ such that $\lambda\xi \in [-1,1]^S$. Standard arguments can be employed to show the extension does not depend on which λ is used to shrink ξ towards the origin.

Claim A.1.5. *I_U^* is increasing, positively homogenous, superadditive, C-additive, and normalized.*

Proof of Claim A.1.5. Let $\xi, \xi' \in \mathbb{R}^S$, and $\lambda > 0$ be such that $\lambda\xi, \lambda\xi' \in [-1,1]^S$ and $\xi \geq \xi'$. Then $\lambda\xi \geq \lambda\xi'$, and by A5 we obtain that $I_U(\lambda\xi) = U(f_{\lambda\xi}) \geq U(f_{\lambda\xi'}) = I_U(\lambda\xi')$, where $f_{\lambda\xi}$ and $f_{\lambda\xi'}$ are the acts associated with $\lambda\xi$ and $\lambda\xi'$, respectively. Hence $I_U^*(\xi) \geq I_U^*(\xi')$, and I_U^* is increasing. It is not difficult to verify I_U^* is positively homogeneous. For any $\xi, \xi' \in \mathbb{R}^S$, $I_U^*\left(\frac{1}{2}\xi + \frac{1}{2}\xi'\right) = \frac{1}{\lambda} I_U\left(\lambda\left(\frac{1}{2}\xi + \frac{1}{2}\xi'\right)\right)$, with $\lambda > 0$ being such that $\lambda\left(\frac{1}{2}\xi + \frac{1}{2}\xi'\right), \lambda\frac{1}{2}\xi, \lambda\frac{1}{2}\xi' \in [-1,1]^S$. By A7 we obtain $I_U\left(\frac{1}{2}\lambda\xi + \frac{1}{2}\lambda\xi'\right) \geq \frac{1}{2} I_U(\lambda\xi) + \frac{1}{2} I_U(\lambda\xi')$, and hence $I_U^*\left(\frac{1}{2}\xi + \frac{1}{2}\xi'\right) \geq \frac{1}{2} I_U^*(\xi) + \frac{1}{2} I_U^*(\xi')$. Using positive homogeneity of I_U^* we conclude that $I_U^*(\xi + \xi') \geq I_U^*(\xi) + I_U^*(\xi')$, and I_U^* is superadditive. Now take $\xi \in \mathbb{R}^S$, $a \in \mathbb{R}$, and let $\lambda > 0$ be such that $\lambda\left(\frac{1}{2}\xi + \frac{1}{2}a\right), \lambda\frac{1}{2}\xi, \lambda\frac{1}{2}a \in [-1,1]^S$ (with abuse of notation, we write a instead of $a\mathbf{1}_S$). Using A6 we know that $I_U\left(\frac{1}{2}\lambda\xi + \frac{1}{2}\lambda a\right) = U\left(\frac{1}{2} f_{\lambda\xi} + \frac{1}{2}\langle p_{\lambda a}\rangle\right) = \frac{1}{2} U(f_{\lambda\xi}) + \frac{1}{2} U(\langle p_{\lambda a}\rangle) = \frac{1}{2} I_U(\lambda\xi) + \frac{1}{2} I_U(\lambda a)$, where $u \circ f_{\lambda\xi} = \lambda\xi$ and $u(p_{\lambda a}) = \lambda a$, with $f_{\lambda\xi}, \langle p_{\lambda a}\rangle \in \mathcal{F}$. Therefore, using positive homogeneity we obtain $I_U^*(\xi + a) = I_U^*(\xi) + I_U^*(a)$, and I_U^* is C-additive. It is clear that I_U^* is normalized, that is, $I_U^*(1) = 1$. □

Because, given any $U \in \mathcal{U}$, I_U^* satisfies all the properties proved in the previous step, we can write $I_U^*(\xi) = \min_{\mu \in M_U} \int \xi d\mu$ for all $\xi \in \mathbb{R}^S$, where M_U is a closed and convex subset of the $|S| - 1$-dimensional simplex (see Gilboa and Schmeidler (1989)). Therefore, for all $f \in \mathcal{F}$, $U(f) = I_U^*(u \circ f) = \min_{\mu \in M_U} \int u(f) \, d\mu$. Now define $\mathcal{M} := \{M_U : U \in \mathcal{U}\}$, and note that the pair $(u, \mathcal{M})$ induces the desired representation of $\succcurlyeq$ on $\Delta(\mathcal{F})$.

A.2 Proof of Proposition 1

The proof of the "if" part is trivial and thus omitted. Let $\mathcal{U}, \mathcal{V} \subseteq C(\mathcal{F})$ be two representations of $\succcurlyeq$ induced, respectively, by the pairs $(u, \mathcal{M})$ and $(v, \mathcal{N})$. Because both u and v represent $\succcurlyeq^{\bullet}$, from the cardinal uniqueness of such a representation it follows that u is a positive affine transformation of v. Also note that, from the uniqueness of the expected multi-utility representation of Dubra et al. (2004), it follows that $cl_{\|\cdot\|_\infty}\left(cone\left(\mathcal{U}\right)+\{\theta\mathbf{1}_{\mathcal{F}}:\theta\in\mathbb{R}\}\right)=cl_{\|\cdot\|_\infty}\left(cone\left(\mathcal{V}\right)+\{\theta\mathbf{1}_{\mathcal{F}}:\theta\in\mathbb{R}\}\right)$.[19] Now we prove two claims, which remain true if we replace $\mathcal{U}$ by $\mathcal{V}$ in their statements.

Claim A.2.1. *For any nonconstant $U\in cone\left(\mathcal{U}\right)+\{\theta\mathbf{1}_{\mathcal{F}}:\theta\in\mathbb{R}\}$, it is possible to find $(U_i)_{i=1}^n\in\mathcal{U}^n$, $\rho\in\Delta(\{1,...,n\})$, and $(\alpha,\beta)\in\mathbb{R}_{++}\times\mathbb{R}$ such that, $U(f)=\min_{\mu\in\Sigma_{i=1}^n\rho_iM_{U_i}}\int(\alpha u(f)+\beta)d\mu$, for all $f\in\mathcal{F}$, where, for all $i\in\{1,...,n\}$, $U_i\left(f\right)=\min_{\mu\in M_{U_i}}\int u\left(f\right)d\mu$, for all $f\in\mathcal{F}$.*

