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Preference for Flexibility and Bayesian Updating*

Gil Riella†

Abstract

Dekel, Lipman, and Rustichini (2001) show that preferences over menus of lotteries can be represented by the use of a unique subjective state space and a prior. We provide foundations for Bayesian updating in such a setup. When the subjective state space is finite, we show that Bayesian updating is linked to a comparative theory of *preference for flexibility.* Without the finiteness of the subjective state space, Bayesian updating is characterized by a more technical condition.

JEL Classification: D11, D81.

Keywords: Preference for Flexibility, Bayesian Updating, Subjective State Space.

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1 Introduction

The issue of updating ones preferences on uncertain alternatives on the face of additional information is a topic that is extensively studied in individual decision theory. In particular, the Bayesian underpinnings of such updating are well understood within the realm of the Savagean theory of decision making, where the state space is regarded as exogenously given. When, however, the decision model at hand regards the state space as endogenous, as in the recently developed dynamic theory of choice over menus, the foundations for the notion of Bayesian updating become much less transparent. In a nutshell, the paper is an attempt to provide such a foundation by using the notion of *preference for flexibility*.

Consider the following situation. At 11 AM Jane has to choose a place for her evening drink with her friends. Suppose the only way places vary is in the menu of drinks they offer and Jane always has only one drink. Let X be the set of all possible drinks and $\mathcal{X}$ be the set of all conceivable drink menus, each menu being represented by capital letters A, B, C, etc.. As in Kreps (1979), we assume that Jane has a well-defined preference relation $\succsim$ on $\mathcal{X}$ and this relation exhibits preference for flexibility in the sense that, for any two menus A and B,

$$A \supseteq B \text{ implies } A \succsim B.$$

Moreover, we assume that her preferences admit the following representation:

$$A \succsim B \text{ iff } \sum_{s \in S} \pi(s) \max_{x \in A} U(x, s) \geq \sum_{s \in S} \pi(s) \max_{x \in B} U(x, s), \tag{1}$$

where S is a finite state-space, π is a probability measure on S and U is a state-dependent utility function. The interpretation is that Jane is uncertain about the future and, in particular, she is not sure what kind of drink she will be in the mood for in the evening. The representation above thus says that she chooses a place that maximizes the expected utility she can get from the place's drink menu, with respect to some prior π about her future tastes.

At lunch, Jane will meet with her friends and one of them is going to be selected as the designated driver for the evening. If Jane gets to be the designated driver, and only in that situation, she would very much appreciate if the place they went had orange juice, with the other drinks being unimportant. We represent the "Jane being the designated driver situation" by the state $s^* \in S$. If we use the letter j

to represent the orange juice alternative, the discussion above can be formalized as follows:

$$U(j, s^*) > U(x, s^*) \text{ and } U(y, s^*) = U(x, s^*),$$

for all $x, y \in X$ distinct from j, and

$$U(j, s) < U(x, s),$$

for all $x \in X$ and $s \in S$, distinct from j and s^*, respectively.

Suppose, now, instead of choosing a place in the morning, Jane first goes for lunch with her friends and at that occasion she is informed that she will not have to drive that evening. In terms of the representation above, this is equivalent to say that she learns that the state s^* will not happen. Finally, we assume that Jane is Bayesian, so, upon learning that state s^* is no longer a possibility, she uses Bayes rule to update her prior about the states of the world. Except for that, she follows (1) to make her decision about where to go for drinks in the evening.

The situation described in the previous paragraphs is entirely standard in economics. Indeed, were the state-space S observable by the modeler, we could easily axiomatize Jane's Bayesian behavior by a Dynamic Consistency like condition. However, in the preference over menus literature, the state-space is not part of the specification of the model, being instead derived as part of the representation of the agent's preferences. For that reason, an axiomatization of Bayesian updating in the present setting is bound to be related to conditions that have a completely different interpretation. The main goal of the present paper is to provide such an axiomatization and discuss the conditions related to Bayesian updating when the primitives of our model consist of preference relations over a space of menus.

Obviously, without an exogenously specified state-space, we cannot write a condition that explicitly deals with the fact that Jane receives new information at lunch. Nonetheless, the fact that she is Bayesian still implies some consistency conditions relating her preferences before and after lunch. To organize the discussion, let $\succsim_1$ represent her preference before lunch and $\succsim_2$ the one after lunch. Now suppose that A and B are two menus such that $A \succ_2 B$, but $B \succsim_1 A$. That is, before lunch she considers menu B at least as good as menu A, but after she learns that she will not have to drive in the evening, menu A becomes strictly more attractive than B. We note that, by (1) and the assumption that Jane is Bayesian, this can happen only if $j \in B$, but $j \notin A$. Intuitively, the only difference between Jane's preferences before and after lunch is that after lunch she no longer cares whether the place she goes has orange juice or not. So, if her before lunch preference relation values menu B more than her after lunch relation, it has to be because B offers exactly the alternative that loses its value once Jane learns she will not have to drive that evening.

Following the insight provided by this example, we investigate in this paper the Bayesian updating behavior when the state-space of the model is subjective. We work in the setup of Dekel et al. (2001) –henceforth DLR– and our main condition is a generalization of the idea discussed in the previous paragraph. In words, our condition says that if menus A and B are such that $A \succ_2 B$, but $B \succsim_1 A$, then it must be the case that $\succsim_1$ sees some gain in flexibility when moving from menu A to menu $A \cup B$ that $\succsim_2$ does not see. In our example, this corresponds to the fact that $\succsim_1$ considers it worthwhile to have the option j in menu B, while for $\succsim_2$ this is entirely irrelevant.

When $\succsim_1$ and $\succsim_2$ satisfy this condition, we say that $\succsim_2$ is a *less flexibility loving version* of $\succsim_1$. The upshot of the present paper is that this condition is intrinsically related to the possibility of representing $\succsim_2$ as a Bayesian update of $\succsim_1$'s representation. If the state-space used in the representation of $\succsim_1$ is finite, we show that this connection is tight, that is, being a less flexibility loving version of $\succsim_1$ is necessary and sufficient to make $\succsim_2$ representable by a Bayesian update of $\succsim_1$'s representation. Without the finiteness of the state-space used in $\succsim_1$'s representation, an additional technical condition is necessary in order to obtain a similar result.

In more abstract terms, this paper can be also seen as relating a comparative theory of preference for flexibility to Bayesian updating. DLR show that for a pair of relations $\succsim_1$ and $\succsim_2$ that can be represented as in (1), if $\succsim_1$ values flexibility more than $\succsim_2$, then the state-space used in the representation of $\succsim_1$ is larger than the one used in the representation of $\succsim_2$. We discuss this result formally in Section 3 bellow, but it is worth noting here that the results in this paper may be seen in the garb of an extension of that analysis. Basically, our "less flexibility loving version" condition may be interpreted as saying that $\succsim_1$ values flexibility more than $\succsim_2$ and, in some sense, this is the only difference between this two relations.

In terms of the literature, the results here are related to two groups of papers. First, they can be seen as contributing to the extensive literature on updating. Of course, it departs from the tradition, since most of that literature works in a standard Savagean setup with an exogenously given state-space.[1]

[1]See Epstein and Le Breton (1993), Ghirardato (2002) and the references therein, for a discus-

On the other hand this paper is related to the preferences over menus literature.[2] This literature had an increase in popularity after the works of DLR and Gul and Pesendorfer (2001). In that literature, however, only a few papers have performed exercises similar to the one performed here, by way of studying how some comparative notion relating a pair of menu preferences affects the properties of a given model. For example, Gul and Pesendorfer (2001) show that a self-control preference $\succsim_1$ has more self-control than another such preference $\succsim_2$ if and only if $\succsim_2$'s temptation ranking is closer to her "temptation free preference" than $\succsim_1$'s. In a similar fashion Sarver (2008) links two different measures of regret attitudes to interesting properties of his regret representation.

The remainder of our paper is organized as follows. We discuss the primitives of the model and revisit some results of DLR, Dekel, Lipman, Rustichini, and Sarver (2007) –henceforth DLRS– and Dekel, Lipman, and Rustichini (2008) –henceforth DLR2– in Section 2. In Section 3, we present the comparative theory of preference for flexibility that will be later related to Bayesian updating. In particular, we define the fundamental notion of a less flexibility loving version. Next, in Section 4, we prove our first result relating the notion of a less flexibility loving version to Bayesian updating in the finite state-space case. Section 5 extends the analysis in Section 4 to the case of an infinite state-space. The Bayesian updating result found in Section 5 characterizes Bayesian updating only when the observed event satisfies a certain topological condition. For completeness, we give a general characterization of Bayesian updating in Section 6, but for that we have to pay the cost of working with further technical conditions. Section 7 concludes. We relegate most of the proofs to the appendix.

2 Preference for Flexibility and Additive EU Representations

In this section we briefly revisit some results from DLR and DLRS that we shall need for our subsequent analysis. We first describe the primitives of their model.

Let X be a finite set of alternatives and $\Delta(X)$ the space of lotteries (probability distributions) on X. We view $\Delta(X)$ as a metric subspace of $\mathbb{R}^{|X|}$ and represent its

sion of Bayesian updating in the classic Savagean framework. For a discussion of non-Bayesian updating rules, see Epstein (2006) and Epstein, Noor, and Sandroni (2008). On updating in the context of the multiple priors model of Gilboa and Schmeidler (1989), see Epstein and Schneider (2003), Gilboa and Schmeidler (1993), Hanany and Klibanoff (2007) and Siniscalchi (2006).

[2] See Kreps (1979), Nehring (1999), Gul and Pesendorfer (2001) and DLR.

elements by p, q, r, etc..[3] Let $\mathcal{X}$ represent the space of nonempty closed subsets of the relative interior of $\Delta(X)$. That is, $\mathcal{X}$ is the set of all nonempty closed subsets of $\Delta(X)$ that include only lotteries with full support. We consider binary relations $\succsim$ on $\mathcal{X}$. As usual, we denote the symmetric part of $\succsim$ by $\sim$ and the asymmetric part by $\succ$. The elements of $\mathcal{X}$ are represented by capital letters A, B, C, etc., and are called *menus*.

The idea here is that an agent whose preference relation is $\succsim$ faces a two-period decision problem. In the first period she chooses a menu knowing that in the next period she will have to make a choice from that menu. Following DLR, we do not explicitly model the agent's second period choice, leaving it as part of the interpretation of the results presented in this section.

2.1 Representations

The uncertainty of the agent about her future tastes is modeled by a probability measure over a set of possible ex post utility functions. As in DLR, we impose the restriction that each ex post utility function be of the expected utility type. Because expected utility functions are only unique up to positive affine transformations, it is convenient to impose a normalization on the set of ex post utility functions we use in our representations. Consequently, we define the set of normalized expected utility functions on $\Delta(X)$ as

$$\mathcal{U} := \left\{ u \in \mathbb{R}^{|X|} : \sum_{i=1}^{|X|} u_i = 0 \text{ and } \sum_{i=1}^{|X|} u_i^2 = 1 \right\}.$$

Just like $\Delta(X)$, we view $\mathcal{U}$ as a metric subspace of $\mathbb{R}^{|X|}$.

We are now ready to introduce the concept of a *Positive Additive Expected Utility representation* that will be extensively used in this paper.

Definition. We say that a binary relation $\succsim$ on $\mathcal{X}$ has a **Positive Additive Expected Utility (PAEU) representation**, μ, if μ is a Borel probability measure on $\mathcal{U}$ such that the function $W : \mathcal{X} \to \mathbb{R}$, which is defined by

$$W(A) := \int_{\mathcal{U}} \max_{p \in A} E_p(u)\, \mu(ds),$$

[3] Since X is finite, this is equivalent to endow $\Delta(X)$ with the topology of weak convergence.

represents $\succsim$.[4]

In what follows we shall also have the opportunity to consider PAEU representations in which only a finite number of states are relevant for the agent's decisions:

Definition. We say that a binary relation $\succsim$ on $\mathcal{X}$ has a **finite PAEU representation**, μ, if μ is a PAEU representation of $\succsim$ with finite support.

2.2 Axioms and Representation Theorems

We now present the postulates that characterize when a binary relation $\succsim$ on $\mathcal{X}$ admits a PAEU or a finite PAEU representation.

Axiom 1 (Preorder). *$\succsim$ is a complete preorder on $\mathcal{X}$.*

Axiom 2 (vNM Continuity). *For any menus A, B, C with $A \succ B \succ C$, there exist two numbers, α and β in $(0,1)$ such that*

$$A \oplus_\alpha C \succ B \succ A \oplus_\beta C.$$ [5]

Axiom 3 (Independence). *For any two menus A and B,*

$$A \succ B \text{ implies } A \oplus_\lambda C \succ B \oplus_\lambda C,$$

for any $\lambda \in (0,1]$ and $C \in \mathcal{X}$.

Axiom 4 (Nontriviality). *There exist two menus A and B such that $A \succ B$.*

These properties are extensively discussed in DLR and DLRS, so we shall not elaborate on them here. In addition to these four postulates, here we will also work with the assumption that the binary relation $\succsim$ satisfies the monotonicity property introduced by Koopmans (1964).

