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Optimal Interest Rate Rules in Inflation Targeting Frameworks

José Alvaro Rodrigues Neto*
Fabio Araújo
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Abstract

This work describes the main characteristics of an inflation targeting regime and derives the optimal solution for interest rates according to an original methodology for two models based on the Phillips and IS curves containing general exogenous variables and a complete loss-function.

Keywords: dynamic programming; stochastic optimization; inflation targeting; monetary policy; fiscal dominance.

JEL Classification: E42, E52, E58.

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Optimal Interest Rate Rules in Inflation Targeting Frameworks

1. Introduction.

There is consensus among economists that keeping inflation at a low and stable level is an essential goal of macroeconomic policy. Empiric evidences[1] suggest that such inflation should be about 2% p.a. Therefore, a monetary policy able to keep[2] inflation at an adequate level, entailing minimum social costs, would be an important contribution of the Central Bank to society at large.

In this context, the inflation targeting regime is being used since the beginning of the nineties by a growing number of countries. New Zealand was the pioneer of this type of monetary policy. Currently, countries using this regime include Brazil, England, Sweden, Mexico and Australia.

However, the adoption of such system requires some preconditions. In the economic field, the main requisite is the lack of fiscal dominance, that is to say, monetary policy shall not be restricted by fiscal considerations. Besides, another important issue is that inflation shall have been already curbed[3].

In the political field, an adequate institutional framework shall be in operation. Therefore, the Central Bank shall enjoy, at least, an operating independence. Ideally, this political independence would be complete, with the Governor and Board of Governors of the Central Bank taking office for a fixed previously established term, not coincident with the terms of office of the President of the Republic or the Prime Minister.

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1 See [2] for more details.

2 If, by the implementation of the inflation targeting regime, the inflation is at a level higher than desired, a reduction movement shall be caused by monetary policy until the target is achieved.

3 The term **curbed** used above may have different meanings, depending on the characteristics of the country where the inflation targeting regime is to be used.

Free international flow of goods and capitals are desirable, since in this case the exchange rate would operate more efficiently in the transmission of monetary policy. This is because, for the principle of non-arbitrage, a change in domestic interest rates may affect the exchange rate, changing immediately the price both of imported goods and of domestic products tradable in international markets. Moreover, the exchange rate channel operates in the same direction as the channel of aggregate demand in the mechanism of price transmission. Actually, most countries adopt the inflation targeting regime after a move in their exchange policies, from more rigid exchange mechanisms, as fixed exchange rate for instance, towards more flexible systems, such as floating exchange rates.

Credibility is a key factor for the good performance of this system, since market expectations have a strong influence in the formation of the inflation rate. In this sense, a transparent monetary policy is a crucial instrument for gaining and maintaining credibility. Its building is a lengthy process, and may be achieved through a series of sound and transparent actions by the monetary authority. On the other hand, it may be easily lost in case of any imprudent action by the Central Bank.

The targets for subsequent periods shall be defined by the Ministry of Finance (or an equivalent organ), being their attainment a responsibility of the monetary authority. In case of non-compliance, the Central Bank shall publicly[4] explain the reasons why the target was not attained, what is being done to resume an adequate path and how long it would take to show results.

The target may be defined as an interval or as a fixed value, associated to a probability distribution of attainment. Furthermore, some countries use an inflation index that purges more volatile, or seasonal, items, such as food and energy prices. Escape clauses are also usual, explaining the lack of compliance with the target due to factors that either are beyond the influence of the monetary authority or imply a high social cost.

[4] This should be done through an Open Letter of the Chairman of the Central Bank to the Minister of Finance.

The inflation targeting policy requires that the Central Bank explain to society the reasons of possible inflationary pressures and the measures adopted to curb them. Then, in case such pressures result from lack of fiscal control, this shall also be made clear, helping the central government to measure the extension of the problem.

The use of a target other than the one publicized is not an efficient strategy, since agents, with time, ends by anticipating the Central Bank actions and nullifying the effects of this sort of policy. Besides, this strategy is not transparent and entails a cost in terms of the massive credibility required to obtain a good performance of the system. Another type of inferior policy is the use of a monetary target[5].