Proof of Claim A.2.1. By definition, there exist $(U_i)_{i=1}^n\in\mathcal{U}^n$, $(\gamma_i)_{i=1}^n\in\mathbb{R}_+^n\setminus\{0\}$, and $\beta\in\mathbb{R}$ such that $U=\sum_{i=1}^n\gamma_iU_i+\beta$, and then $U=\alpha\sum_{i=1}^n\rho_iU_i+\beta$, where $\alpha=\sum_{i=1}^n\gamma_i$ and $\rho_i=\frac{\gamma_i}{\alpha}$ for all $i\in\{1,...,n\}$. Because every U_i can be written as $U_i\left(f\right)=-\sigma_{M_{U_i}}\left(-u\left(f\right)\right)$, where $\sigma_{M_{U_i}}$ stands for the support function of M_{U_i}, it follows that $U\left(f\right)=\alpha\min_{\Sigma_{i=1}^n\rho_iM_{U_i}}\int u\left(f\right)d\mu+\beta$ (see, e.g., section 5.19 of Aliprantis and Border (1999)). □

Claim A.2.2. *For any nonconstant $U\in cl_{\|\cdot\|_\infty}\left(cone\left(\mathcal{U}\right)+\{\theta\mathbf{1}_{\mathcal{F}}:\theta\in\mathbb{R}\}\right)$, there exist $(\alpha,\beta)\in\mathbb{R}_{++}\times\mathbb{R}$, and $M\in cl_{d_H}\left(co\left(\mathcal{M}\right)\right)$ such that, for all $f\in\mathcal{F}$, $U\left(f\right)=\min_{\mu\in M}\int\left(\alpha u\left(f\right)+\beta\right)d\mu$.*

Proof of Claim A.2.2. We can take $(U_n)\in\left(cone\left(\mathcal{U}\right)+\{\theta\mathbf{1}_{\mathcal{F}}:\theta\in\mathbb{R}\}\right)^\infty$, where each U_n is nonconstant w.l.o.g., and such that $U_n\to U$. For all $n\in\mathbb{N}$, $f\in\mathcal{F}$, $U_n\left(f\right)=\alpha_n\left(-\sigma_{M_n}\left(-u\left(f\right)\right)\right)+\beta_n$. Let $p,q\in\Delta\left(X\right)$ be such that $u\left(p\right)>u\left(q\right)$. Because (U_n) also converges pointwise, $\lim_n\left[\alpha_nu\left(p\right)+\beta_n\right]=U\left(\langle p\rangle\right)$ and $\lim_n\left[\alpha_nu\left(q\right)+\beta_n\right]=U\left(\langle q\rangle\right)$, which implies $\lim_n\alpha_n\left[u\left(p\right)-u\left(q\right)\right]=U\left(\langle p\rangle\right)-U\left(\langle q\rangle\right)$. Hence $\alpha_n\to\alpha\geq 0$, and indeed $\alpha>0$ because U is nonconstant. Therefore $\beta_n\to\beta$, for some $\beta\in\mathbb{R}$. Now use the fact $\mathfrak{M}$ is compact to obtain a convergent subsequence (M_{n_k}), and clearly $M_{n_k}\to_{d_H}M\in cl_{d_H}\left(co\left(\mathcal{M}\right)\right)$. Each σ_{M_n} is a real-valued function on $u\left(\Delta\left(X\right)\right)^S$, which is compact. Then $\left(\sigma_{M_{n_k}}\right)$ converges uniformly to σ_M.[20] □

From claims A.2.1 and A.2.2, it follows that $cl\left(cone\left(\mathcal{U}\right)+\{\theta\mathbf{1}_{\mathcal{F}}:\theta\in\mathbb{R}\}\right)=\{U\in C\left(\mathcal{F}\right):U\left(f\right)=\min_{\mu\in M}\int\left(\alpha u\left(f\right)+\beta\right)d\mu,\alpha\geq 0,\beta\in\mathbb{R},M\in cl_{d_H}\left(co\left(\mathcal{M}\right)\right)\}$, where a similar equality holds if $\mathcal{U}$ is replaced by $\mathcal{V}$. Now use the uniqueness results of Dubra et al. (2004) and Gilboa and Schmeidler (1989) to conclude that $cl_{d_H}\left(co\left(\mathcal{M}\right)\right)=cl_{d_H}\left(co\left(\mathcal{N}\right)\right)$.

[19] For any subset F of a vector space, $cone\left(F\right)$ is the smallest convex cone which contains F.

[20] This last part follows from Hirirart-Urruty and Lemarechal (2001, Corollary 3.3.8, p.156)

A.3 Proof of Proposition 2

Claim A.3.1. *Fix any $f, g \in \mathcal{F}$. There exists $s^* \in S$ such that $\frac{1}{2}\langle g(s)\rangle + \frac{1}{2}\langle f(s^*)\rangle \succcurlyeq \frac{1}{2}\langle f(s)\rangle + \frac{1}{2}\langle g(s^*)\rangle$ for all $s \in S$.*

Proof of Claim A.3.1. Assume by way of contradiction this is not the case. Then, using Reduction of Lotteries and Partial Completeness, for any $s_i \in S$ there exists s_j such that

$$\frac{1}{2}\langle g(s_j)\rangle + \frac{1}{2}\langle f(s_i)\rangle \prec \frac{1}{2}\langle f(s_j)\rangle + \frac{1}{2}\langle g(s_i)\rangle .$$

Enumerate $S = \{s_1, ..., s_n\}$ and let $s_{n_1} := s_1$. If $k \geq 1$, let n_{k+1} be such that

$$\frac{1}{2}\left\langle g\left(s_{n_{k+1}}\right)\right\rangle + \frac{1}{2}\langle f(s_{n_k})\rangle \prec \frac{1}{2}\left\langle f\left(s_{n_{k+1}}\right)\right\rangle + \frac{1}{2}\langle g(s_{n_k})\rangle .$$

Let $l > 1$ be the smallest integer to satisfy $s_{n_{l+1}} \in \{s_{n_1}, ..., s_{n_l}\}$. Then $s_{n_{l+1}} = s_{n_k}$ for some $k \in \{1, ..., l-1\}$, and by the repeated application of A2, A4, and A6', one obtains

$$\sum_{o=k+1}^{l+1} \frac{1}{N_{l,k}}\langle g(s_{n_o})\rangle + \sum_{o=k}^{l} \frac{1}{N_{l,k}}\langle f(s_{n_o})\rangle \prec \sum_{o=k+1}^{l+1} \frac{1}{N_{l,k}}\langle f(s_{n_o})\rangle + \sum_{o=k}^{l} \frac{1}{N_{l,k}}\langle g(s_{n_o})\rangle ,$$

where $N_{l,k} := 2(l+1-k)$ and the summation symbol $\sum$ operates w.r.t. the mixture operation "+". This contradicts reflexivity as the lotteries of acts on both sides are the same. □

Now use A5' to obtain $\frac{1}{2}g + \frac{1}{2}\langle f(s^*)\rangle \succcurlyeq \frac{1}{2}f + \frac{1}{2}\langle g(s^*)\rangle$.