Axiom 5 (Monotonicity). *For any two menus A and B,*

$$B \subseteq A \text{ implies } A \succsim B.$$

[4]Here $E_p(u)$ represents the expectation of the random variable on $\mathbb{R}^{|X|}$ that takes value u_i with probability p_i, $i = 1, ..., |X|$.

[5]*Notation:* For any two menus A, B and $\lambda \in [0,1]$, by $A \oplus_\lambda B$ we mean the set $\{p \in \Delta(X) : p = \lambda q + (1-\lambda) r \text{ for some } q \in A \text{ and } r \in B\}$.

This property is what Kreps (1979) refers to as *preference for flexibility.* The interpretation comes from the idea that the agent chooses today a menu from which she will have to make a choice tomorrow. With regards to this interpretation, the Monotonicity axiom says that the agent always likes the flexibility of having more options to choose from in the future.

In their seminal contributions, DLR and DLRS prove that an individual whose preference relation on $\mathcal{X}$ abides by the above five postulates is guaranteed to have a PAEU representation.

Theorem 1 (DLRS). *A binary relation $\succsim$ on $\mathcal{X}$ satisfies Preorder, vNM Continuity, Independence, Nontriviality and Monotonicity if and only if it has a unique PAEU representation.*[6]

Motivated by this result, we call $\succsim$ a PAEU preference whenever it satisfies Preorder, vNM Continuity, Independence, Nontriviality and Monotonicity.

Some of the results in this paper will be derived under the assumption that $\succsim$ has, in fact, a *finite* PAEU representation. The following condition, found by DLR2, characterizes this case.

Axiom 6 (Finiteness). *Every menu A has a finite subset C such that $A \sim C$.*

This property is powerful enough to guarantee that a PAEU preference admits a finite PAEU representation.

Theorem 2 (DLR2). *A binary relation $\succsim$ on $\mathcal{X}$ satisfies Preorder, vNM Continuity, Independence, Nontriviality, Monotonicity and Finiteness if and only if it has a unique finite PAEU representation.*

Throughout the present paper we refer to a binary relation on $\mathcal{X}$ that satisfies the six postulates in the statement of Theorem 2 as a finite PAEU preference.

3 Comparative Desire for Flexibility

In this section we discuss some comparative notions of desire for flexibility. We begin with the following definition due to DLR.

[6] Actually, DLRS do not state this result in terms of the normalized space $\mathcal{U}$ as we do here. Instead, they write it in terms of a generic measurable space and a generic state-dependent utility function. As it is clear from the proof in their paper, however, a binary relation $\succsim$ on $\mathcal{X}$ has a representation in terms of generic measurable state and state-dependent utility function if and only if it has a representation in terms of the normalized state-space used here.

Definition. We say that a binary relation $\succsim_1$ on $\mathcal{X}$ **values flexibility more than** some other binary relation $\succsim_2$ on $\mathcal{X}$ if and only if, for any two menus A and B with $B \subseteq A$,

$$A \succ_2 B \text{ implies } A \succ_1 B.$$

In words, $\succsim_1$ values flexibility more than $\succsim_2$ if in any situation where $\succsim_2$ strictly prefers the flexibility of having more options, so does $\succsim_1$. When $\succsim_1$ and $\succsim_2$ are PAEU preferences, one can show that this property has a rather intuitive characterization.

Lemma 1. *Suppose $\succsim_1$ and $\succsim_2$ are PAEU preferences with representations μ_1 and μ_2, respectively. Then, $\succsim_1$ values flexibility more than $\succsim_2$ if and only if*

$$supp\,(\mu_2) \subseteq supp\,(\mu_1)\,.$$[7]

This result says that $\succsim_1$ values flexibility more than $\succsim_2$ if and only if every future state considered possible by $\succsim_2$ is also considered possible by $\succsim_1$. The "if" part of this fact is, of course, straightforward. We prove its "only if" in the Appendix, but, in passing, we note that the needed argument basically consists of showing that if $supp\,(\mu_2) \setminus supp\,(\mu_1) \neq \emptyset$, then it is possible to find two menus A and B with $B \subseteq A$ such that $A \succ_2 B$, but $A \sim_1 B$.

In DLR we find a more general version of Lemma 1. First, instead of writing the result in terms of the normalized set $\mathcal{U}$, they state it in terms of a generic state space and a suitable topology on the space of expected utility preferences on $\Delta\,(X)$. Moreover, they show that the assertion is true for a class of binary relations larger than the class of PAEU preferences. The simpler version of the result stated here, however, will be enough for our purposes.

Now, let A and B be any two menus such that $A \subseteq B$. We want to be able to say that $\succsim_1$ values the flexibility provided by the extra options in B more than $\succsim_2$. We are going to represent such a situation by the binary relation $\rhd$ defined below.

Definition. For any two menus A and B, we say that $B \rhd A$, if and only if $A \subseteq B$ and there exists a menu C such that

$$A \cup C \sim_2 B \cup C, \text{ but } A \cup C \prec_1 B \cup C.$$

[7] *Notation:* For a given probability measure μ we use $supp\,(\mu)$ to represent the support of μ.

We call $\rhd$ the **Extra-flexibility relation** and note that it is an asymmetric binary relation on $\mathcal{X}$ (The dependence of $\rhd$ on $\succsim_1$ and $\succsim_2$ is suppressed in our notation for simplicity.)

In the definition above, the presence of the options in C makes the flexibility gained with the extra options in B worthless for $\succsim_2$. However, $\succsim_1$ sees the opportunity of choosing from that larger menu in the future as a strict improvement.

When $\succsim_1$ and $\succsim_2$ are PAEU preferences, the Extra-flexibility relation has an interesting characterization.

Lemma 2. *Suppose $\succsim_1$ and $\succsim_2$ are preferences with PAEU representations μ_1 and μ_2, respectively, and define $S_i := supp\,(\mu_i)$ for $i = 1, 2$. For any two menus A and B such that $A \subseteq B$, we have $B \rhd A$ if and only if there exists some $u \in S_1 \setminus S_2$ such that*

$$\max_{p \in B} E_p\,(u) > \max_{p \in A} E_p\,(u)\,.$$

This result says that the existence of a situation where $\succsim_2$ does not see any gain in having the possibility of choosing from the larger menu, B, while $\succsim_1$ does, is equivalent to the existence of some state, considered possible only by $\succsim_1$, at which it is strictly better to make a choice from B than from A.

Above, we learned how to characterize when $\succsim_1$ values flexibility more than $\succsim_2$. That definition is valid even if the differences between $\succsim_1$ and $\succsim_2$ go far beyond the way they value flexibility. In some cases, it might be interesting to be able to say that the only difference between $\succsim_1$ and $\succsim_2$ is the fact that $\succsim_1$ values flexibility more than $\succsim_2$. One way to capture this idea is to require that any disagreement between $\succsim_1$ and $\succsim_2$ be only a consequence of $\succsim_1$'s higher desire for flexibility. The definition below formalizes this discussion.

Definition. We say that $\succsim_2$ is **a less flexibility loving version** of $\succsim_1$ if for any menus A and B,

$$A \succ_2 B \text{ and } B \succsim_1 A \text{ imply } A \cup B \rhd A.$$

Thus, if $\succsim_2$ is a less flexibility loving version of $\succsim_1$, and we have $A \succ_2 B$, but $B \succsim_1 A$, then it must be the case that there exists at least one situation where $\succsim_2$ sees no value in adding the options in B to A, but $\succsim_1$ still sees that as a strict improvement. In this sense, whenever the two preferences disagree in the way described above, we can blame the disagreement on the fact that $\succsim_1$ sees more value in the flexibility achieved by adding B to A than $\succsim_2$.

We note that if $\succsim_1$ and $\succsim_2$ are PAEU preferences and $\succsim_2$ is a less flexibility loving version of $\succsim_1$, then, clearly, $\succsim_1$ values flexibility more than $\succsim_2$. Intuitively, if $B \subseteq A$, then neither $\succsim_1$ nor $\succsim_2$ see any gain in flexibility when B is added to A. Therefore, we will never be able to find a situation where $\succsim_1$ sees a strict gain in adding the options in B to A and $\succsim_2$ does not. But then, it is immediate from the assumption that $\succsim_2$ is a less flexibility loving version of $\succsim_1$ that $A \succ_1 B$ if $A \succ_2 B$. We summarize this discussion with the following lemma:

Lemma 3. *Let $\succsim_1$ and $\succsim_2$ be two PAEU preferences. If $\succsim_2$ is a less flexibility loving version of $\succsim_1$, then $\succsim_1$ values flexibility more than $\succsim_2$.*

The next two sections will be dedicated to the characterization of the less flexibility loving versions of a given PAEU preference, $\succsim_1$. We are going to see that if $\succsim_1$ is a finite PAEU preference, then they correspond exactly to the preferences which can be represented as a Bayesian update of the representation of $\succsim_1$. When $\succsim_1$ does not have a PAEU representation with a finite support an additional condition will be needed in order to obtain a similar result.

4 Finite PAEU Preferences and Bayesian Updating

In the previous section we introduced the notion of a less flexibility loving version of a given PAEU preference $\succsim_1$. That property captured the idea of a relation differing from $\succsim_1$ only because of its weaker desire for flexibility. We now show that when $\succsim_1$ has a representation with a finite support, its less flexibility loving versions correspond exactly to the relations that can be obtained as a Bayesian update of $\succsim_1$'s representation. The formal result is the following:

Theorem 3. *Let $\succsim_1$ be a finite PAEU preference. A PAEU preference $\succsim_2$ is a less flexibility loving version of $\succsim_1$ if and only if there exists a Borel subset T of $\mathcal{U}$ such that the PAEU representation, μ_2, of $\succsim_2$ is the Bayesian update of the PAEU representation, μ_1, of $\succsim_1$ after the observation of T.*

The intuition for the result above is simple. Suppose first that the representation, μ_2, of $\succsim_2$ is a Bayesian update of the representation, μ_1, of $\succsim_1$. Let $S_i := supp\,(\mu_i)$, for $i = 1, 2$. It is easy to see that this implies that, for any two menus A and B,

$$\begin{array}{ccc} \sum_{u \in S_2} \mu_2(u) \max_{p \in A} E_p(u) & \geq & \sum_{u \in S_2} \mu_2(u) \max_{p \in B} E_p(u) \\ & \Longleftrightarrow & \\ \sum_{u \in S_2} \mu_1(u) \max_{p \in A} E_p(u) & \geq & \sum_{u \in S_2} \mu_1(u) \max_{p \in B} E_p(u). \end{array}$$

So, if $A \succ_2 B$ and $B \succsim_1 A$ for some pair of menus A and B, it must be the case that there exists $u^* \in S_1 \setminus S_2$ such that

$$\max_{p \in B} E_p(u^*) > \max_{p \in A} E_p(u^*).$$

By Lemma 2, we know that this implies that $A \cup B \rhd A$. We conclude that $\succsim_2$ is a less flexibility loving version of $\succsim_1$. Conversely, suppose that the representation, μ_2, of $\succsim_2$ is not a Bayesian update of the representation, μ_1, of $\succsim_1$. In the appendix we show that in this case we can always find two menus A and B such that

$$\sum_{u \in S_2} \mu_2(u) \max_{p \in A} E_p(u) > \sum_{u \in S_2} \mu_2(u) \max_{p \in B} E_p(u),$$

but

$$\sum_{u \in S_1} \mu_1(u) \max_{p \in A} E_p(u) \leq \sum_{u \in S_1} \mu_1(u) \max_{p \in B} E_p(u)$$

and

$$\max_{p \in A} E_p(u) \geq \max_{p \in B} E_p(u) \text{ for all } u \in S_1 \setminus S_2.$$

Again, by Lemma 2, this implies that $A \cup B \rhd A$ is false and, therefore, $\succsim_2$ is not a less flexibility loving version of $\succsim_1$.

In brief, the proof of Theorem 3 consists of showing that if it is not the case that the representation of $\succsim_2$ is a Bayesian update of the representation of $\succsim_1$, then we can always find two menus A and B such that $\succsim_1$ and $\succsim_2$ disagree about the ranking of these two menus, but the reason for that is not $\succsim_1$'s stronger desire for flexibility. This makes the behavioral implications of the Bayesian updating result in Theorem 3 intuitive and easy to understand.

The tight connection between the concept of a less flexibility loving version and Bayesian updating for finite PAEU preferences naturally makes one wonder if such a result remains true when the finiteness assumption is dropped. Unfortunately, as Example 1 bellow shows, this is not the case.