Another issue discussed in the literature refers to the so-called loss function, to be defined in the next section. It commonly considers an infinite number of periods. In this paper, the use of a loss function with a finite and arbitrary number of periods was preferred[6].

Given the permanent change of the economic structure, the parameters of the model shall change with time[7]. Since the path of interest rates calculated assumes validity of such factors for all periods, a solution for infinite periods would not be consistent.

An instance of this was observed by the Reserve Bank of Australia: after adopting the inflation targeting, the Bank realized that, with time and increased monetary policy credibility, the economic agents came to act in a way in which it was possible to reduce inflation without resorting to interest rates as high as they were at the time the system has been adopted[8]

Another criticism to the use of an infinite number or periods in the loss function hinges to the need of foreseeing the behavior of exogenous variables in all periods. As

[5] For further details, see [2].

[6] The use of a finite number of periods in a loss function is a more general procedure than the solution for infinite periods, since the latter is the limit of solutions for finite periods, according to [10].

[7] Lucas' critique.

[8] For more details, see [4].

uncertainty increases, no more than noise would be added to the solution. To eschew this effect or simplify the mathematical treatment, most models in the literature do not consider exogenous variables. In this paper, the first model analyzed uses exogenous variables.

Another characteristic of the loss function presented in this paper is that it takes into account not only the deviation of inflation from the target, as it is common in the literature, but also considers unemployment and changes in interest rates.

Besides, the mathematical technique used is simpler, straightforward and works for different types of models. Two examples are examined: an original model, and one already studied to some extent[9]. It was unnecessary to resort to sophisticated mathematical techniques, as dynamic programming and others[10], as customary.

The paper is divided into six sections and two appendices. The problem of conducting monetary policy is described in the second section; in the third, a model and a loss function are proposed and an optimal rule for interest rate is derived; the fourth analyzes the specific case of a pure inflation targeting regime; the fifth studies a model common in the literature; and in the last sections the conclusions of the work are drawn. The appendix A exhibit the mathematical details to facilitate the derivation of results shown in Equations 3. In the appendix B, the interest rate optimal rule for the model described in section 5 is derived.

2. The Problem of Conducting Monetary Policy.

The main objective of this paper is determining a monetary policy rule that would make possible the achievement of given inflation target with minimal loss for society as a whole. It is worth stressing that one does not intend, with this, automating the interest

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9 This model is shown in [2].

10 An example of mathematical complexity involved in this type of problem may be found in [1].

rates setting, but just providing more inputs to the monetary authority's decision-making.

With this purpose, it shall be considered an economic model that intends to describe, the most precisely possible, the transmission mechanism of monetary policy, which enables to quantify the intensity of impacts of instruments, assumed here as being solely the interest rate set by the Central Bank, on the variables of interest, mainly inflation.

Normally, such models, estimated through econometric techniques, show a lagged influence of the instrument on the observed variables. Therefore, a change in interest rate of a given period would affect inflation only after some periods, the number of which is determined by the chain of lags in the transmission mechanism modeled. With this, it is necessary to act preventively, in a way to make a contemporary action able to curb an inflationary pressure still not realized. This makes this type of monetary policy eminently prospective and the main actions of the Central Bank pro-active rather than reactive.

Besides, a function shall be established to quantify social losses, contemplating behaviors considered as harmful to the selected macroeconomic variables. With this purpose, one selected to work with a generic loss function, considering the deviation of inflation from the established target, the deviation of unemployment from NAIRU[11], and the variability of interest rates, as harmful. The term associated to inflation is naturally given by the considered monetary policy framework. The term of unemployment aims at assessing the social cost resulting from actions of the monetary authority. Finally, the inclusion of a term accounting for interest rate variability tries to measure the loss related to a possible lower credibility, stemming from sudden movements of interest rates that may be interpreted by the economic agents as lack of consistence in the conduction of monetary policy[12]. Inclusion of such additional factors brings a larger degree of complexity to the problem.

[11] NAIRU (Non Accelerating Inflation Rate of Unemployment) is an economic indicator intending to express the level of unemployment below which there are not inflationary pressures. A better discussion of the matter may be found in [7].