A.4 Proof of Theorem 2

The proof of the direction (b)⇒(a) is standard, and thus omitted. We now prove (a)⇒(b).

Use claims A.1.1, A.1.2 and A.1.3 to obtain a set $\mathcal{U} \subseteq C(\mathcal{F})$ such that, for all $P, Q \in \Delta(\mathcal{F})$, $P \succcurlyeq Q$ iff $\int_{\mathcal{F}} UdP \geq \int_{\mathcal{F}} UdQ$ for all $U \in \mathcal{U}$, and each U satisfies $U|_{\Delta(X)} = u$, for some affine $u \in C(\Delta(X))$ with $u(\Delta(X)) = [-1, 1]$. For all U, let the functional $I_U : [-1, 1]^S \to \mathbb{R}$ be defined as $I_U(\xi_f) = U(f)$.

Take any $U \in \mathcal{U}$, $\xi_f, \xi_g \in [-1, 1]^S$. Using A1, A2, A4, A5' and A6', Proposition 2 implies the existence of $s^* \in S$ such that $\frac{1}{2}I_U(\xi_g) + \frac{1}{2}\xi_f(s^*) \geq \frac{1}{2}I_U(\xi_f) + \frac{1}{2}\xi_g(s^*)$, which is the case iff $I_U(\xi_f) - I_U(\xi_g) \leq \max_{s \in S}[\xi_f(s) - \xi_g(s)]$. Moreover, A7 implies that for all $f, g \in \mathcal{F}$, $\lambda \in (0,1)$, $I_U(\lambda\xi_f + (1-\lambda)\xi_g) \geq \lambda I_U(\xi_f) + (1-\lambda) I_U(\xi_g)$. Therefore I_U is a concave niveloid. Moreover, for any $a \in [-1,1]$, we have, for some $p \in \Delta(X)$, $I_U(a) = U(\langle p \rangle) = u(\langle p \rangle) = a$. Hence I_U is also normalized.

By putting together Lemma 24, Corollary 28 and Remark 3 of Maccheroni et al. (2004), we obtain that, for all $U \in \mathcal{U}$, there exists a l.s.c., grounded and convex function $c_U : \Delta \to \overline{\mathbb{R}}_+$ such that, for all ξ_f, $I_U(\xi_f) = \min_{\mu\in\Delta} \left[\int \xi_f d\mu + c_U(\mu)\right]$. Define $\mathcal{C} := \{c_U : U \in \mathcal{U}\}$, and note that the pair $(u, \mathcal{C})$ yields the desired representation of $\succcurlyeq$ on $\Delta(\mathcal{F})$. The proof that, for each c_U, there exists a minimal c_U^* defined as $c_U^*(\mu) := -\min_{f\in\mathcal{F}} \left\{\int u(f)\, d\mu - U(f)\right\}$ is a consequence of Lemma 27 of Maccheroni et al. (2004).

A.5 Proof of Proposition 3

The proof of the "if" part is trivial and thus omitted. Let $\mathcal{U}, \mathcal{V} \subseteq C(\mathcal{F})$ be two representations of $\succcurlyeq$ induced, respectively, by the pairs $(u, \mathcal{C})$ and $(v, \mathcal{E})$ with mininal cost functions. Therefore, $U|_{\Delta(X)} = u$ and $V|_{\Delta(X)} = v$ for all $(U, V) \in \mathcal{U} \times \mathcal{V}$. (In this case we say that $\mathcal{U}$ and $\mathcal{V}$ are normalized.)

Claim A.5.1. *For any nonconstant $V \in cone(\mathcal{V}) + \{\theta\mathbf{1}_{\mathcal{F}} : \theta \in \mathbb{R}\}$, there exist $(\overline{\alpha}, \overline{\beta}) \in \mathbb{R}_{++} \times \mathbb{R}$ and a nonconstant $\overline{V} \in co(\mathcal{V})$ such that $V = \overline{\alpha}\overline{V} + \overline{\beta}$.*

Proof of Claim A.5.1. For some $n \in \mathbb{N}$, there exist $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$, $V_1, ..., V_n \in \mathcal{V}$, and $\theta \in \mathbb{R}$ such that $V = \sum_{i=1}^n \lambda_i V_i + \theta$. Now define $\overline{\alpha} := \sum_{i=1}^n \lambda_i > 0$ and $\overline{\beta} := \theta$, and note that $V = \overline{\alpha}\overline{V} + \overline{\beta}$, where $\overline{V} := \frac{1}{\overline{\alpha}} V \in co(\mathcal{V})$ is nonconstant. □

Claim A.5.2. *For any nonconstant $V \in cl_{\|\cdot\|_\infty}(cone(\mathcal{V}) + \{\theta\mathbf{1}_{\mathcal{F}} : \theta \in \mathbb{R}\})$, there exist $(\overline{\alpha}, \overline{\beta}) \in \mathbb{R}_{++} \times \mathbb{R}$ and a nonconstant $\overline{V} \in cl_{\|\cdot\|_\infty}(co(\mathcal{V}))$ such that $V = \overline{\alpha}\overline{V} + \overline{\beta}$.*

Proof of Claim A.5.2. Let $(V_n) \in (cone(\mathcal{V}) + \{\theta\mathbf{1}_{\mathcal{F}} : \theta \in \mathbb{R}\})^\infty$ be such that $V_n \to V$. Using Claim 1, each $V_n = \overline{\alpha}_n \overline{V}_n + \overline{\beta}_n$, for some $(\overline{\alpha}_n, \overline{\beta}_n) \in \mathbb{R}_{++} \times \mathbb{R}$ and $\overline{V}_n \in co(\mathcal{V})$. Let $p, q \in \Delta(X)$ be such $v(p) > v(q)$, and note that $V_n \to V$ implies that $V_n(\langle p \rangle) - V_n(\langle q \rangle) \to V(\langle p \rangle) - V(\langle q \rangle)$, which is equivalent to $\overline{\alpha}_n [v(p) - v(q)] \to V(\langle p \rangle) - V(\langle q \rangle)$. Therefore, there exists $\overline{\alpha} \geq 0$ such that $\overline{\alpha}_n \to \overline{\alpha}$. Because V is nonconstant, we have $\overline{\alpha} > 0$. Using the fact that $\overline{\alpha}_n v(p) + \overline{\beta}_n \to V(\langle p \rangle)$, we conclude that $\overline{\beta}_n \to \beta$, for some $\beta \in \mathbb{R}$. Conclusion: $V = \lim_n (\overline{\alpha}_n \overline{V}_n + \overline{\beta}_n) = \overline{\alpha}\overline{V} + \overline{\beta}$, and $\overline{V}$ is nonconstant. □