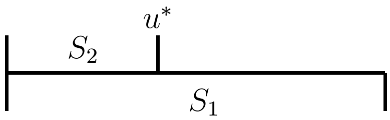

Figure 1

Example 1. Suppose $|X| = 3$. In this case $\mathcal{U}$ is simply a circle of radius one and center $(0,0,0)$ located on the hyperplane that is parallel to the simplex and touches the origin. Let μ_1 be a prior over $\mathcal{U}$ whose support, S_1, is an arc of this circle (for simplicity we represent this arc as a line segment in Figure 1). We also assume that μ_1 has a unique mass point, u^*, located somewhere between the two extremities of the arc S_1. Now consider another prior, μ_2, whose support, S_2, is an arc that goes from one of the extremities of the arc S_1 to the point u^* (See Figure 1). Finally, we assume that μ_1 and μ_2 satisfy

$$\frac{\mu_2(V)}{\mu_2(S_2 \setminus \{u^*\})} = \frac{\mu_1(V)}{\mu_1(S_2 \setminus \{u^*\})}, \text{ for all } V \subseteq S_2 \setminus \{u^*\},$$

and

$$\frac{\mu_2(\{u^*\})}{\mu_2(S_2 \setminus \{u^*\})} = \frac{1}{2}\frac{\mu_1(\{u^*\})}{\mu_1(S_2 \setminus \{u^*\})}.$$

Now consider the PAEU preferences $\succsim_1$ and $\succsim_2$ induced by the priors μ_1 and μ_2, and let A and B be any two menus such that $A \cup B \rhd A$ is false. By Lemma 2, we know that this is equivalent to say that

$$\max_{p \in A} E_p(u) \geq \max_{p \in B} E_p(u) \text{ for all } u \in S_1 \setminus S_2.$$

It is not hard to see that this implies

$$\max_{p \in A} E_p(u^*) \geq \max_{p \in B} E_p(u^*).$$

But then,

$$\begin{aligned}
\int_{S_1} \left(\max_{p \in A} E_p(u) - \max_{p \in B} E_p(u) \right) \mu_1(du) &\geq \int_{S_2} \left(\max_{p \in A} E_p(u) - \max_{p \in B} E_p(u) \right) \mu_1(du) \\
&\geq \alpha \int_{S_2} \left(\max_{p \in A} E_p(u) - \max_{p \in B} E_p(u) \right) \mu_2(du),
\end{aligned}$$

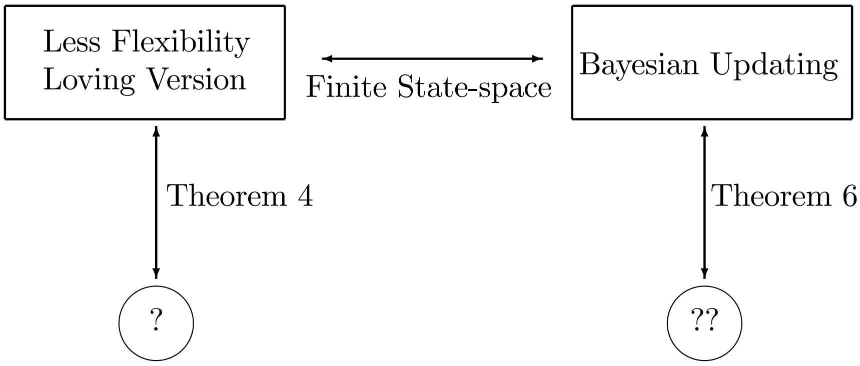

Figure 2

where

$$\alpha := \frac{\mu_1 \left(S_2 \setminus \{u^*\} \right)}{\mu_2 \left(S_2 \setminus \{u^*\} \right)}.$$

It is now clear that if $A \succ_2 B$, which is equivalent to say that

$$\int_{S_2} \left(\max_{p \in A} E_p \left(u \right) - \max_{p \in B} E_p \left(u \right) \right) \mu_2 \left(du \right) > 0,$$

then

$$\int_{S_1} \left(\max_{p \in A} E_p \left(u \right) - \max_{p \in B} E_p \left(u \right) \right) \mu_1 \left(du \right) > 0,$$

which is equivalent to $A \succ_1 B$. That is, $\succsim_2$ is a less flexibility loving version of $\succsim_1$. ■

Example 1 shows that being a less flexibility loving version of $\succsim_1$ is not enough to make $\succsim_2$ representable as a Bayesian update of $\succsim_1$'s representation when $\succsim_1$ does not necessarily satisfy Finiteness. This leaves us with two open questions, represented by the diagram in Figure 2. First of all, we may investigate the consequences of the less flexibility loving version concept when the finiteness assumption is dropped. This is the subject of Section 5, where we will see that although such a concept is not powerful enough to deliver a full Bayesian updating result in the infinite state-space case, it is still related to some quasi-bayesian property linking the representations of $\succsim_1$ and $\succsim_2$.

Of course, alternatively, one could abandon the concept of a less flexibility loving version and simply focus on the characterization of Bayesian updating for generic PAEU preferences. We do that in Section 6. Unfortunately, the conditions that deliver such a result are more technical and harder to interpret than the property of being a less flexibility loving version.

5 Less Flexibility Loving Versions of an Infinite PAEU Preference

We now characterize the less flexibility loving versions of a generic, not necessarily finite, PAEU preference, $\succsim_1$. Although it is no longer true that any less flexibility loving version, $\succsim_2$, of $\succsim_1$ can be represented as a Bayesian update of $\succsim_1$'s representation, we will see that this property is still related to some quasi-bayesian behavior from the part of $\succsim_2$. We first present the formal result and discuss it afterwards.

Theorem 4. *Suppose $\succsim_1$ and $\succsim_2$ have PAEU representations μ_1 and μ_2, respectively. Define $S_1 := supp\,(\mu_1)$. Then, $\succsim_2$ is a less flexibility loving version of $\succsim_1$ if and only if there exists a closed subset, T, of S_1 such that $\mu_2(T) = 1$ and either $int_{S_1}(T) = \emptyset$ or, for any Borel subset V of $\mathcal{U}$,*

$$\frac{\mu_2(V)}{\mu_2(int_{S_1}(T))} \leq \frac{\mu_1(V)}{\mu_1(int_{S_1}(T))}, \text{ with equality if } V \subseteq int_{S_1}(T).^{8} \tag{2}$$

So, the theorem above says that if $\succsim_2$ is a less flexibility loving version of $\succsim_1$, then we can find a closed subset, T, of the state-space used to represent $\succsim_1$ such that $\succsim_2$'s representation will act as a Bayesian update of $\succsim_1$'s representation in the interior of T. We note that if μ_2 were a Bayesian update of μ_1 after the observation of $int_{S_1}(T)$, then, for events $V \in S_1 \setminus int_{S_1}(T)$ we would have $\mu_2(V) = 0$. On the other hand, if μ_2 were a Bayesian update of μ_1 after the observation of T, then we would have (2) satisfied with equality even for the events $V \subseteq T \setminus int_{S_1}(T)$. So, for events in $T \setminus int_{S_1}(T)$, μ_2 assigns probabilities that lie in between what a Bayesian updater that had observed $int_{S_1}(T)$ and what a Bayesian updater that had observed T would assign.

We note that Example 1 can be perfectly mapped into the conditions in Theorem 4. If we define T to be S_2 in that example, then $int_{S_1}(T) = S_2 \setminus \{u^*\}$ and it can be easily checked that condition (2) in Theorem 4 is satisfied.

In fact, Example 1 points out to the main reason why the concept of a less flexibility loving version does not imply a full Bayesian updating result when $\succsim_1$ is

[8] *Notation:* By $int_{S_1}(T)$ we mean the interior of T relative to the metric subspace S_1.

not a finite PAEU preference. This is a consequence of the continuity of

$$\max_{p \in A} E_p(.),$$

when viewed as a function from $\mathcal{U}$ to $\mathbb{R}$, for any menu A. Because of that, whenever we have two menus A and B such that $A \cup B \rhd A$ is false, which, by Lemma 2, is equivalent to

$$\max_{p \in A} E_p(u) \geq \max_{p \in B} E_p(u) \text{ for all } u \in S_1 \setminus S_2,$$

we in fact have

$$\max_{p \in A} E_p(u) \geq \max_{p \in B} E_p(u) \text{ for all } u \in cl(S_1 \setminus S_2).^9$$

We can now see that if μ_2 acts as a Bayesian update of μ_1 inside $int_{S_1}(S_2)$, we are no longer capable of finding two menus A and B that contradict the fact that $\succsim_2$ is a less flexibility loving version of $\succsim_1$ the same way we did in the proof of Theorem 3.

We now discuss a possible way to strengthen Theorem 4 to a full Bayesian updating result. The idea here will be to impose a condition that guarantees that the left-hand side of (2) is null whenever $V \subseteq \mathcal{U} \setminus int_{S_1}(T)$. It turns out that this is related to a sort of independence condition between the relations $\succsim_2$ and $\rhd$. Formally, we consider the following property:

Definition. For any two PAEU preferences, $\succsim_1$ and $\succsim_2$, we say that $\succsim_2$ is **strongly independent** from $\rhd$ if and only if for any two menus A and B, $A \succ_2 B$ implies that there exists a set $D \in \mathcal{X} \cup \{\emptyset\}$ such that

$$A \succ_2 B \cup D, \text{ but } A \cup B \cup D \rhd B \cup D \text{ is false.}$$

Loosely speaking, the condition above says that it can never be the case that the reason for $\succsim_2$ to prefer a menu A to a menu B is the extra flexibility that $\succsim_1$ sees in A, when compared to $\succsim_2$. The axiom presents a strong version of this idea. It asks that whenever $\succsim_2$ strictly prefers a menu A to a menu B, it must be possible to add options to menu B in a way that it eliminates any extra flexibility $\succsim_1$ might see in A, but it still does not reverse $\succsim_2$'s preference. It turns out that this postulate is too strong for our objectives, so we need to weaken it by instead of asking the above to be immediately true, we allow first for the mixing of A and B with some other menu

[9] *Notation:* For any set T, by $cl(T)$ we mean the closure of T.

C. Formally, we work with the following weaker version of the definition above:

Definition. For any two PAEU preferences, $\succsim_1$ and $\succsim_2$, we say that $\succsim_2$ is **independent** from $\rhd$ if and only if for any two menus A and B, $A \succ_2 B$ implies that there exist menus C and D and number $\lambda \in [0, 1)$ such that

$$C \oplus_\lambda A \succ_2 (C \oplus_\lambda B) \cup D, \text{ but } [C \oplus_\lambda (A \cup B)] \cup D \rhd (C \oplus_\lambda B) \cup D \text{ is false.}$$

Given the representations of $\succsim_1$ and $\succsim_2$, we have that for any menu C and $\lambda \in [0, 1)$,

$$A \succsim_i B \iff C \oplus_\lambda A \succsim_i (C \oplus_\lambda B) \text{ for } i = 1, 2.$$

So, this new condition, besides being a genuine weakening of the previous one, still carries the same interpretation. As we have pointed out before, when added to the requirement that $\succsim_2$ be a less flexibility loving version of $\succsim_1$, the condition above delivers a Bayesian updating result in the spirit of Theorem 3.

Theorem 5. *Suppose $\succsim_1$ and $\succsim_2$ have PAEU representations μ_1 and μ_2, respectively, and let $S_1 := supp(\mu_1)$. Then $\succsim_2$ is a less flexibility loving version of $\succsim_1$ that is independent from $\rhd$ if and only if there exists a set T that is regularly open in the subspace S_1 such that μ_2 is the Bayesian update of μ_1 after the observation of T.*[10]

In the appendix (Lemma 7) we prove that $\succsim_2$ is independent from $\rhd$ if and only if $\mu_2(S_2 \cap cl(S_1 \setminus S_2)) = 0$.[11] If $S_2 \subseteq S_1$, this is equivalent to say that $\mu_2(int_{S_1}(S_2)) = 1$. The theorem above is, therefore, an easy corollary of this fact and Theorem 4.

Theorem 5 uses an auxiliary condition to relate the concept of a less flexibility loving version to a particular Bayesian updating result relating two PAEU preferences. It leaves open the question about which condition, if any, would generically characterize Bayesian updating. In the next section we provide such a condition. Unfortunately, that condition is more technical and far less intuitive than the property of being a less flexibility loving version.

6 Infinite PAEU Preferences and Bayesian Updating

So far, our main goal has been to characterize the less flexibility loving versions of a given PAEU preference, $\succsim_1$. We now depart from that goal and, instead, aim at

[10]Given a topological space Z, a subset O of Z is regularly open if and only if $O = int(cl(O))$.

[11]In the language of Example 1 this means simply $\mu_2(\{u^*\}) = 0$.

finding a condition that generically characterizes when a PAEU preference $\succsim_2$ can be represented as a Bayesian update of the PAEU representation of $\succsim_1$. We begin with a definition.

Definition. Consider a PAEU preference, $\succsim$. We say that a pair of menus C and D **preserve $\succsim$ in the limit** if, for any two menus A and B with $A \succ B$, there exists $\bar{\lambda} \in (0,1)$ such that

$$C \oplus_\lambda A \cup [D \oplus_\lambda (A \cup B)] \succ (C \oplus_\lambda B) \cup [D \oplus_\lambda (A \cup B)],$$

for all $\lambda \in (\bar{\lambda}, 1)$.