[12] In addition, greater variability of interest rates implies greater risk in the market, entailing unnecessary increase of the risk premium for the economy as a whole.

It shall be stressed that the inclusion, in the loss function, of factors other than the deviation of inflation from the target, characterizes the operation of a mixed regime. In such system, non-zero weights are attributed to the factors taken into account, which leads to softening not only the inflation response, but also the other selected variables. The weights make convergence of inflation towards the target slower, both for unexpected deviations (shocks) and for changes in the inflation level. However, with this, more favorable paths for the indicators included in the loss function are obtained. One shall, however, exercise care so that the magnitude of the chosen relative weights does not disfigure the system.

3. General Solution for the First Model.

Let us consider an economic model in its reduced form, given by an investment-savings curve (IS curve) and a Phillips curve, depicted by Equation 1, and a loss function, which is the object function of the problem, described in its general form by Equation 2. Then, the problem boils down to minimize the loss function, constrained by the equations of the model[13]

$$\begin{cases} u_{k+1} = \beta . i_k + e_{k+1} + \varepsilon_{k+1} \\ \pi_{k+2} = \alpha\pi_{k+1} + \gamma u_{k+1} + et_{k+2} + \eta_{k+2} \end{cases} \qquad \text{(Eq. 1)}$$

where:

ε, η i.i.d. white noise independent between them;

e, et exogenous variables[14]

π inflation;

u deviation of unemployment from NAIRU;

i interest rate.

---

[13] Notice that the model coefficients belong to the set of real numbers, i.e., they may be either positive or negative.

[14] The exogenous variables may be, for instance, associated to external and monetary environment. In addition, they incorporate constants that may be obtained in regressions.

$$L = \sum_{j=1}^{N} \delta^{j} \{E[(\pi_j - \pi^*)^2] + \lambda E[u_j^2] + \mu[i_{j-1} - i_{j-2}]^2\} \qquad \text{(Eq. 2)}$$

where:

δ time discount rate[15];

N number of periods considered in the optimization;

π^* inflation target;

λ, μ weights given to unemployment and interest rate variability, respectively, as related to the deviation from the inflation target.

This is a linear-quadratic problem. A problem is said to be linear-quadratic if the objective function is quadratic while the constraints are linear, or while the objective function is linear and the restrictions quadratic[16].

By the principle of equivalence in certainty[17], the optimization of a linear-quadratic problem is equivalent to solving its corresponding deterministic problem. Despite this equivalence, the solution of a stochastic problem does not pose additional effort, with the exception of pure inflation targeting ($\lambda = \mu = 0$), for which a solution is shown in the next section. Therefore, one selected the solution of the stochastic case.

For the sake of simplifying the notation, the reference to current expectations with respect to exogenous variables is made without using the expectation operator. With this, references in the text to future values of such variables are references to expectations. Thus, finding the first order conditions for minimization[18] of L one has:

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15 The use of a time discount rate lower than one makes deviations in a more distant future more tolerable. This is consistent with the fact that econometric forecasts, reflected in the conditional expectations used in the loss function, have uncertainties that increase with the number of periods considered.

16 See the principle of duality in [6].

17 Whenever the objective function is linear, only the average will be relevant for minimization purposes. Higher order momentums (as the variance, for instance), may be discarded. See [5] for greater details on the principle of equivalence in certainty.

18 Minimization is granted by the fact that the quadratic form considered is positive definite.

$$\frac{\partial L}{\partial i_k} = \sum_{j=k+2}^{N} \delta^{j} \frac{\partial E[\pi_j^2]}{\partial i_k} - 2\pi^* \sum_{j=k+2}^{N} \delta^{j} \frac{\partial E\pi_j}{\partial i_k} + \lambda\delta^{k+1} \frac{\partial E(u_{k+1}^2)}{\partial i_k}$$

$$+ 2\mu\delta^{k+1}(i_k - i_{k-1}) - 2\mu\delta^{k+2}(i_{k+1} - i_k)$$

To solve the problem, one needs to study the general dependency relations of interest variables with respect to the instrument of monetary policy. Considering the causality relations described in the model, after some algebraic manipulation[19], one obtains:

$$\frac{\partial E[\pi_j^2]}{\partial i_k} = \begin{cases} 2\alpha^{j-k-2}\beta\gamma E[\pi_j] & , j \geq k+2 \\ 0 & , \text{otherwise} \end{cases}$$

$$\frac{\partial E[\pi_j]}{\partial i_k} = \begin{cases} \alpha^{j-k-2}\beta\gamma & , j \geq k+2 \\ 0 & , \text{otherwise.} \end{cases} \qquad \text{(Eq. 3)}$$

$$\frac{\partial E[u_j^2]}{\partial i_k} = \begin{cases} 2\beta^2 i_{j-1} + 2\beta e & , j = k+1 \\ 0 & , \text{otherwise} \end{cases}$$

The auxiliary variables w_k and z_k are defined in Equation 4 below:

$$w_n = \begin{cases} (\alpha^2\delta)^{n+1} \dfrac{1-(\alpha^2\delta)^{N-n}}{1-(\alpha^2\delta)} & , \text{n} = 1, 2, ..., \text{N - 1} \\ 0 & , \text{otherwise} \end{cases}$$

$$z_n = \begin{cases} i_n \alpha^{-n-1} w_{n+1} & , n = 0, 1, ..., N-2 \\ i_n & , n = N-1 \\ 0 & , otherwise \end{cases} \qquad \text{(Eq. 4)}$$

In the first order condition, the terms depending on z_k (and, consequently, on i_k) may be separated from the independent terms.

Then, $\frac{\partial L}{\partial i_k} = H_k + G_k$, being H_k the term dependent from z_k, and G_k the independent term, where:

---

19 See the appendix A for more details.

$$H_k = 2\beta\gamma E_k + C1_k - 2\alpha^{-k-3}\beta^2\gamma^2 \frac{w_{k+1}}{w_0} z_{-1}$$

$$+2\mu\delta^{k+1}\left(-\frac{\delta\alpha^{k+2}}{w_{k+2}} z_{k+1} + \frac{(1+\delta)\alpha^{k+1}}{w_{k+1}} z_k - \frac{\alpha^k}{w_k} z_{k-1}\right),$$

with:

$$E_k = \alpha^{-3-k}\beta\gamma \sum_{n=0}^{k+1}\left(\frac{w_{k+1}}{w_n} z_{n-1}\right) + \alpha^{-3-k}\beta\gamma \sum_{n=k+2}^{N-1}\left(z_{n-1}\right)$$

$$C1_k = \left(\frac{2\beta^2\lambda\left[1-(\alpha^2\delta)\right]}{\alpha^{k+3}\delta\left[1-(\alpha^2\delta)^{N-k-1}\right]}\right) z_k$$

$$G_k = 2\beta\gamma\left[D_k + F_k\right] + B_k + C2_k + 2\alpha^{-k-3}\beta^2\gamma^2 \frac{w_{k+1}}{w_0} z_{-1},$$

with:

$$B_k = -2\pi^*\alpha^{-k-2}\beta\gamma \sum_{j=k+2}^{N} (\alpha\delta)^j$$

$$D_k = \pi_0\alpha^{-2-k} w_{k+1}$$

$$F_k = \sum_{n=0}^{k+1}\left(\alpha^{-3-k-n}(\gamma e_n + et_{n+1}) w_{k+1}\right) + \sum_{n=k+2}^{N-1}\left(\alpha^{-3-k-n}(\gamma e_n + et_{n+1}) w_n\right)$$

$$C2_k = 2\beta\lambda\delta^{k+1} e_{k+1}$$

Equating the first order condition to zero to determine the critical point and making k to vary between 0 and $N-1$, one has a system of N linear algebraic equations with N unknowns, as described by Equation 5.