Claim A.5.3. *If $\mathcal{U}, \mathcal{V} \subseteq C(\mathcal{F})$ are normalized, then they represent $\succcurlyeq$ iff there exists $(\alpha, \beta) \in \mathbb{R}_{++} \times \mathbb{R}$ such that $cl_{\|\cdot\|_\infty}(co(\mathcal{U})) = \alpha \left[cl_{\|\cdot\|_\infty}(co(\mathcal{V}))\right] + \beta$.*

Proof of Claim A.5.3. The proof of the "if" part is trivial and thus omitted. Now assume that $\mathcal{U}, \mathcal{V} \subseteq C(\mathcal{F})$ are normalized representations of $\succcurlyeq$. It follows from the uniqueness theorem of Dubra et al. (2004) that $cl_{\|\cdot\|_\infty}(cone(\mathcal{U}) + \{\theta \mathbf{1}_\mathcal{F} : \theta \in \mathbb{R}\}) = cl_{\|\cdot\|_\infty}(cone(\mathcal{V}) + \{\theta \mathbf{1}_\mathcal{F} : \theta \in \mathbb{R}\})$. The (cardinal) uniqueness of the standard Expected Utility theorem implies the existence of $(\alpha, \beta) \in \mathbb{R}_{++} \times \mathbb{R}$ such that, for all $(U, V) \in \mathcal{U} \times \mathcal{V}$, $U|_{\Delta(X)} = \alpha V|_{\Delta(X)} + \beta$. Pick any $U \in cl_{\|\cdot\|_\infty}(co(\mathcal{U}))$, so that $U \in cl_{\|\cdot\|_\infty}(cone(\mathcal{V}) + \{\theta \mathbf{1}_\mathcal{F} : \theta \in \mathbb{R}\})$. Since $\mathcal{U}$ is normalized and $\succcurlyeq$ is nontrivial and monotonic, U is nonconstant. Claim 2 implies the existence of $(\overline{\alpha}, \overline{\beta}) \in \mathbb{R}_{++} \times \mathbb{R}$ and a nonconstant $V \in cl_{\|\cdot\|_\infty}(co(\mathcal{V}))$ such that $U = \overline{\alpha} V + \overline{\beta}$. Because $U|_{\Delta(X)} = \alpha V|_{\Delta(X)} + \beta$, we obtain $\overline{\alpha} = \alpha$ and $\overline{\beta} = \beta$. Therefore $cl_{\|\cdot\|_\infty}(co(\mathcal{U})) \subseteq \alpha \left[cl_{\|\cdot\|_\infty}(co(\mathcal{V}))\right] + \beta$. A symmetric argument can be employed to show that $\alpha \left[cl_{\|\cdot\|_\infty}(co(\mathcal{V}))\right] + \beta \subseteq cl_{\|\cdot\|_\infty}(co(\mathcal{U}))$. □

For each $c \in \mathcal{C}$, define the function U_c by $U_c(f) := \min_{\mu \in \Delta(S)} \left[\int u(f)\, d\mu + c(\mu)\right]$, for all $f \in \mathcal{F}$. Similarly define functions V_e. We note that the collections $\mathcal{U} := \{U_c : c \in \mathcal{C}\}$ and $\mathcal{V} := \{V_e : e \in \mathcal{E}\}$ are both normalized expected multi-utility representations of $\succcurlyeq$. By Claim 3, there exists $(\alpha, \beta) \in \mathbb{R}_{++} \times \mathbb{R}$ such that $cl_{\|\cdot\|_\infty}(co(\mathcal{U})) = \alpha cl_{\|\cdot\|_\infty}(co(\mathcal{V})) + \beta$.

Claim A.5.4. *$co(\mathcal{U}) = \{U_c : epi(c) \in co_{epi}(\mathcal{C})\}$, and each cost function $c \in co_{epi}(\mathcal{C})$ is minimal for some $U_c \in co(\mathcal{U})$.*

Proof of Claim A.5.4. Let $(a, b) := \left(\min_{p \in \Delta(X)} u(p), \max_{p \in \Delta(X)} u(p)\right)$. For each $c \in \mathcal{C}$, define $I_c : [a, b]^S \to \mathbb{R}$ by

$$I_c(\xi) = \min_{\mu \in \Delta(S)} \left[\int \xi d\mu + c(\mu)\right],$$

for all $\xi \in [a, b]$. Let $\mathcal{I} := \{I_c : c \in \mathcal{C}\}$. We note that in order to prove the claim it is enough to show that $co(\mathcal{I}) = \{I_c : c \in epi(c)\}$. Now, fix $I, J \in \mathcal{I}$ and let $c_I, c_J \in \mathcal{C}$ be the cost functions in the definition of I and J. Put $L := \lambda I + (1 - \lambda) J$, for some $\lambda \in (0, 1)$, and define c_L such that

$$c_L(\mu) := \max_{\xi \in [a,b]^S} \left[L(\xi) - \int \xi d\mu\right].$$

Lemma 27 of Maccheroni et al. (2004) shows that c_L is the minimal function satisfying

$$L(\xi) = \min_{\mu \in \Delta(S)} \left[\int \xi d\mu + c_L(\mu)\right], \text{ for all } \xi \in [a, b]^S.$$

Let c_λ be the function that satisfies $epi\,(c_\lambda) = \lambda epi\,(c_I) + (1-\lambda)\,epi\,(c_J)$. We want to prove that $c_L = c_\lambda$.

Following Maccheroni et al. (2004), for $A = I, J, L$, we extend A to $[a,b]^S + \mathbb{R}$ using vertical invariance of A. Call this extension $\widetilde{A}$. Now, we further extend A to $\mathbb{R}^S$ by

$$\widehat{A}\,(\xi) = \max\left(\kappa \in \mathbb{R} : \exists\tilde{\xi} \in [a,b]^S + \mathbb{R} \text{ with } \xi - \kappa \geq \widetilde{\xi} \text{ and } \widetilde{A}\left(\widetilde{\xi}\,\right) \geq 0\right).$$

Lemma 24 of Maccheroni et al. (2004) shows that $\widehat{A}$ is the minimum niveloid that extends A to $\mathbb{R}^S$. For each $\xi \in \mathbb{R}^S$, define $\widetilde{\xi}$ such that

$$\widetilde{\xi}_s := \min\left\{\xi_s, \min_{s\in S}\{\xi_s\} + (b-a)\right\}.$$

Note that $\xi \geq \widetilde{\xi}$ and that $\widetilde{\xi} \in [a,b]^S + \mathbb{R}$. Moreover, $\xi - \widetilde{A}(\widetilde{\xi}) \geq \widetilde{\xi} - \widetilde{A}(\widetilde{\xi})$ and $\widetilde{A}(\widetilde{\xi} - \widetilde{A}(\widetilde{\xi})) = A(\widetilde{\xi}) - A(\widetilde{\xi}) = 0$. For any $\varepsilon > 0$ and $\zeta \in [a,b]^S + \mathbb{R}$, if $\xi - \widetilde{A}(\widetilde{\xi}) - \varepsilon \geq \zeta$, then $\widetilde{\xi} - \widetilde{A}(\widetilde{\xi}) \geq \zeta$. Therefore $\widehat{A}\,(\xi) = \widetilde{A}(\widetilde{\xi})$, for $A = I, J, L$.