Note that, for any $\lambda \in (0,1)$,

$$A \succ B \iff C \oplus_\lambda A \succ C \oplus_\lambda B,$$

but, of course, the addition of the term $[D \oplus_\lambda (A \cup B)]$ could, in general, reverse the preference above. The definition above says that C and D preserve $\succsim$ in the limit if the term $[D \oplus_\lambda (A \cup B)]$ is always inconsequential when λ is large enough.

We now use the definition above to write the condition that generically characterizes Bayesian updating in the present setting.

Definition. For a pair of PAEU preferences, $\succsim_1$ and $\succsim_2$, we say that $\succsim_2$ **can be extracted from** $\succsim_1$ if for any pair of menus A and B with $A \succ_2 B$ we can find a number $\bar{\lambda} \in (0,1)$ and menus C and D that preserve $\succsim_2$ in the limit such that

$$(C \oplus_\lambda A) \cup [D \oplus_\lambda (A \cup B)] \succ_1 (C \oplus_\lambda B) \cup [D \oplus_\lambda (A \cup B)], \tag{3}$$

for all $\lambda \in (\bar{\lambda}, 1)$.

As we have already mentioned, this condition gives a general characterization of Bayesian updating for PAEU preferences.

Theorem 6. *Suppose $\succsim_1$ and $\succsim_2$ have PAEU representations μ_1 and μ_2, respectively. Then $\succsim_2$ can be extracted from $\succsim_1$ if and only if there exists a Borel subset T of supp (μ_1) such that either $\mu_1(T) = 0$ and $\mu_2(T) = 1$ or μ_2 is the Bayesian update of μ_1 after the observation of T.*

The interpretation of Theorem 6 is simple. Whenever $\succsim_2$ can be extracted from $\succsim_1$, we can represent $\succsim_2$ as being an instance of $\succsim_1$ after having learned that the event T occurred. If $\succsim_1$ assigns probability zero to the event T, then the theorem does not impose any restriction on $\succsim_2$'s beliefs over T. However, whenever $\mu_1(T) > 0$ and, therefore, Bayesian updating is well-defined, $\succsim_2$ makes use of the Baye's rule to update μ_1.

7 Conclusion

We studied updating in the absence of an exogenously specified state-space. We worked in the framework of Dekel et al. (2001), where the state-space is endogenously obtained as part of the representation of a preference relation over menus. In that environment, we proved results that connected Bayesian updating to a comparative theory of preference for flexibility.

The analysis in this paper is essentially static, so a natural way to extend the results here is to embed them in a truly dynamic model. Also, we only work with the model that can be considered to be the correspondent, in the preference over menus literature, to the standard subjective expected utility model in a Savagean world. The same way the updating theory is extended to other models when the state-space is exogenously given, it would also be interesting to see which implications the conditions here would have for some of the alternative models that have been studied in the preferences over menus literature.[12]

A Proofs

A.1 Preliminaries

In this section we collect a series of results that will be useful for the proof of the theorems in the main text. As we did in the main text, define

$$\mathcal{U} := \left(u \in \mathbb{R}^{|X|} : \sum_{i=1}^{|X|} u_i = 0 \text{ and } \sum_{i=1}^{|X|} u_i^2 = 1 \right).$$

For any menu A, the support function $\sigma_A : \mathcal{U} \to \mathbb{R}$ is defined by

$$\sigma_A(u) := \max_{p \in A} E_p(u), \text{ for each } u \in \mathcal{U}.$$

[12] A natural candidate would be the menu preferences version of the maxmim model proved by Epstein, Marinacci, and Seo (2007).

It is easy to show that the map that associates to each menu its support function is injective. If $\succsim$ has a PAEU representation with prior μ, then, for any two menus A and B,

$$A \succsim B \iff \int_{\mathcal{U}} \sigma_A(u)\,\mu(du) \geq \int_{\mathcal{U}} \sigma_B(u)\,\mu(du).$$

The following lemma collect some standard results about support functions:

Lemma 4. *For any two menus A and B, the following conditions are satisfied:*[13]

1. $\sigma_{\lambda A+(1-\lambda)B} = \lambda\sigma_A + (1-\lambda)\,\sigma_B$ *for any* $\lambda \in [0,1]$;
2. $\sigma_{A\cup B} = \sigma_A \vee \sigma_B$;[14]
3. $d_{Hausdorff}(A,B) = d_{Supnorm}(\sigma_A, \sigma_B)$.

Now, let $C(\mathcal{U})$ be the space of continuous functions over $\mathcal{U}$ endowed with the supnorm distance. It is well known that

$$C_\sigma := \{\sigma_A : A \text{ is a menu}\} \subseteq C(\mathcal{U}).$$

We are also going to work with the following subsets of $C(\mathcal{U})$:

$$H := \bigcup_{r\geq 0} rC_\sigma$$

and

$$H^* := H - H.$$

DLR prove that H^* satisfies the following properties:

Lemma 5. *H^* satisfies the following conditions:*

1. *H^* is a linear subspace of $C(\mathcal{U})$;*
2. *For any $f \in H^*$, there exists $r > 0$ and $\sigma_1, \sigma_2 \in C_\sigma$ such that $f = r(\sigma_1 - \sigma_2)$;*
3. *The set H^* is dense in $C(\mathcal{U})$.*

[13] All properties can be easily checked from the definition of a support function. The first two can be found in Rockafellar (1997), chapter 13.

[14] That is, for any $u \in \mathcal{U}$, $\sigma_{A\cup B}(u) = \max\{\sigma_A(u), \sigma_B(u)\}$.

A.2 A Useful Lemma

Let C and D be any two menus and let $\succsim$ be a preference that has a PAEU representation with probability measure μ. Define a set $T \subseteq \mathcal{U}$ by

$$T := \{u \in \mathcal{U} : \sigma_C(u) > \sigma_D(u)\}.$$

We can prove that:

Lemma 6. *For any two menus A and B, define*

$$f^\lambda := \sigma_{(C \oplus_\lambda A) \cup [D \oplus_\lambda (A \cup B)]} - \sigma_{(C \oplus_\lambda B) \cup [D \oplus_\lambda (A \cup B)]}$$

and

$$\hat{f}^\lambda := \sigma_{(C \oplus_\lambda A)} - \sigma_{(C \oplus_\lambda B) \cup [D \oplus_\lambda (A \cup B)]}.$$

It must be the case that

$$\lim_{\lambda \uparrow 1} \frac{1}{1-\lambda} \int_{\mathcal{U}} f^\lambda(u)\, \mu(du) = \int_T (\sigma_A - \sigma_B)(u)\, \mu(du) = \lim_{\lambda \uparrow 1} \frac{1}{1-\lambda} \int_T \hat{f}^\lambda(u)\, \mu(du).$$

To prove the lemma, first note that for any λ,

$$\int_{\mathcal{U}} f^\lambda(u)\, \mu(du) = \int_T f^\lambda(u)\, \mu(du).$$

Now let $\varepsilon > 0$ and pick any closed subset T_ε of T such that $\mu(T \setminus T_\varepsilon) < \varepsilon/2$.[15] We need the following claim:

Claim 1. *There exists $\bar{\lambda} \in (0,1)$ such that for any $\lambda \in (\bar{\lambda}, 1)$ and any $u \in T_\varepsilon$,*

$$\sigma_{D \oplus_\lambda (A \cup B)}(u) < \min\{\sigma_{C \oplus_\lambda A}(u), \sigma_{C \oplus_\lambda B}(u)\}.$$

Proof of Claim. Since T_ε is a compact set and σ_C and σ_D are continuous functions that satisfy $\sigma_C(u) > \sigma_D(u)$ for any $u \in T_\varepsilon$, we know that there exists $\delta > 0$ such that

$$\sigma_C(u) > \sigma_D(u) + \delta,$$

for all $u \in T_\varepsilon$. Now note that for any two menus A and B and any $u \in \mathcal{U}$,

[15]Since μ is a Borel probability measure over the metric space $\mathcal{U}$ it is regular, so we can always find such a set T_ε.

$|\sigma_A(u) - \sigma_B(u)| \leq 2$. So if we make $\bar{\lambda} = 2/(2+\delta)$, for any $\lambda > \bar{\lambda}$,

$$\begin{aligned}
\sigma_{D\oplus_\lambda(A\cup B)}(u) &= \lambda\sigma_D(u) + (1-\lambda)\sigma_{A\cup B}(u) \\
&< \lambda(\sigma_C(u) - \delta) + (1-\lambda)\sigma_A(u) + (1-\lambda)(\sigma_{A\cup B}(u) - \sigma_A(u)) \\
&\leq \sigma_{C\oplus_\lambda A}(u) - \lambda\delta + 2(1-\lambda) \\
&< \sigma_{C\oplus_\lambda A}(u).
\end{aligned}$$

Of course, the very same reasoning shows that $\sigma_{D\oplus_\lambda(A\cup B)}(u) < \sigma_{C\oplus_\lambda B}(u)$, so the claim is proved. ‖

So, for any $\lambda \in (\bar{\lambda}, 1)$,

$$\begin{aligned}
\int_T \hat{f}^\lambda(u)\,\mu(du) &\leq \int_T f^\lambda(u)\,\mu(du) \\
&= \int_{T_\varepsilon} (\sigma_{C\oplus_\lambda A} - \sigma_{C\oplus_\lambda B})(u)\,\mu(du) + \int_{T\setminus T_\varepsilon} f^\lambda(u)\,\mu(du) \\
&\leq \int_{T_\varepsilon} (\sigma_{C\oplus_\lambda A} - \sigma_{C\oplus_\lambda B})(u)\,\mu(du) \\
&\quad + \int_{T\setminus T_\varepsilon} \left(\sigma_{C\oplus_\lambda(A\cup B)} - \sigma_{(C\oplus_\lambda B)}\right)(u)\,\mu(du) \\
&= (1-\lambda)\left[\int_{T_\varepsilon} (\sigma_A - \sigma_B)(u)\mu(du) + \int_{T\setminus T_\varepsilon} (\sigma_{A\cup B} - \sigma_B)(u)\mu(du)\right] \\
&= (1-\lambda)\left[\int_T (\sigma_A - \sigma_B)(u)\mu(du) + \int_{T\setminus T_\varepsilon} (\sigma_{A\cup B} - \sigma_A)(u)\mu(du)\right] \\
&\leq (1-\lambda)\left[\int_T (\sigma_A - \sigma_B)(u)\,\mu(du) + 2\mu(T\setminus T_\varepsilon)\right] \\
&< (1-\lambda)\left[\int_T (\sigma_A - \sigma_B)(u)\,\mu(du) + \varepsilon\right].
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_T f^\lambda(u)\,\mu(du) &\geq \int_T \hat{f}^\lambda(u)\,\mu(du) \\
&= \int_{T_\varepsilon} \left(\sigma_{(C\oplus_\lambda A)} - \sigma_{(C\oplus_\lambda B)}\right)(u)\,\mu(du) + \int_{T\setminus T_\varepsilon} \hat{f}^\lambda(u)\,\mu(du) \\
&\geq \int_{T_\varepsilon} \left(\sigma_{(C\oplus_\lambda A)} - \sigma_{(C\oplus_\lambda B)}\right)(u)\,\mu(du) \\
&\quad + \int_{T\setminus T_\varepsilon} \left(\sigma_{(C\oplus_\lambda A)} - \sigma_{(C\oplus_\lambda (A\cup B))}\right)(u)\,\mu(du) \\
&= (1-\lambda)\left[\int_{T_\varepsilon} (\sigma_A - \sigma_B)(u)\mu(du) + \int_{T\setminus T_\varepsilon} (\sigma_A - \sigma_{A\cup B})(u)\mu(du)\right] \\
&= (1-\lambda)\left[\int_{T} (\sigma_A - \sigma_B)(u)\mu(du) + \int_{T\setminus T_\varepsilon} (\sigma_B - \sigma_{A\cup B})(u)\mu(du)\right] \\
&\geq (1-\lambda)\left[\int_{T} (\sigma_A - \sigma_B)(u)\,\mu(du) - 2\mu(T\setminus T_\varepsilon)\right] \\
&> (1-\lambda)\left[\int_{T} (\sigma_A - \sigma_B)(u)\,\mu(du) - \varepsilon\right].
\end{aligned}$$

But this means that for any $\lambda \in \left(\bar{\lambda}, 1\right)$ we have

$$\begin{aligned}
\int_T (\sigma_A - \sigma_B)(u)\,\mu(du) + \varepsilon &> \frac{1}{1-\lambda}\int_{\mathcal{U}} f^\lambda(u)\,\mu(du) \\
&\geq \frac{1}{1-\lambda}\int_T \hat{f}^\lambda(u)\,\mu(du) \\
&> \int_T (\sigma_A - \sigma_B)(u)\,\mu(du) - \varepsilon.
\end{aligned}$$