Now we show that $\widehat{L} = \lambda\widehat{I} + (1-\lambda)\,\widehat{J}$. First note that for all $(\xi,\kappa) \in [a,b]^S \times \mathbb{R}$,

$$\begin{aligned}
\widetilde{L}\,(\xi+\kappa) &= L\,(\xi) + \kappa \\
&= \lambda\,(I\,(\xi)+\kappa) + (1-\lambda)\,(J\,(\xi)+\kappa) \\
&= \lambda\widetilde{I}\,(\xi+\kappa) + (1-\lambda)\,\widetilde{J}\,(\xi+\kappa)\,.
\end{aligned}$$

Using the fact $\widehat{A}\,(\xi) = \widetilde{A}(\widetilde{\xi})$ for all $\xi \in [a,b]^S$, $A = I, J, L$, we obtain

$$\begin{aligned}
\widehat{L}\,(\xi) &= \widetilde{L}\left(\widetilde{\xi}\right) \\
&= \lambda\widetilde{I}\left(\widetilde{\xi}\right) + (1-\lambda)\,\widetilde{J}\left(\widetilde{\xi}\right) \\
&= \lambda\widehat{I}\,(\xi) + (1-\lambda)\,\widehat{J}\,(\xi)\,.
\end{aligned}$$

From Lemma 27 of Maccheroni et al. (2004), we know that c_A, for $A = I, J, L$ is the unique l.s.c. and convex function such that

$$\widehat{A}\,(\xi) = \min_{\mu\in\Delta(S)}\left[\int \xi d\mu + c_A\,(\mu)\right], \text{ for all } \xi \in \mathbb{R}^S.$$

It can be easily checked that

$$\lambda\widehat{I}(\xi)+(1-\lambda)\widehat{J}(\xi)=\min_{\mu\in\Delta(S)}\left[\int\xi d\mu+c_{\lambda}(\mu)\right],\text{ for all }\xi\in\mathbb{R}^{S}.$$

Conclusion: $c_L = c_\lambda$. A simple inductive argument completes the proof of the claim. $\square$

Claim A.5.5. $cl_{\|\cdot\|_\infty}(co(\mathcal{U})) = \left\{U_c : c \in cl_{\|\cdot\|_\infty}(co_{epi}(\mathcal{C}))\right\}$, *and each cost function* $c \in cl_{\|\cdot\|_\infty}(co_{epi}(\mathcal{C}))$ *is minimal for some* $U_c \in cl_{\|\cdot\|_\infty}(co(\mathcal{U}))$.

Proof of Claim A.5.5. By the previous claim, $co(\mathcal{U}) = \{U_c : c \in co_{epi}(\mathcal{C})\}$ and all $c \in co_{epi}(\mathcal{C})$ are minimal, so it is enough to show that for any sequence $(c_n) \in (co_{epi}(\mathcal{C}))^{\infty}$, $c_n \to c$ if and only if $U_{c_n} \to U_c$ and c is the minimal cost function associated to U_c. Suppose that $U_{c_n} \to U_c$, where c is the minimal cost function associated to U_c. Fix $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that, for all $f \in \mathcal{F}$, $|U_{c^n}(f) - U_c(f)| < \varepsilon$ for all $n > N$. Each $e \in \{c\} \cup \{c_n\}_{n=1}^{\infty}$ satisfies:

$$e(\mu)=\max_{f\in\mathcal{F}}\left[U_e(f)-\int u(f)\,d\mu\right].$$

Fix some $\mu \in \Delta(S)$, and let f_c and $\{f_n\}_{n=1}^{\infty}$ be the maximizers associated to $c(\mu)$ and $\{c_n\}_{n=1}^{\infty}$, respectively, in the expression above. We note that, for all $n > N$, $U_c(f_c) - U_{c_n}(f_c) < \varepsilon$. This implies that, for all $n > N$, $c_n(\mu) > c(\mu) - \varepsilon$. Similarly, for all $n > N$, $U_{c_n}(f_n) - U_c(f_n) < \varepsilon$. Again, this implies that, for all $n > N$, $c(\mu) > c_n(\mu) - \varepsilon$. We conclude that, for all $n > N$, $|c(\mu) - c_n(\mu)| < \varepsilon$. Since μ is arbitrary, (c_n) converges uniformly to c. We can perform a similar analysis using the fact that for each c,

$$U_c(f)=\min_{\mu\in\Delta(S)}\left[\int u(f)\,d\mu-c(\mu)\right]$$

to show that uniform convergence of the functions (c_n) implies uniform convergence of the functions U_{c_n}. By what we have proved before this will, in turn, imply that c is minimal, which completes the proof of the claim. $\square$

To complete the proof of the proposition, we simply observe that for any U with variational representation (u, c), for all $(\alpha, \beta) \in \mathbb{R}_{++} \times \mathbb{R}$, the variational representation of $\alpha U + \beta$ is $(\alpha u + \beta, \alpha c)$.

A.6 Proof of Proposition 5

Assume that there exists some $\mu^* \in \bigcap_{c\in\mathcal{C}} M_c$, where $M_c := \{\mu \in \Delta(S) : c(\mu) = 0\}$. If $(p, f) \in \Delta(X)\times\mathcal{F}$ is such that $u(p) \geq \int u(f)\, d\mu^*$, then $u(p) \geq \int u(f)\, d\mu^* + c(\mu^*) \geq \min_{\mu\in\Delta(S)} \left\{\int u(f)\, d\mu + c(\mu)\right\}$ for all $c \in \mathcal{C}$.

Now suppose that $\bigcap_{c\in\mathcal{C}} M_c = \emptyset$, and assume w.l.o.g. that $u(\Delta(X)) = [-1, 1]$. For all $\varepsilon > 0$, define $M_c^\varepsilon := \{\mu \in \Delta(S) : c(\mu) \leq \varepsilon\}$.

Claim A.6.1. *If $\bigcap_{c\in\mathcal{C}} M_c = \emptyset$, then there exists $\varepsilon > 0$ such that $\bigcap_{c\in\mathcal{C}} M_c^\varepsilon = \emptyset$.*

Proof of Claim A.6.1. Suppose that $\bigcap_{c\in\mathcal{C}} M_c^\varepsilon \neq \emptyset$ for all $\varepsilon > 0$. Then for all $n \in \mathbb{N}$ there exists $\mu_n \in \Delta(S)$ such that $c(\mu_n) \leq \frac{1}{n}$ for all $c \in \mathcal{C}$. Use compactness of $\Delta(S)$ to extract a subsequence (μ_{n_k}) such that $\mu_{n_k} \to \tilde{\mu}$ for some $\tilde{\mu} \in \Delta(S)$. For any fixed $c \in \mathcal{C}$ we use the l.s.c. of c to obtain $c(\tilde{\mu}) \leq \liminf c(\mu_{n_k}) \leq \liminf \frac{1}{n_k} = 0$, thus contradicting $\bigcap_{c\in\mathcal{C}} M_c = \emptyset$. □

From the previous claim, we know there exists some $\varepsilon > 0$ such that $\bigcap_{c\in\mathcal{C}} M_c^\varepsilon = \emptyset$. Fix any μ^* and note that $\mu^* \notin M_{\hat{c}}^\varepsilon$ for some $\hat{c} \in \mathcal{C}$. Because $\hat{c}$ is convex and l.s.c., the nonempty set $M_{\hat{c}}^\varepsilon$ is closed and convex. Using the Separating Hyperplane Theorem we can find $u_f \in [-1, 1]^S$ such that $\int u_f d\mu^* < \int u_f d\hat{\mu}$ for all $\hat{\mu} \in M_{\hat{c}}^\varepsilon$. We can also assume w.l.o.g. that $\left|\int u_f d\mu\right| < \frac{\varepsilon}{3}$ for all $\mu \in \Delta(S)$.

Now pick $p \in \Delta(X)$ such that $u(p) = \int u_f d\mu^*$ and note that, by construction, $u(p) < \int u_f d\hat{\mu} + c(\hat{\mu})$ for all $\hat{\mu} \in M_{\hat{c}}^\varepsilon$. Hence $u(p) < \min_{\mu\in M_{\hat{c}}^\varepsilon} \left\{\int u_f d\mu + c(\mu)\right\}$, where in the last inequality we used the fact $M_{\hat{c}}^\varepsilon$ is compact. For all $\mu \in \Delta(X) \setminus M_{\hat{c}}^\varepsilon$ we have $u(p) < \frac{\varepsilon}{3} < \frac{2\varepsilon}{3} < \int u_f d\mu + c(\mu)$. As a consequence, since $\Delta(S)$ is compact, we must have $u(p) < \min_{\mu\in\Delta(S)} \left\{\int u_f d\mu + c(\mu)\right\}$. Let $f \in \mathcal{F}$ be such that $u(f) = u_f$, and $\succcurlyeq^*$ be the Anscombe-Aumann preference relation induced by the pair (u, μ^*). Therefore $\langle p\rangle \sim^* f$, but $\neg \langle p\rangle \succcurlyeq f$. Because μ^* was arbitrary, this implies $\succcurlyeq$ is not ambiguity averse.

A.7 Proof of Theorem 3

Claim A.7.1. *Let $a, b \in \mathbb{R}$, $b > a$, and $V : [a, b]^S \to \mathbb{R}$. The following are equivalent:*

(i) *V is increasing, u.s.c., and quasiconcave.*

(ii) *There exists an u.s.c. function $G : [a,b] \times \Delta(S) \to \mathbb{R}$ such that, for all $\xi \in [a,b]^S$,*

$$V(\xi) = \inf_{\mu \in \Delta(S)} G\left(\int \xi d\mu, \mu\right),$$

and, for all $\mu \in \Delta(S)$, $G(\cdot,\mu)$ is increasing.

Proof of Claim A.7.1. (i)⇒(ii). Define $\widetilde{V} : \mathbb{R}^S \to \mathbb{R} \cup \{-\infty\}$ by $\widetilde{V}(\xi) := \sup\{V(\zeta) : \zeta \in [a,b]^S \text{ and } \zeta \le \xi\}$. It can be checked that $\widetilde{V}$ is an increasing, u.s.c., and quasiconcave extension of V. Now define the function $\widetilde{V} : \mathbb{R} \times \Delta(S) \to \mathbb{R} \cup \{-\infty\}$ as $\widetilde{G}(r,\mu) := \sup_{\xi \in \mathbb{R}^S}\{\widetilde{V}(\xi) : \int \xi d\mu \le r\}$. By construction, for any fixed $\xi \in \mathbb{R}^S$, $\widetilde{V}(\xi) \le \widetilde{G}\left(\int \xi d\mu, \mu\right)$ for all $\mu \in \Delta(S)$; hence $\widetilde{V}(\xi) \le \inf_{\mu \in \Delta(S)} \widetilde{G}\left(\int \xi d\mu, \mu\right)$. If $\{\zeta \in \mathbb{R}^S : \widetilde{V}(\zeta) \ge \widetilde{V}(\xi)\} = \emptyset$, then $\inf_{\mu \in \Delta(S)} \widetilde{G}\left(\int \xi d\mu, \mu\right) \le \widetilde{V}(\xi)$. Otherwise, there exists $\bar{\varepsilon} > 0$ such that $\Gamma_\varepsilon := \{\zeta \in \mathbb{R}^S : \widetilde{V}(\zeta) \ge \widetilde{V}(\xi) + \varepsilon\} \ne \emptyset$ for all $\varepsilon \in (0,\bar{\varepsilon}]$. Because Γ_ε is closed and convex, and $\xi \notin \Gamma_\varepsilon$, by the Separating Hyperplane Theorem there exists $q \in \mathbb{R}^S \setminus \{0\}$ such that $\int \zeta dq > \int \xi dq$ for all $\zeta \in \Gamma_\varepsilon$. Since $\widetilde{V}$ is increasing, $q \in \mathbb{R}^S_+ \setminus \{0\}$. Therefore, it is w.l.o.g. to take $\nu \in \Delta(S)$ such that $\int \zeta d\nu > \int \xi d\nu$ for all $\zeta \in \Gamma_\varepsilon$. This implies that $\widetilde{G}\left(\int \xi d\nu, \nu\right) \le \widetilde{V}(\xi) + \varepsilon$ and, consequently, $\inf_{\mu \in \Delta(S)} \widetilde{G}\left(\int \xi d\mu, \mu\right) \le \widetilde{V}(\xi) + \varepsilon$. Since $\varepsilon \in (0,\bar{\varepsilon}]$ was arbitrary, we obtain $\inf_{\mu \in \Delta(S)} \widetilde{G}\left(\int \xi d\mu, \mu\right) \le \widetilde{V}(\xi)$. Conclusion: $\widetilde{V}(\xi) = \inf_{\mu \in \Delta(S)} \widetilde{G}\left(\int \xi d\mu, \mu\right)$ for all $\xi \in \mathbb{R}^S$.