This completes the proof of the lemma. ■

A.3 Proof of Lemma 1

Obviously, if $supp(\mu_2) =: S_2 \subseteq S_1 := supp(\mu_1)$, then $\succsim_1$ values flexibility more than $\succsim_2$, so we only need to show the converse. But suppose that $S_2 \setminus S_1 \neq \emptyset$ and let E be any closed sphere in the interior of $\Delta(X)$.[16] Define menus A and B by

$$B := \bigcup_{u\in S_1} \arg\max_{p\in E} E_p(u)$$

[16] That is, let $E \subseteq \Delta(X)$ be such that $E = \left\{q \in \mathbb{R}^{|X|} : d(p,q) \leq \alpha\right\}$ for some $p \in int(\Delta(X))$ and $\alpha > 0$ small enough.

and $A := E$.[17] It is easy to see that $\sigma_A(u) = \sigma_B(u)$ for all $u \in S_1$, which implies that $B \sim_1 A$. On the other hand, $\sigma_B(u) < \sigma_A(u)$ for all $u \in S_2 \setminus S_1$. This implies that $B \prec_2 A$ and, therefore, it is not true that $\succsim_1$ values flexibility more than $\succsim_2$. ■

A.4 Proof of Lemma 2

Suppose that there exists $u^* \in S_1 \setminus S_2$ with $\sigma_B(u^*) > \sigma_A(u^*)$. We first note that for any $u, v \in \mathcal{U}$ and $p \in \Delta(X)$,

$$|E_p(u) - E_p(v)| \leq d(u, v). \tag{4}$$

We can now prove the following claim:

Claim 1. *Let $\tilde{\varepsilon} := (\sigma_B(u^*) - \sigma_A(u^*))/4$. There exists $0 < \delta < 2$ such that for any $u \in \mathcal{U}$ satisfying $d(u, -u^*) \leq \delta$ we can find $q_u \in \Delta(X)$ with*

$$E_{q_u}(u^*) < \sigma_B(u^*) - \tilde{\varepsilon}$$

and

$$E_{q_u}(u) \geq \sigma_B(u).$$

Proof of Claim. Since support functions are continuous, we know that there exists $0 < \delta_1 < 2$ such that $\sigma_B(u^*) - 2\tilde{\varepsilon} > \sigma_A(u)$ for any u such that $d(u, u^*) \leq \delta_1$. Define $\delta := \min\{\delta_1, \tilde{\varepsilon}\}$. For any u such that $d(-u, u^*) \leq \delta$, pick

$$q_u \in \arg\max_{q \in \Delta(X)} E_q(u).$$

By construction we have

$$E_{q_u}(u) \geq -\sigma_A(-u),$$

which implies that

$$\begin{aligned} E_{q_u}(-u) &= -E_{q_u}(u) \\ &\leq \sigma_A(-u) \\ &< \sigma_B(u^*) - 2\tilde{\varepsilon}. \end{aligned}$$

[17] Since E is a sphere, it is easy to see that each expected-utility function has a unique maximizer in E. Moreover, no $q \in E$ maximizes two different expected-utility functions in $\mathcal{U}$.

By (4) we have

$$|E_{q_u}(-u) - E_{q_u}(u^*)| \leq d(-u, u^*) \leq \tilde{\varepsilon}.$$

Combining the two conditions above we get

$$E_{q_u}(u^*) < \sigma_B(u^*) - \tilde{\varepsilon}.$$

Since obviously $E_{q_u}(u) \geq \sigma_B(u)$, this completes the proof of the claim. ‖

Now consider the following set

$$T := \{u \in S_2 : \sigma_B(u) > \sigma_A(u)\}.$$

We also need the following claim:

Claim 2. *There exists $\hat{\varepsilon} > 0$ such that for any $u \in T$ such that $d(-u, u^*) > \delta$, where the δ here is the one found in the claim above, there exists $q_u \in int(\Delta(X))$ with the property that*

$$E_{q_u}(u^*) < \sigma_B(u^*) - \hat{\varepsilon}$$

and

$$E_{q_u}(u) \geq \sigma_B(u).$$

Proof of Claim. Since $B \subseteq int(\Delta(X))$, we know that there exists $\delta_1 > 0$ such that for any $p \in B$ and any point $r \in aff(\Delta(X))$ with $(d(p,r))^2 \leq \delta_1$ necessarily has to be in $int(\Delta(X))$.[18] Moreover, since $u^* \in S_1 \setminus S_2$, there exists $0 < \delta_2 < 2$ such that for any $u \in S_2$,

$$d(u, u^*) > \delta_2,$$

or, equivalently,

$$u \cdot u^* < 1 - \frac{\delta_2^2}{2}.$$

Now observe that $d(-u, u^*) > \delta$ is equivalent to say that

$$-(u \cdot u^*) < 1 - \frac{\delta^2}{2}.$$

[18] By $aff(\Delta(X))$ we mean the affine hull of $\Delta(X)$. More specifically, $aff(\Delta(X)) := \left\{r \in \mathbb{R}^{|X|} : \sum_{i=1}^{|X|} r_i = 1\right\}$.

So, for any $u \in T$ with $d(-u, u^*) > \delta$ we have

$$(u \cdot u^*)^2 < \delta_3 := \left(\max\left\{1 - \frac{\delta_2^2}{2}, 1 - \frac{\delta^2}{2}\right\}\right)^2 < 1.$$

Now pick an arbitrary $u \in T$ satisfying the condition above and choose some $q \in \arg\max_{p \in B} E_p(u)$. Define q_u to be

$$q_u := q - \sqrt{\frac{\delta_1}{1 - (u \cdot u^*)^2}} u^* + (u \cdot u^*) \sqrt{\frac{\delta_1}{1 - (u \cdot u^*)^2}} u.$$

First observe that $q_u \cdot 1 = 1$ and $(d(q, q_u))^2 = \delta_1$, so $q_u \in \Delta(X)$. Also, observe that $q_u \cdot u = q \cdot u$ and

$$\begin{aligned}
q_u \cdot u^* &= \left(q - \sqrt{\frac{\delta_1}{1 - (u \cdot u^*)^2}} u^* + (u \cdot u^*) \sqrt{\frac{\delta_1}{1 - (u \cdot u^*)^2}} u\right) \cdot u^* \\
&= q \cdot u^* - \left(\sqrt{\left(1 - (u^* \cdot u)^2\right) \delta_1}\right) \\
&< q \cdot u^* - \left(\sqrt{(1 - \delta_3) \delta_1}\right) \\
&\leq \sigma_B(u^*) - \left(\sqrt{(1 - \delta_3) \delta_1}\right).
\end{aligned}$$

So if we make $\hat{\varepsilon} := \sqrt{(1 - \delta_3) \delta_1}$ we have the claim. $\|$

Combining the two claims above we have the following result:

Claim 3. *There exists $\varepsilon > 0$ such that for any $u \in T$, there exists $q_u \in \Delta(X)$ with the property that*

$$E_{q_u}(u^*) < \sigma_B(u^*) - \varepsilon$$

and

$$E_{q_u}(u) \geq \sigma_B(u).$$

Now fix ε satisfying the condition in the claim above. For each $u \in T$ pick some $q_u \in \Delta(X)$ such that $E_{q_u}(u^*) < \sigma_B(u^*) - \varepsilon$ and $E_{q_u}(u) \geq \sigma_B(u)$. From the argument used in the proof of the previous claim we see that we can, in fact, choose such lotteries q_u's in a way that they be uniformly distanced away from the boundary of the simplex. Define C to be the closure of the set of all q_u's found this way. By what we have just discussed, $C \subseteq int(\Delta(X))$. It is now clear that for all $u \in S_2$ we must have

$$\sigma_{A \cup C}(u) = \sigma_{A \cup B \cup C}(u),$$

and this implies that $A \cup C \sim_2 A \cup B \cup C$. On the other hand, we have, by construction, that

$$\sigma_{A \cup C}(u^*) < \sigma_{A \cup B \cup C}(u^*),$$

which implies that $A \cup C \prec_1 A \cup B \cup C$. We conclude that $A \cup B \rhd A$.

Conversely, suppose that A is such that $\sigma_A(u) \geq \sigma_B(u)$ for all $u \in S_1 \setminus S_2$. From the representation of $\succsim_2$ we know that for any menu C such that $A \cup C \sim_2 A \cup B \cup C$ we must necessarily have $\sigma_{A \cup C}(u) \geq \sigma_{B \cup C}(u)$ for all $u \in S_2$. The fact that $\sigma_A(u) \geq \sigma_B(u)$ for all $u \in S_1 \setminus S_2$ implies that $\sigma_{A \cup C}(u) \geq \sigma_{B \cup C}(u)$ for all $u \in S_1 \setminus S_2$. But then we have that $\sigma_{A \cup C}(u) \geq \sigma_{B \cup C}(u)$ for all $u \in S_1$ which implies that $A \cup C \sim_1 A \cup B \cup C$. We conclude that $A \cup B \ntriangleright A$ and this completes the proof of the lemma. ■

A.5 Proof of Theorem 3

[Necessity] Suppose that there exists S, T, μ_1 and μ_2 as in the statement of the theorem. Now suppose that A and B are such that $A \succ_2 B$ and $B \succsim_1 A$. Let $S_1 := supp(\mu_1)$. Given the representations of $\succsim_1$ and $\succsim_2$, it is clear that there must exist $u^* \in S_1 \setminus S_2$ such that

$$\sigma_B(u^*) > \sigma_A(u^*).$$

By Lemma 2, we know that this implies that $B \rhd A$.

[Sufficiency] Let μ_1 be the prior used in representation of $\succsim_1$ and μ_2 be the one used in the representation of $\succsim_2$. By Lemmas 1 and 3, we know that $supp(\mu_2) =: S_2 \subseteq S_1 := supp(\mu_1)$. Suppose, then, that μ_2 is not the Bayesian updating of μ_1 after the observation of S_2. By a separation argument we can show that this implies that there exists $f \in C(\mathcal{U})$ such that

$$\sum_{u \in S_2} \mu_2(u) f(u) > 0 > \sum_{u \in S_2} \mu_1(u) f(u).$$

Moreover, we can find such a function f that in addition to the above also satisfies:

$$\sum_{u \in S_1} \mu_1(u) f(u) < 0$$

and $f(u) = \varepsilon$ for all $u \in S_1 \setminus S_2$ where $\varepsilon > 0$.[19] But then, Lemma 5 implies that

[19] For the details about how to find such a function, see the argument in the proof of Theorem 4. Alternatively, one can derive this direction of the proof as a straightforward corollary of that

there exist menus A and B such that

$$\sum_{u \in S_2} \mu_2(u)\,\sigma_A(u) > \sum_{u \in S_2} \mu_2(u)\,\sigma_B(u),$$

$$\sum_{u \in S_1} \mu_1(u)\,\sigma_A(u) < \sum_{u \in S_1} \mu_1(u)\,\sigma_B(u)$$

and $\sigma_A(u) > \sigma_B(u)$ for all $u \in S_1 \setminus S_2$. The conditions above imply that $A \succ_2 B$, $B \succ_1 A$. But now notice that for any menu C such that $A \cup C \sim_2 A \cup B \cup C$, we must necessarily have $\sigma_{A\cup C}(u) \geq \sigma_{B\cup C}(u)$ for all $u \in S_2$. Since $\sigma_A(u) > \sigma_B(u)$ for all $u \in S_1 \setminus S_2$, it must be the case that $\sigma_{A\cup C}(u) \geq \sigma_{B\cup C}(u)$ for all $u \in S_1$, which implies that $A \cup C \sim_1 A \cup B \cup C$. This contradicts the fact that $\succsim_2$ is a less flexibility loving version of $\succsim_1$. We conclude that μ_2 must be the Bayesian update of μ_1 after the observation of S_2. ■

A.6 Proof of Theorem 4

[Necessity] Suppose $\succsim_1, \succsim_2$ have representations μ_1, μ_2 that satisfy the conditions in the statement of the theorem and let $S_i := supp(\mu_i)$ for $i = 1, 2$. It is easy to see that $S_2 = T$ and $int_{S_1}(T) = S_2 \setminus cl(S_1 \setminus S_2)$. Now pick any two menus A and B such that $A \succ_2 B$ and $B \succsim_1 A$. Suppose first that $S_2 \subseteq cl(S_1 \setminus S_2)$. Since $A \succ_2 B$ and $B \succsim_1 A$, there must exist $u \in S_1$ such that $\sigma_B(u) > \sigma_A(u)$. If $u \notin S_1 \setminus S_2$, then $u \in S_2 \subseteq cl(S_1 \setminus S_2)$. But then, for $u^* \in S_1 \setminus S_2$ sufficiently close to u it must still be true that $\sigma_B(u^*) > \sigma_A(u^*)$. By Lemma 2, this implies that $A \cup B \rhd A$. Now assume that $S_2 \setminus cl(S_1 \setminus S_2) \neq \emptyset$ and suppose that $\sigma_A(u) \geq \sigma_B(u)$ for all $u \in S_1 \setminus S_2$. Since support functions are continuous, this implies that

$$\sigma_B(u) \leq \sigma_A(u)$$

for all $u \in cl(S_1 \setminus S_2)$ as well. We can now show that our representation implies that

$$\int_{S_2 \cap cl(S_1 \setminus S_2)} (\sigma_A - \sigma_B)(u)\mu_1(du) \geq \frac{\mu_1(S_2 \setminus cl(S_1 \setminus S_2))}{\mu_2(S_2 \setminus cl(S_1 \setminus S_2))} \int_{S_2 \cap cl(S_1 \setminus S_2)} (\sigma_A - \sigma_B)(u)\mu_2(du).$$