Now let $\alpha \in \mathbb{R}$ be such that $A := \{(r,\mu) \in \mathbb{R} \times \Delta(S) : \widetilde{G}(r,\mu) \ge \alpha\} \ne \emptyset$. Let $(r_n,\mu_n) \in A^\infty$ satisfy $(r_n,\mu_n) \to (r,\mu)$. For all n, pick $\xi_n \in [a,b]^S$ such that $\int \xi_n d\mu_n \le r_n$ and $\widetilde{V}(\xi_n) \ge \widetilde{V}(\zeta)$. The existence of ξ_n follows from the way $\widetilde{V}$ and $\widetilde{G}$ were constructed. Note that, we can assume w.l.o.g. that $\xi_n \to \xi$, by passing to a subsequence if necessary. Clearly, $\int \xi d\mu \le r$, so that $\widetilde{G}(r,\mu) \ge \widetilde{V}(\xi)$. Because $\widetilde{V}(\xi_n) \ge \alpha$ for all n, and $\widetilde{V}$ is u.s.c., we conclude that $\widetilde{V}(\xi) \ge \alpha$, implying that $\widetilde{G}(r,\mu) \ge \alpha$. Therefore $\widetilde{G}$ is u.s.c.. It is also increasing in the first argument, as it can be easily checked. Put $G := \widetilde{G}|_{[a,b]\times\Delta(S)}$ and note that $V(\xi) = \inf_{\mu \in \Delta(S)} G\left(\int \xi d\mu, \mu\right)$ for all $\xi \in [a,b]^S$.

(ii)⇒(i). Let $\xi, \zeta \in [a,b]^S$ be such that $\xi \ge \zeta$. For all $\mu \in \Delta(S)$, $V(\zeta) \le G\left(\int \zeta d\mu, \mu\right) \le G\left(\int \xi d\mu, \mu\right)$, where the last inequality follows from the fact $G(\cdot,\mu)$ is increasing. Therefore $V(\zeta) \le \inf_{\mu \in \Delta(S)} G\left(\int \xi d\mu, \mu\right) = V(\xi)$, and V must be increasing. Now let $\lambda \in (0,1)$ and ξ and ζ be any two elements in $[a,b]^S$. For all

$\mu \in \Delta(S)$,

$$\begin{aligned} G\left(\lambda \int \xi d\mu + (1-\lambda)\int \zeta d\mu, \mu\right) &\\ \geq \min\left\{G\left(\int \xi d\mu, \mu\right), G\left(\int \zeta d\mu, \mu\right)\right\} &\\ \geq \min\left\{\inf_{\mu\in\Delta(S)} G\left(\int \xi d\mu, \mu\right), \inf_{\mu\in\Delta(S)} G\left(\int \zeta d\mu, \mu\right)\right\}. \end{aligned}$$

Hence $V(\lambda\xi+(1-\lambda)\zeta) = \inf_{\mu\in\Delta(S)} G(\lambda\int\xi d\mu+(1-\lambda)\int\zeta d\mu, \mu) \geq \min\{V(\xi), V(\zeta)\}$, implying V is quasiconcave. Finally, let $\alpha \in \mathbb{R}$ be such that $B := \{\zeta \in [a,b]^S : V(\zeta) \geq \alpha\} \neq \emptyset$, and take a sequence $(\xi_n) \in B^\infty$ such that $\xi_n \to \xi$. By construction, $G(\int \xi_n d\mu, \mu) \geq V(\xi_n) \geq \alpha$ for all $n \in \mathbb{N}$, for all $\mu \in \Delta(S)$. Because G is u.s.c., we must have $G\left(\int \xi d\mu, \mu\right) \geq \alpha$, and hence $V(\xi) \geq \alpha$. □

Claim A.7.2. *Every upper semicontinuous and convex preorder can be represented by a set of upper semicontinuous and quasiconcave utility functions.*

Proof of Claim A.7.2. Adapt the arguments of Evren and Ok (2007) and Kochov (2007). (The representation is induced by the set of indicator functions of the upper contour sets of all elements on the domain of preferences.) □

(a)⇒(b). Standard arguments can be employed to show the existence of a continuous and affine function $u : \Delta(X) \to \mathbb{R}$ such that, for all $p, q \in \Delta(X)$, $p \trianglerighteq^\bullet q$ iff $u(p) \geq u(q)$. Now every act $f \in \mathcal{F}$ can mapped into a vector of utils $\xi_f := u(f) \in u(\Delta(X))^S$. We can also define a binary relation $\succsim \subseteq u(\Delta(X))^S \times u(\Delta(X))^S$ so that, for all $\xi_f, \xi_g \in u(\Delta(X))^S$, $\xi_f \succsim \xi_g$ iff $f \trianglerighteq g$. The monotonicity axiom B4 guarantees $\succsim$ is well-defined. It is easy to see that $\succsim$ is a monotonic preorder. Now take any sequence (ξ_{f_n}) in $u(\Delta(X))^S$ such that $\xi_{f_n} \succsim \xi_g$ for all $n \in \mathbb{N}$, some $g \in \mathcal{F}$, and $\xi_{f_n} \to \xi$. Because $\mathcal{F}$ is a compact metric space, we may assume, by passing to a subsequence if necessary, that $f_n \to f$, for some $f \in \mathcal{F}$. As a consequence, using the continuity axiom B2, we conclude $\xi = \xi_f \succsim \xi_g$. Therefore $\succsim$ is upper semicontinuous. It is a standard exercise to show $\succsim$ is also convex. Now apply claim A.7.2 to find a set $\mathcal{V}$ of u.s.c. and quasiconcave function such that, for all $f, g \in \mathcal{F}$, $f \trianglerighteq g$ iff $\xi_f \succsim \xi_g$ iff $V(\xi_f) \geq V(\xi_g)$ for all $V \in \mathcal{V}$. Monotonicity of $\succsim$ implies each $V \in \mathcal{V}$ must be increasing.

For all $c \in u(\Delta(X))$, let V_c denote the function in $\mathcal{V}$ which takes the value 1 when evaluated at ξ_f with $\xi_f \succsim c\mathbf{1}_S$, and 0 otherwise. Consider the enumeration $\{d_1, d_2, ...\}$ of the set $D := u(\Delta(X)) \cap \mathbb{Q}$, and define the function $W := \sum_{i=1}^\infty \frac{1}{2^i} V_{d_i}$.