To see that, let f^m be an increasing sequence of non-negative simple functions over $S_2 \cap cl(S_1 \setminus S_2)$ converging to the restriction of $\sigma_A - \sigma_B$ to $S_2 \cap cl(S_1 \setminus S_2)$. But for

Theorem.

any simple function f^m,

$$\begin{aligned}
\int\limits_{S_2 \cap cl(S_1 \setminus S_2)} f^m(u)\,\mu_1(du) &= \sum_{c \in f^m(S_2 \cap cl(S_1 \setminus S_2))} c \cdot \mu_1\left((f^m)^{-1}(c)\right) \\
&\geq \frac{\mu_1(S_2 \setminus cl(S_1 \setminus S_2))}{\mu_2(S_2 \setminus cl(S_1 \setminus S_2))} \sum_{c \in f^m(S_2 \cap cl(S_1 \setminus S_2))} \mu_2\left((f^m)^{-1}(c)\right) \\
&= \int\limits_{S_2 \cap cl(S_1 \setminus S_2)} f^m(u)\,\mu_2(du),
\end{aligned}$$

where

$$(f^m)^{-1}(c) := \{u \in S_2 \cap cl(S_1 \setminus S_2) : f^m(u) = c\}.$$

An application of the monotone convergence theorem shows that

$$\int\limits_{S_2 \cap cl(S_1 \setminus S_2)} (\sigma_A - \sigma_B)(u)\mu_1(du) \geq \frac{\mu_1(S_2 \setminus cl(S_1 \setminus S_2))}{\mu_2(S_2 \setminus cl(S_1 \setminus S_2))} \int\limits_{S_2 \cap cl(S_1 \setminus S_2)} (\sigma_A - \sigma_B)(u)\mu_2(du).$$

But then

$$\begin{aligned}
\int\limits_{S_1} (\sigma_A - \sigma_B)(u)\,\mu_1(du) &\geq \int\limits_{S_2 \setminus cl(S_1 \setminus S_2)} (\sigma_A - \sigma_B)(u)\,\mu_1(du) \\
&\quad + \int\limits_{S_2 \cap cl(S_1 \setminus S_2)} (\sigma_A - \sigma_B)(u)\,\mu_1(du) \\
&\geq \frac{\mu_1(S_2 \setminus cl(S_1 \setminus S_2))}{\mu_2(S_2 \setminus cl(S_1 \setminus S_2))} \int_{S_2} (\sigma_A - \sigma_B)(u)\,\mu_2(du) \\
&> 0,
\end{aligned}$$

where the first line uses the fact that $\sigma_A - \sigma_B$ is nonnegative in $S_1 \setminus S_2$ and the second uses what we have just proved together with the representation. But this contradicts the assumption that $B \succsim_1 A$. We conclude that there must exist $u^* \in S_1 \setminus S_2$ with $\sigma_B(u^*) > \sigma_A(u^*)$ and, by Lemma 2 again, we know that this implies that $A \cup B \rhd A$.

[Sufficiency] Now suppose that $\succsim_2$ is a less flexibility loving version of $\succsim_1$ and let μ_1 and μ_2 be their PAEU representations. By Lemmas 1 and 3 we know that $supp(\mu_2) =: S_2 \subseteq S_1 := supp(\mu_1)$. Suppose that $S_2 \setminus cl(S_1 \setminus S_2) \neq \emptyset$. We can prove the following claim.

Claim 1. *For any Borel set* $V \subseteq S_2^o := S_2 \setminus cl\left(S_1 \setminus S_2\right),$

$$\mu_2\left(V\right) = \frac{\mu_1\left(V\right)}{\mu_1\left(S_2^o\right)}.$$

Proof of Claim. Define the auxiliary measure $\tilde{\mu}_2$ by

$$\tilde{\mu}_2\left(V\right) = \mu_1\left(S_2^o\right)\mu_2\left(V\right), \text{ for any Borel set } V \subseteq \mathcal{U}.$$

Clearly, the claim is equivalent to say that $\tilde{\mu}_2\left(V\right) = \mu_1\left(V\right)$ whenever $V \subseteq S_2^o$. Suppose this is not true. By a separation argument we can find a function $f \in C\left(\mathcal{U}\right)$ such that

$$\int_{S_2^o} f\left(u\right)\mu_1\left(du\right) > \int_{S_2^o} f\left(u\right)\tilde{\mu}_2\left(du\right).$$

We can assume without loss of generality that

$$\int_{S_2^o} f\left(u\right)\mu_1\left(du\right) > 0 > \int_{S_2^o} f\left(u\right)\tilde{\mu}_2\left(du\right).$$

Let $\alpha > 0$ be such that $\left|f\left(u\right)\right| < \alpha$ for any $u \in \mathcal{U}$. For each $i \in \mathbb{N}$ consider the sets

$$O_i := \left\{u \in \mathcal{U} : d\left(u, cl\left(S_1 \setminus S_2\right)\right) < \frac{1}{i}\right\}.$$

Observe that $O_i \searrow cl\left(S_1 \setminus S_2\right)$, so there exists i^* such that

$$\mu_1\left(S_2^o \cap O_{i^*}\right), \tilde{\mu}_2\left(S_2^o \cap O_{i^*}\right) < \frac{\min\left\{\int_{S_2^o} f\left(u\right)\mu_1\left(du\right), -\int_{S_2^o} f\left(u\right)\tilde{\mu}_2\left(du\right)\right\}}{\alpha}.$$

Now notice that $\mathcal{U} \setminus O_{i^*}$ and $cl\left(S_1 \setminus S_2\right)$ are disjoint closed sets, so we can use Urysohn's Lemma to find a continuous function $g : \mathcal{U} \to [0, 1]$ such that $g\left(u\right) = 1$ if

$u \in \mathcal{U} \setminus O_{i^*}$ and $g(u) = 0$ if $u \in cl(S_1 \setminus S_2)$. Moreover,

$$\begin{aligned}
\int_{\mathcal{U}} g(u) f(u) \mu_1(du) &= \int_{S_2 \setminus O_{i^*}} g(u) f(u) \mu_1(du) + \int_{S_2^o \cap O_{i^*}} g(u) f(u) \mu_1(du) \\
&= \int_{S_2 \setminus O_{i^*}} f(u) \mu_1(du) + \int_{S_2^o \cap O_{i^*}} g(u) f(u) \mu_1(du) \\
&= \int_{S_2^o} f(u) \mu_1(du) + \int_{S_2^o \cap O_{i^*}} (g(u) - 1) f(u) \mu_1(du) \\
&\geq \int_{S_2^o} f(u) \mu_1(du) - \int_{S_2^o \cap O_{i^*}} \alpha \mu_1(du) \\
&= \int_{S_2^o} f(u) \mu_1(du) - \alpha \mu_1(S_2^o \cap O_{i^*}) \\
&> \int_{S_2^o} f(u) \mu_1(du) - \alpha \left(\frac{\int_{S_2^o} f(u) \mu_1(du)}{\alpha} \right) \\
&= 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_{\mathcal{U}} g(u) f(u) \tilde{\mu}_2(du) &= \int_{S_2^o \setminus O_{i^*}} g(u) f(u) \tilde{\mu}_2(du) + \int_{S_2^o \cap O_{i^*}} g(u) f(u) \tilde{\mu}_2(du) \\
&= \int_{S_2 \setminus O_{i^*}} f(u) \tilde{\mu}_2(du) + \int_{S_2^o \cap O_{i^*}} g(u) f(u) \tilde{\mu}_2(du) \\
&= \int_{S_2^o} f(u) \tilde{\mu}_2(du) + \int_{S_2^o \cap O_{i^*}} (g(u) - 1) f(u) \tilde{\mu}_2(du) \\
&\leq \int_{S_2^o} f(u) \tilde{\mu}_2(du) + \int_{S_2^o \cap O_{i^*}} \alpha \tilde{\mu}_2(du) \\
&= \int_{S_2^o} f(u) \tilde{\mu}_2(du) + \alpha \tilde{\mu}_2(S_2^o \cap O_{i^*}) \\
&< \int_{S_2^o} f(u) \tilde{\mu}_2(du) + \alpha \left(\frac{-\int_{S_2^o} f(u) \tilde{\mu}_2(du)}{\alpha} \right) \\
&= 0.
\end{aligned}$$

So, there exists $fg =: h \in C(\mathcal{U})$ such that $h(u) = 0$ for all $u \in cl(S_1 \setminus S_2)$ and

$$\int_{\mathcal{U}} h(u) \mu_1(du) > 0 > \int_{\mathcal{U}} h(u) \tilde{\mu}_2(du).$$

Let $\tilde{h} := h - \varepsilon$ for some $\varepsilon > 0$ such that it is still true that

$$\int_{\mathcal{U}} (h(u) - \varepsilon) \mu_1(du) > 0 > \int_{\mathcal{U}} (h(u) - \varepsilon) \tilde{\mu}_2(du).$$

Since H^* is dense in $C(\mathcal{U})$, we can use $\tilde{h}$ to find $h^* \in H^*$ such that $h^*(u) < 0$ for all $u \in cl(S_1 \setminus S_2)$ and

$$\int_{\mathcal{U}} h^*(u)\,\mu_1(du) > 0 > \int_{\mathcal{U}} h^*(u)\,\tilde{\mu}_2(du).$$

This implies that there exist menus $A, B \in 2^{\Delta(X)} \setminus \{\emptyset\}$ such that

$$\int_{\mathcal{U}} (\sigma_B - \sigma_A)(u)\,\mu_1(du) > 0 > \int_{\mathcal{U}} (\sigma_B - \sigma_A)(u)\,\tilde{\mu}_2(du),$$

and $\sigma_A(u) > \sigma_B(u)$ for all $u \in cl(S_1 \setminus S_2)$. Of course, the condition above implies that $B \succ_1 A$ and $A \succ_2 B$. By Lemma 2, this implies that $B \not\triangleright A$, which contradicts the fact that $\succsim_2$ is a less flexibility loving version of $\succsim_1$. We conclude that $\mu_1(V) = \tilde{\mu}_2(V)$ for any $V \subseteq S_2^o$. $\|$

To complete the proof of the theorem we need one last claim.

Claim 2. *For any Borel subset V of $\mathcal{U}$,*

$$\frac{\mu_2(V)}{\mu_2(S_2^o)} \leq \frac{\mu_1(V)}{\mu_1(S_2^o)}.$$

Proof of Claim. Suppose that there exists Borel set $V \subseteq \mathcal{U}$ with

$$\frac{\mu_2(V)}{\mu_2(S_2^o)} > \frac{\mu_1(V)}{\mu_1(S_2^o)}.$$

Define $\tilde{\mu}_2$ as in the previous claim. The condition above is equivalent to $\tilde{\mu}_2(V) > \mu_1(V)$. Since μ_1 and $\tilde{\mu}_2$ are positive Borel measures over the metric space $\mathcal{U}$ and, therefore, are regular, we can assume without loss of generality that V is closed. Define $\delta := (\tilde{\mu}_2(V) - \mu_1(V))/2$ and let O_V be some open subset of $\mathcal{U}$ such that $V \subseteq O_V$ and $\mu_1(O_V \setminus V) < \delta$. By the Urysohn's Lemma, there exists a continuous function $f : S \to [\delta, 1]$ such that $f(u) = 1$ for all $u \in V$ and $f(u) = \delta$ for all

$u \in \mathcal{U} \setminus O_V$. This implies that

$$\begin{aligned}
\int_{\mathcal{U}} f(u)\, \tilde{\mu}_2(du) &\geq \int_V f(u)\, \tilde{\mu}_2(du) \\
&= \tilde{\mu}_2(V) \\
&= \mu_1(V) + 2\delta \\
&> \int_V f(u)\, \mu_1(du) + \int_{O_V \setminus V} 1 \mu_1(du) + \int_{\mathcal{U} \setminus O_V} \delta \mu_1(du) \\
&\geq \int_V f(u)\, \mu_1(du) + \int_{O_V \setminus V} f(u)\, \mu_1(du) + \int_{\mathcal{U} \setminus O_V} f(u)\, \mu_1(du) \\
&= \int_{\mathcal{U}} f(u)\, \mu_1(du).
\end{aligned}$$

Let $\varepsilon := \left(\int_{\mathcal{U}} f(u)\, \tilde{\mu}_2(du) - \int_{\mathcal{U}} f(u)\, \mu_1(du)\right)$. Now let F be any closed set in $S_2 \setminus cl(S_1 \setminus S_2)$ with $\mu_1(F) > 0$ and let O_F be an open subset of $\mathcal{U}$ such that

$$\frac{\left(\int_{\mathcal{U}} f(u)\, \mu_1(du) + \varepsilon/2\right)}{\mu_{1,2}(F)} \mu_{1,2}(O_F \setminus F) < \varepsilon/2,$$

and

$$F \subseteq O_F \subseteq \mathcal{U} \setminus cl(S_1 \setminus S_2).[20]$$

Again, we can make use of Urysohn's Lemma to find a continuous function $g : \mathcal{U} \to \left[-\left(\int_{\mathcal{U}} f(u)\, \mu_1(du) + \varepsilon/2\right)/\mu_{1,2}(F), 0\right]$ such that, for any $u \in F$, $g(u) = -\left(\int_{\mathcal{U}} f(u)\, \mu_1(du) + \varepsilon/2\right)/\mu_{1,2}(F)$ and, for any $u \in \mathcal{U} \setminus O_F$, $g(u) = 0$. Now observe that

$$\begin{aligned}
\int_{\mathcal{U}} g(u)\, \tilde{\mu}_2(du) &\geq -\int_{O_F} \left[\left(\int_{\mathcal{U}} f(u)\, \mu_1(du) + \varepsilon/2\right)/\mu_{1,2}(F)\right] \tilde{\mu}_2(du) \\
&= -\left[\left(\int_{\mathcal{U}} f(u)\, \mu_1(du) + \varepsilon/2\right)/\mu_{1,2}(F)\right] \mu_{1,2}(O_F) \\
&= -\left[\left(\int_{\mathcal{U}} f(u)\, \mu_1(du) + \varepsilon/2\right)/\mu_{1,2}(F)\right] (\mu_{1,2}(F) + \mu_{1,2}(O_F \setminus F)) \\
&> -\left(\int_{\mathcal{U}} f(u)\, \mu_1(du) + \varepsilon/2\right) - \varepsilon/2 \\
&= -\int_{\mathcal{U}} f(u)\, \mu_1(du) - \varepsilon \\
&= -\int_{\mathcal{U}} f(u)\, \tilde{\mu}_2(du).
\end{aligned}$$

[20]Notation: We write $\mu_{1,2}(F)$ and $\mu_{1,2}(O_F \setminus F)$ to indicate that μ_1 and $\tilde{\mu}_2$ agree for this sets. This is a consequence of the previous claim.

and

$$\begin{aligned}\int_{\mathcal{U}} g(u)\,\mu_1(du) &\leq -\int_F \left[\left(\int_{\mathcal{U}} f(u)\,\mu_1(du)+\varepsilon/2\right)/\mu_{1,2}(F)\right]\mu_1(du)\\ &= -\int_{\mathcal{U}} f(u)\,\mu_1(du)-\varepsilon/2.\end{aligned}$$

But then

$$\begin{aligned}\int_{\mathcal{U}} (f+g)(u)\,\tilde{\mu}_2(du) &= \int_{\mathcal{U}} f(u)\,\tilde{\mu}_2(du)+\int_{\mathcal{U}} g(u)\,\tilde{\mu}_2(du)\\ &> \int_{\mathcal{U}} f(u)\,\tilde{\mu}_2(du)-\int_{\mathcal{U}} f(u)\,\tilde{\mu}_2(du)\\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\int_{\mathcal{U}} (f+g)(u)\,\mu_1(du) &= \int_{\mathcal{U}} f(u)\,\mu_1(du)+\int_{\mathcal{U}} g(u)\,\mu_1(du)\\ &\leq \int_{\mathcal{U}} f(u)\,\mu_1(du)-\int_{\mathcal{U}} f(u)\,\mu_1(du)-\varepsilon/2\\ &= -\varepsilon/2.\end{aligned}$$

Define $h := f+g$ and observe that $h(u) = f(u) \geq \delta$, for all $u \in cl(S_1 \setminus S_2)$. By Lemma 5, we know that this implies the existence of menus A, B such that

$$\int_{\mathcal{U}} (\sigma_A - \sigma_B)(u)\,\tilde{\mu}_2(du) > 0 > \int_{\mathcal{U}} (\sigma_A - \sigma_B)(u)\,\mu_1(du)$$

and $\sigma_A(u) > \sigma_B(u)$ for all $u \in cl(S_1 \setminus S_2)$. But the conditions above imply that $A \succ_2 B$, $B \succ_1 A$ and $A \cup B \not\rhd A$, which contradicts the fact that $\succsim_2$ is a less flexibility loving version of $\succsim_1$. We conclude that we must have $\tilde{\mu}_2(V) \leq \mu_1(V)$ for all Borel subsets V of $\mathcal{U}$. ‖

Making $T := S_2$ and using the two claims above concludes the proof of the theorem. ■

A.7 Proof of Theorem 5

The theorem will be a consequence of the following lemma:

Lemma 7. *Let $\succsim_1, \succsim_2$ be PAEU preferences with representations μ_1, μ_2 and define $S_i := supp(\mu_i)$, $i = 1, 2$. Then, $\succsim_2$ is independent from $\rhd$ if and only if*

$\mu_2(cl(S_1 \setminus S_2)) = 0$.

Proof of Lemma. Suppose $\mu_2(cl(S_1 \setminus S_2)) = 0$ and let A and B be two menus such that $A \succ_2 B$. Let C be any closed sphere in the interior of $\Delta(X)$ and define

$$D := \bigcup_{u \in cl(S_1 \setminus S_2)} \arg\max_{p \in C} E_p(u).$$

We note that, by construction, $\{u \in \mathcal{U} : \sigma_C(u) > \sigma_D(u)\} = \mathcal{U} \backslash cl(S_1 \setminus S_2)$. For each $\lambda \in (0,1)$, define $\hat{f}^\lambda := \sigma_{(C \oplus_\lambda A)} - \sigma_{(C \oplus_\lambda B) \cup [D \oplus_\lambda (A \cup B)]}$. By Lemma 6, we know that

$$\lim_{\lambda \uparrow 1} \frac{1}{1-\lambda} \int_{\mathcal{U} \backslash cl(S_1 \setminus S_2)} \hat{f}^\lambda(u)\, \mu_2(du) = \int_{\mathcal{U} \backslash cl(S_1 \setminus S_2)} (\sigma_A - \sigma_B)(u)\, \mu_2(du).$$

Since $\mu_2(cl(S_1 \setminus S_2)) = 0$, the condition above implies that

$$\lim_{\lambda \uparrow 1} \frac{1}{1-\lambda} \int_{\mathcal{U} \backslash cl(S_1 \setminus S_2)} \hat{f}^\lambda(u)\, \mu_2(du) = \int_{\mathcal{U}} (\sigma_A - \sigma_B)(u)\, \mu_2(du) > 0.$$

So, for λ sufficiently close to one, we have $C \oplus_\lambda A \succ_2 (C \oplus_\lambda B) \cup [D \oplus_\lambda (A \cup B)]$. Since obviously $\sigma_{(C \oplus_\lambda B) \cup [D \oplus_\lambda (A \cup B)]}(u) \geq \sigma_{(C \oplus_\lambda A)}(u)$ for all $u \in cl(S_1 \setminus S_2)$, we also have that $[C \oplus_\lambda (A \cup B)] \cup [D \oplus_\lambda (A \cup B)] \not\triangleright (C \oplus_\lambda B) \cup [D \oplus_\lambda (A \cup B)]$.[21] We conclude that $\succsim_2$ is indeed independent from $\triangleright$.

Conversely, suppose now that $\mu_2(cl(S_1 \setminus S_2)) > 0$. Now pick any $\varepsilon > 0$ such that $\varepsilon\mu_2(S_2 \setminus cl(S_1 \setminus S_2)) < \mu_2(cl(S_1 \setminus S_2) \cap S_2)$. Now pick a closed subset T of $S_2 \setminus cl(S_1 \setminus S_2)$ such that

$$\mu_2(S_2 \setminus cl(S_1 \setminus S_2)) - \mu_2(T) \leq \frac{\varepsilon}{1+\varepsilon}\mu_2(S_2 \setminus cl(S_1 \setminus S_2)).$$

We can now use Urysohn's Lemma to find a continuous function $f : \mathcal{U} \to [-\varepsilon, 1]$ such that $f(u) = -\varepsilon$ for all $u \in T$ and $f(u) = 1$ for all $u \in S_2 \cap cl(S_1 \setminus S_2)$. By construction we have

$$\int_{S_2 \backslash cl(S_1 \backslash S_2)} f(u)\, \mu_2(du) \leq 0,$$

with equality only if $S_2 \subseteq cl(S_1 \setminus S_2)$, and

$$\int_{S_2} f(u)\, \mu_2(du) > 0.$$

[21] Note that $cl(A) \subseteq int(\Delta(X))$ implies that $cl(C \oplus_\lambda A) \subseteq int(\Delta(X))$ and $cl([C \oplus_\lambda (A \cup B)] \cup [D \oplus_\lambda (A \cup B)] \setminus (C \oplus_\lambda B) \cup [D \oplus_\lambda (A \cup B)]) \subseteq cl(C \oplus_\lambda A)$.

We can now use Lemma 5 to find menus A and B such that

$$\int_{S_2 \setminus cl(S_1 \setminus S_2)} (\sigma_A - \sigma_B)(u)\, \mu_2(du) \leq 0$$

and

$$\int_{S_2} (\sigma_A - \sigma_B)(u)\, \mu_2(du) > 0.$$

By Lemma 2, we know that for any menus C, D and $\lambda \in (0,1]$ such that

$$[C \oplus_\lambda (A \cup B)] \cup D \not\triangleright (C \oplus_\lambda B) \cup D$$

we must necessarily have $\sigma_{(C \oplus_\lambda B) \cup D}(u) \geq \sigma_{(C \oplus_\lambda A)}(u)$ for all $u \in cl(S_1 \setminus S_2)$. But this implies that

$$\begin{aligned} \int_{S_2} (\sigma_{(C\oplus_\lambda B)\cup D} - \sigma_{(C\oplus_\lambda A)})(u)\mu_2(du) &\geq \int_{S_2 \setminus cl(S_1 \setminus S_2)} (\sigma_{(C\oplus_\lambda B)\cup D} - \sigma_{(C\oplus_\lambda A)})(u)\mu_2(du) \\ &\geq \int_{S_2 \setminus cl(S_1 \setminus S_2)} (\sigma_{(C\oplus_\lambda B)} - \sigma_{(C\oplus_\lambda A)})(u)\mu_2(du) \\ &= \lambda \int_{S_2 \setminus cl(S_1 \setminus S_2)} (\sigma_B - \sigma_A)(u)\, \mu_2(du) \\ &\geq 0. \end{aligned}$$

We conclude that $\succsim_2$ is not independent from $\triangleright$ and this finishes the proof of the lemma. $\|$

Now note that theorem 5 is a straightforward corollary of Theorem 4 and the lemma above. ■

A.8 Proof of Theorem 6

The proof will make use of the following lemmas:

Lemma 8. *Let $\succsim$ be a PAEU preference with representation μ. If C, D preserve μ in the limit, then $\mu(\{\sigma_C > \sigma_D\}) = 1$.*

Proof of Lemma. Suppose $\mu(\{\sigma_D \geq \sigma_C\}) > 0$ and let $T := \{\sigma_D \geq \sigma_C\}$ and $\alpha := \mu(T)$. Find a closed subset S of $\{\sigma_C > \sigma_D\}$ with $\mu(\{\sigma_C > \sigma_D\}) - \mu(S) < \delta\mu(S)$, where $\delta := \frac{\mu(T)}{2\mu(\{\sigma_C > \sigma_D\})}$. Now use Urysohn's Lemma to find a continuous function

$f : \mathcal{U}\, [-\delta, 1]$ such that $f(u) = -\delta$ for $u \in S$ and $f(u) = 1$ for $u \in T$. Note that

$$\begin{aligned}\int_{\mathcal{U}} f(u)\, \mu(du) &> \mu(T) - \delta\mu(\{\sigma_C > \sigma_D\}) \\ &= \mu(T) - \frac{\mu(T)}{2} \\ &> 0,\end{aligned}$$

but

$$\begin{aligned}\int_{\{\sigma_C > \sigma_D\}} f(u)\, \mu(du) &< -\delta\mu(S) + (\mu(\{\sigma_C > \sigma_D\}) - \mu(S)) \\ &< 0.\end{aligned}$$

By lemma 5, we can find menus A and B such that

$$\int_{\mathcal{U}} \sigma_A(u) - \sigma_B(u)\, \mu(du) > 0,$$

but

$$\int_{\{\sigma_C > \sigma_D\}} \sigma_A(u) - \sigma_B(u)\, \mu(du) < 0.$$

By Lemma 6, the second inequality implies that $(C \oplus_\lambda B) \cup [D \oplus_\lambda (A \cup B)] \succ (C \oplus_\lambda A) \cup [D \oplus_\lambda (A \cup B)]$ when λ is large, while, by assumption, the first is equivalent to say that $A \succ B$. This contradicts the assumption that C and D preserve $\succsim$ in the limit and, therefore, we conclude that we must have $\mu(\{\sigma_C > \sigma_D\}) = 1$. $\|$

Lemma 9. *Let* (Ω, Σ) *be a measurable space and let* μ_1, μ_2 *be two probability measures over* (Ω, Σ). *There exists* $S \in \Sigma$ *such that* $\mu_2(S) = 1$ *and for any* $T \subseteq S$, $\mu_2(T) = 0 \implies \mu_1(T) = 0$.