Because W is the uniform limit of a sequence of u.s.c. functions, it is itself a u.s.c. function. Now we show W is quasiconcave. For all $j \in \mathbb{N}$, define $W_j := \sum_{i=1}^{j} \frac{1}{2^i} V_{d_i}$ and $\varepsilon_j := \sum_{i=j+1}^{\infty} \frac{1}{2^i} > 0$. Let $\alpha \in (0,1]$, and $f, g \in \mathcal{F}$ be such that $W(\xi_f) \geq \alpha$ and $W(\xi_g) \geq \alpha$. Note $W_j(\xi_f) \geq \alpha - \varepsilon_j$ and $W_j(\xi_g) \geq \alpha - \varepsilon_j$. For some j such that $\alpha - \varepsilon_j > 0$, put $d_j^* := \left\{ d \in \{d_1, ..., d_j\} : \sum_{i=1}^{j} \frac{1}{2^i} V_{d_i}(d) \geq \alpha - \varepsilon_j \right\}$. By construction, d_j^* is well-defined. Because $W_j(\xi_f) \geq \alpha - \varepsilon_j$, we must have $\xi_f \succsim d_j^* \mathbf{1}_S$, for otherwise $\neg \xi_f \succsim d_j^* \mathbf{1}_S$ implies $V_d(f) = 0$ for all $d \geq d_j^*$. As a consequence, in order to attain $W_j(\xi_f) \geq \alpha - \varepsilon_j$, we must have $\sum_{i=1}^{j} \frac{1}{2^i} V_{d_i}(d_*) \geq \alpha - \varepsilon_j$ for some $d_* < d_j^*$, a contradiction with the definition of d_j^*. A similar argument can be employed to show $\xi_g \succsim d_j^* \mathbf{1}_S$. Because $\succsim$ is convex, for all $\lambda \in (0,1)$, we have $\lambda \xi_f + (1-\lambda) \xi_f \succsim d_j^* \mathbf{1}_S$, which in turn implies $W(\lambda \xi_f + (1-\lambda)\xi_f) \geq W(d_j^*) \geq \alpha - \varepsilon_j + \sum_{i=j+1}^{\infty} \frac{1}{2^i} V_{d_i}(d_j^*)$. If we let $j \to \infty$, we obtain $W(\lambda \xi_f + (1-\lambda)\xi_f) \geq \alpha$. Therefore, W is quasiconcave. Also note that, for all $c, d \in u(\Delta(X))$, $c \geq d$ iff $W(c) \geq W(d)$. Moreover, we can consider $\mathcal{V} \cup \{W\}$ instead of $\mathcal{V}$, and $\inf_{\mu \in \Delta(X)} H(\cdot, \mu)$ is strictly increasing for all $\mu \in \Delta(S)$, where $H : u(\Delta(X)) \times \Delta(S) \to \mathbb{R}$ is defined as in the proof of claim A.7.1.

(b)⇒(a). First note that, for all $p, q \in \Delta(X)$, $p \trianglerighteq^\bullet q$ iff $u(p) \geq u(q)$. Clearly $u(p) \geq u(q)$ implies $G(u(p), \mu) \geq G(u(q), \mu)$. As a consequence $\inf_{\mu \in \Delta(S)} G(u(p)), \mu \geq \inf_{\mu \in \Delta(S)} G(u(q), \mu)$ and $p \trianglerighteq^\bullet q$. Now assume $u(p) > u(q)$, then $\inf_{\mu \in \Delta(S)} H(u(p), \mu) > \inf_{\mu \in \Delta(S)} H(u(q), \mu)$, and $\inf_{\mu \in \Delta(S)} G(u(p), \mu) \geq \inf_{\mu \in \Delta(S)} G(u(q), \mu)$ for all other $G \in \mathcal{G}$. This implies $p \triangleright q$. Therefore, u is an expected utility representation of $\trianglerighteq^\bullet$, and $\trianglerighteq^\bullet$ must satisfy B5-B7. Second, let $(f_n) \in \mathcal{F}^\infty$ be such that $f_n \succcurlyeq g \in \mathcal{F}$ for all $n \in \mathbb{N}$. Fix any $G \in \mathcal{G}$, and note that $\inf_{\mu \in \Delta(S)} G\left(\int u(f_n) d\mu, \mu\right) \geq \inf_{\mu \in \Delta(S)} G\left(\int u(g) d\mu, \mu\right)$. As the pointwise infimum of a family of u.s.c. functions, $\xi \mapsto \inf_{\mu \in \Delta(S)} G\left(\int \xi d\mu, \mu\right)$ is itself a continuous function. Since $u(f_n) \to u(f)$, we obtain

$$\inf_{\mu \in \Delta(S)} G\left(\int u(f) d\mu, \mu\right) = \lim_n \inf_{\mu \in \Delta(S)} G\left(\int u(f_n) d\mu, \mu\right)$$
$$\geq \inf_{\mu \in \Delta(S)} G\left(\int u(g) d\mu, \mu\right)).$$

Therefore, $\succcurlyeq$ is u.s.c. Third, let $f, g \succcurlyeq h$, $\lambda \in (0,1)$, and $G \in \mathcal{G}$. Because $G(\cdot, \mu)$ is increasing and $\lambda u(f) + (1-\lambda) u(g) \geq \min\{u(f), u(g)\}$, one obtains, for all

$\mu \in \Delta(S)$,

$$\begin{aligned}
&G\left(\int u(\lambda f + (1-\lambda)\, g) d\mu, \mu\right) \\
&\geq \min\{G\left(\int u(f)\, d\mu, \mu\right), G\left(\int u(g)\, d\mu, \mu\right)\} \\
&\geq \min\{\inf_{\mu \in \Delta(S)} G\left(\int u(f)\, d\mu, \mu\right), \inf_{\mu \in \Delta(S)} G\left(\int u(g)\, d\mu, \mu\right)\} \\
&\geq \inf_{\mu \in \Delta(S)} G\left(\int u(h)\, d\mu, \mu\right).
\end{aligned}$$

Hence $\inf_{\mu \in \Delta(S)} G\left(\int u(\lambda f + (1-\lambda)\, g)\, d\mu, \mu\right) \geq \inf_{\mu \in \Delta(S)} G\left(\int u(h)\, d\mu, \mu\right)$. Since $G \in \mathcal{G}$ was arbitrary, then $\lambda f + (1-\lambda)\, g \succcurlyeq h$, and $\succcurlyeq$ must be convex. Finally, let $f, g \in \mathcal{F}$ be such that $f(s) \trianglerighteq^{\bullet} g(s)$ for all $s \in S$. Therefore $\int u(f)\, d\mu \geq \int u(g)\, d\mu$ for all $\mu \in \Delta(S)$ and, since $G(\cdot, \mu)$ is an increasing function, we obtain that $\inf_{\mu \in \Delta(S)} G\left(\int u(f)\, d\mu, \mu\right) \geq \inf_{\mu \in \Delta(S)} G\left(\int u(g)\, d\mu, \mu\right)$ for all $G \in \mathcal{G}$.

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