Proof of Lemma. By the Lebesgue decomposition theorem, there exists signed measures $\tilde{\mu}_1, \hat{\mu}_1$ such that $\mu_1 = \tilde{\mu}_1 + \hat{\mu}_1$, $\tilde{\mu}_1 << \mu_2$ and $\hat{\mu}_1 \perp \mu_2$. Since $\hat{\mu}_1 \perp \mu_2$, we know that there exists disjoint T', T'' such that $T' \cup T'' = \Omega$, $\hat{\mu}_1$ is zero on all measurable subsets of T' and $\mu_2(T'') = 0$. It is clear now that the lemma is satisfied for $S := T'$. $\|$

Lemma 10. *Let* (Ω, Σ) *be a measurable space and let* μ_1, μ_2 *be two probability measures over* (Ω, Σ). *Fix a measurable set* S *such that* $\mu_2(S) = 1$ *and for any measurable set* $T \subseteq S$, $\mu_2(T) = 0 \implies \mu_1(T) = 0$. μ_2 *is the Bayesian update of* μ_1 *after the observation of some set only if* μ_2 *is the Bayesian update of* μ_1 *after the observation of* S.

Proof of Lemma. Suppose that μ_2 is the Bayesian update of μ_1 after the observation of a set $\tilde{S}$. First note that

$$1 = \mu_2 (S) = \frac{\mu_1 \left(S \cap \tilde{S} \right)}{\mu_1 \left(\tilde{S} \right)},$$

which implies that $\mu_1 \left(\tilde{S} \setminus S \right) = 0$. Also note that $\mu_2 \left(S \setminus \tilde{S} \right) = 0$, which by our assumptions imply that $\mu_1 \left(S \setminus \tilde{S} \right) = 0$. We learn that $\mu_1 (S) = \mu_1 \left(\tilde{S} \right)$. Now fix any measurable set T. Note that:

$$\mu_2 (T) = \mu_2 (T \cap S) = \frac{\mu_1 \left(T \cap S \cap \tilde{S} \right)}{\mu_1 \left(\tilde{S} \right)} = \frac{\mu_1 (T \cap S)}{\mu_1 (S)}.$$

This concludes the proof of the lemma. $\|$

Now we are ready to finish the proof of the theorem.

[Necessity] Suppose first that $\succsim_1$ and $\succsim_2$ have PAEU representations μ_1, μ_2, respectively, and that there exists a Borel set $T \subseteq S_1 := \mathit{supp} (\mu_1)$ satisfying one of the conditions in the statement of the theorem. Fix some pair of menus A, B such that $A \succ_2 B$. We first need the following claim:

Claim 1. *There exists a set* $\hat{T} \supseteq T$ *such that*

$$\int_{\hat{T}} \sigma_A (u) \mu_1 (du) > \int_{\hat{T}} \sigma_B (u) \mu_1 (du) .$$

Proof of Claim. If μ_2 is the Bayesian update of μ_1 after the observation of T, then we can simply make $\hat{T} := T$. Suppose, then, that $\mu_1 (T) = 0$ and $\mu_2 (T) = 1$. Since $A \succ_2 B$, we know that that $\sigma_A (u^*) > \sigma_B (u^*)$ for some $u^* \in T \subseteq S_1$. By continuity of support functions, we know that in fact we must have $\sigma_A (u) > \sigma_B (u)$ for all u in some neighborhood of u^*. This implies that

$$\begin{aligned} \int_{\{\sigma_A > \sigma_B\} \cup T} (\sigma_A - \sigma_B) (u) \mu_1 (du) &= \int_{\{\sigma_A > \sigma_B\}} (\sigma_A - \sigma_B) (u) \mu_1 (du) \\ &> 0. \end{aligned}$$

Defining $\hat{T} := \{\sigma_A > \sigma_B\} \cup T$ completes the proof of the claim. $\|$

Now let $\hat{T}$ be as in the claim above. By regularity of μ_1, we can find an open superset O of $\hat{T}$ such that

$$\int_O \sigma_A(u)\,\mu_1(du) > \int_O \sigma_B(u)\,\mu_1(du).$$

Now let E be any closed sphere in the interior of $\Delta(X)$. Define the following sets:

$$C := E \text{ and } D := \left\{ p \in E : \{p\} = \arg\max_{q \in E} E_q(u) \text{ for some } u \in \mathcal{U} \backslash O \right\}.$$

By construction, $\{\sigma_C > \sigma_D\} = O$ and, by Lemma 6, it must be the case that $(C \oplus_\lambda A) \cup [D \oplus_\lambda (A \cup B)] \succ_1 (C \oplus_\lambda B) \cup [D \oplus_\lambda (A \cup B)]$ when λ is large. Again, by Lemma 6, it is clear that for any two menus A', B' with $A' \succ_2 B'$ we must necessarily have $(C \oplus_\lambda A') \cup [D \oplus_\lambda (A' \cup B')] \succ_2 (C \oplus_\lambda B') \cup [D \oplus_\lambda (A' \cup B')]$ when λ is large, so that C and D preserve $\succsim_2$ in the limit.

[Sufficiency] Suppose that $\succsim_1$ and $\succsim_2$ have PAEU representations with priors μ_1 and μ_2, respectively, and suppose that $\succsim_2$ can be extracted from $\succsim_1$. Fix any Borel set $T \subseteq \mathcal{U}$ such that $\mu_2(T) = 1$ and for any Borel subset $\hat{T}$ of T, $\mu_2(\hat{T}) = 0 \implies \mu_1(\hat{T}) = 0$. By Lemma 9 above, we know that such a set always exists. We need the following claim:

Claim 2. *Let $S_1 := supp(\mu_1)$. It must be the case that $\mu_2(T \cap S_1) = 1$.*

Proof of Claim. Suppose $\mu_2(T \setminus S_1) > 0$. Let E be any closed sphere in the interior of $\Delta(X)$ and define

$$A := E \text{ and } B := \left\{ p \in E : \{p\} = \arg\max_{q \in E} E_q(u) \text{ for some } u \in S_1 \right\}.$$

Notice that, by construction, $\{\sigma_A > \sigma_B\} = \mathcal{U} \backslash S_1$ and $\{\sigma_A = \sigma_B\} = S_1$. This implies that

$$\int_{\mathcal{U}} (\sigma_A - \sigma_B)(u)\,\mu_2(du) = \int_{T \backslash S_1} (\sigma_A - \sigma_B)(u)\,\mu_2(du) > 0,$$

or, equivalently, $A \succ_2 B$. Now pick any two menus C, D. For each $\lambda \in (0,1)$, define $f^\lambda := \sigma_{(C \oplus_\lambda A) \cup [D \oplus_\lambda (A \cup B)]} - \sigma_{(C \oplus_\lambda B) \cup [D \oplus_\lambda (A \cup B)]}$. Now note that, for any $\lambda \in (0,1)$,

$$\begin{aligned} \int_{\mathcal{U}} f^\lambda(u)\,\mu_1(du) &= \int_{S_1} f^\lambda(u)\,\mu_1(du) \\ &= \int_{S_1} 0\mu_1(du) \\ &= 0. \end{aligned}$$

Since C, D were entirely arbitrary, this contradicts the assumption that $\succsim_2$ can be extracted from $\succsim_1$ and, therefore, we conclude that $\mu_2(T \cap S_1) = 1$. $\|$

Given the claim above, we now assume, without loss of generality, that $T \subseteq S_1$. If $\mu_1(T) = 0$, the proof is complete, so assume that $\mu_1(T) > 0$. If μ_2 is not the Bayesian update of μ_1 after the observation of T, by a separation argument, we can find a continuous function $f : \mathcal{U} \to \mathbb{R}$ such that

$$\int_T f(u)\,\mu_2(du) > 0, \text{ but } \int_T f(u)\,\mu_1(du) < 0.$$

Let

$$\alpha := \max_{u \in \mathcal{U}} |f(u)|,$$

and let $\varepsilon > 0$ be such that

$$\varepsilon < \frac{1}{\alpha} \min\left\{\left|\int_T f(u)\,\mu_1(du)\right|, \int_T f(u)\,\mu_2(du)\right\}.$$

Since μ_1 and μ_2 are Borel probability measures over a metric space, they are both regular. Therefore, we can find closed sets $T' \subseteq T$ and $T'' \subseteq (\mathcal{U} \backslash T)$ such that

$$\mu_i(T) - \mu_i(T') < \varepsilon/3, \text{ for } i = 1, 2$$

and

$$\mu_i(\mathcal{U} \setminus T) - \mu_i(T'') < \varepsilon/3, \text{ for } i = 1, 2.$$

Using Urysohn's lemma we can find a continuous function $g : \mathcal{U} \to [0, 1]$ such that $g(u) = 0$ for all $u \in T''$ and $g(u) = 1$ for all $u \in T'$. Finally, define the function h by $h := f \cdot g$. We note that h is a continuous function, so there exists menus A and B and $r > 0$ such that

$$|r(\sigma_A(u) - \sigma_B(u)) - h(u)| < \frac{\alpha\varepsilon}{3} \text{ for all } u \in \mathcal{U}.$$

This implies that

$$\begin{aligned}
r\int_{\mathcal{U}}\left(\sigma_A(u)-\sigma_B(u)\right)\mu_2(du) &> \int_{\mathcal{U}} h(u)\,\mu_2(du)-\frac{\alpha\varepsilon}{3}\\
&= \int_T f(u)\,\mu_2(du)\\
&\quad+\int_{T\setminus T'}\left(f(u)\,g(u)-f(u)\right)\mu_2(du)-\frac{\alpha\varepsilon}{3}\\
&\geq \int_T f(u)\,\mu_2(du)-\alpha\mu_2(T\setminus T')-\frac{\alpha\varepsilon}{3}\\
&> \int_T f(u)\,\mu_2(du)-\alpha\varepsilon\\
&> 0.
\end{aligned}$$

So $A\succ_2 B$. Finally, let C and D be any two menus that preserve $\succsim_2$ in the limit. By Lemma 8, this implies that $\mu_2(T\cap\{\sigma_C>\sigma_D\})=1$, or, equivalently, $\mu_2(T\setminus\{\sigma_C>\sigma_D\})=0$. By construction, this implies that $\mu_1(T\setminus\{\sigma_C>\sigma_D\})=0$ and, therefore, $\mu_1(T)=\mu_1(T\cap\{\sigma_C>\sigma_D\})$. For each $\lambda\in(0,1)$ define $f^\lambda:=\sigma_{(C\oplus_\lambda A)\cup[D\oplus_\lambda(A\cup B)]}-\sigma_{(C\oplus_\lambda B)\cup[D\oplus_\lambda(A\cup B)]}$. By Lemma 6, we know that

$$\lim_{\lambda\uparrow 1}\frac{1}{1-\lambda}\int_{\mathcal{U}} f^\lambda(u)\,\mu_1(du)=\int_{\{\sigma_C>\sigma_D\}}\left(\sigma_A(u)-\sigma_B(u)\right)\mu_1(du).$$

But now note that

$$\begin{aligned}
\int_{\{\sigma_C>\sigma_D\}} r(\sigma_A(u)-\sigma_B(u))\mu_1(du) &< \int_{\{\sigma_C>\sigma_D\}} h(u)\,\mu_1(du)+\frac{\alpha\varepsilon}{3}\\
&= \int_T h(u)\,\mu_1(du)-\int_{T\setminus\{\sigma_C>\sigma_D\}} h(u)\,\mu_1(du)\\
&\quad+\int_{\{\sigma_C>\sigma_D\}\setminus T} h(u)\,\mu_1(du)+\frac{\alpha\varepsilon}{3}\\
&= \int_T h(u)\,\mu_1(du)\\
&\quad+\int_{\{\sigma_C>\sigma_D\}\setminus T} h(u)\,\mu_1(du)+\frac{\alpha\varepsilon}{3}\\
&< \int_T f(u)\,\mu_1(du)+\alpha\mu_1(T\setminus T')\\
&\quad+\alpha\mu_1\left((\{\sigma_C>\sigma_D\}\setminus T)\setminus T''\right)+\frac{\alpha\varepsilon}{3}\\
&< \int_T f(u)\,\mu_1(du)+\alpha\varepsilon\\
&< 0.
\end{aligned}$$

So, for λ large enough we must necessarily have

$$(C \oplus_\lambda B) \cup [D \oplus_\lambda (A \cup B)] \succ_1 (C \oplus_\lambda A) \cup [D \oplus_\lambda (A \cup B)],$$

which contradicts the assumption that $\succsim_2$ can be extracted from $\succsim_1$. We conclude that μ_2 must be the Bayesian update of μ_1 after the observation of T. ■

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