$$\frac{\partial L}{\partial i_k} = 0 = H_k + G_k \quad \therefore \quad H_k = -G_k \quad , \quad k = 0, 1, \ldots, N-1$$

$$\left(\sum_{n=1}^{k+1}\left[\frac{w_{k+1}}{w_n}z_{n-1}\right]+\sum_{\substack{n=k+2\\k\le N-3}}^{N-1}z_{n-1}\right)+\rho_k z_k+b_{k+1}z_{k+1}+a_k z_k+c_{k-1}z_{k-1}=d_k \qquad \text{(Eq. 5)}$$

or, in matrix form:

$$\left(\begin{bmatrix}\rho_0 & 0 & 0 & \cdots & 0 & 0\\ 0 & \rho_1 & 0 & \cdots & 0 & 0\\ 0 & 0 & \rho_2 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \rho_{N-2} & 0\\ 0 & 0 & 0 & \cdots & 0 & \rho_{N-1}\end{bmatrix}+\begin{bmatrix}\frac{w_1}{w_1} & 1 & 1 & \cdots & 1 & 0\\ \frac{w_2}{w_1} & \frac{w_2}{w_2} & 1 & \cdots & 1 & 0\\ \frac{w_3}{w_1} & \frac{w_3}{w_2} & \frac{w_3}{w_3} & \cdots & 1 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{w_{N-1}}{w_1} & \frac{w_{N-1}}{w_2} & \frac{w_{N-1}}{w_3} & \cdots & \frac{w_{N-1}}{w_{N-1}} & 0\\ 0 & 0 & 0 & \cdots & 0 & 0\end{bmatrix}+\begin{bmatrix}a_0 & b_0 & 0 & 0 & \cdots & 0\\ c_1 & a_1 & b_1 & 0 & \cdots & 0\\ 0 & c_2 & a_2 & b_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & c_{N-2} & a_{N-2} & b_{N-2}\\ 0 & 0 & 0 & \cdots & c_{N-1} & a_{N-1}\end{bmatrix}\right)\bullet\begin{bmatrix}z_0\\ z_1\\ z_2\\ \vdots\\ z_{N-2}\\ z_{N-1}\end{bmatrix}=\begin{bmatrix}d_0\\ d_1\\ d_2\\ d_3\\ \vdots\\ d_{N-1}\end{bmatrix}$$

where:

$$a_k=\begin{cases}\dfrac{\alpha^{2k+4}\delta^{k+1}(1+\delta)\mu}{\beta^2\gamma^2 w_{k+1}} & ,k=0,1,\dots,N-2\\ -\dfrac{\alpha^{N+2}\delta^{N+1}\mu}{\beta^2\gamma^2} & ,k=N-1\\ 0 & ,\text{otherwise}\end{cases},\qquad b_k=\begin{cases}-\dfrac{\alpha^{2k+5}\delta^{k+2}\mu}{\beta^2\gamma^2 w_{k+2}} & ,k=0,1,\dots,N-3\\ -\dfrac{\alpha^{N+1}\delta^{N}\mu}{\beta^2\gamma^2} & ,k=N-2\\ 0 & ,\text{otherwise}\end{cases},$$

$$c_k=\begin{cases}-\dfrac{\alpha^{2k+3}\delta^{k+1}\mu}{\beta^2\gamma^2 w_k} & ,k=1,2,\dots,N-1\\ 0 & ,\text{otherwise}\end{cases}\qquad \rho_k=\begin{cases}\dfrac{\lambda\left[1-\alpha^2\delta\right]}{\gamma^2\delta\left[1-\left(\alpha^2\delta\right)^{N-k-1}\right]} & ,k=0,1,\dots,N-2\\ \dfrac{\alpha^{N+2}\lambda\delta^{N}}{\gamma^2} & ,k=N-1\\ 0 & ,\text{otherwise}\end{cases} \quad \text{and}$$

$$d_k = \begin{cases} = \dfrac{\pi^* \alpha}{\beta\gamma} \displaystyle\sum_{j=k+2}^{N} (\alpha\delta)^j - \dfrac{\lambda \alpha^{k+3} \delta^{k+1}}{\beta\gamma^2} e_{k+1} - w_{k+1} i_{-1} - \dfrac{\pi_0 \alpha w_{k+1}}{\beta\gamma} \\ - \dfrac{1}{\beta\gamma} \displaystyle\sum_{n=0}^{k+1} \left(\alpha^{-n} (\gamma y_n + et_{n+1}) w_{k+1}\right) - \dfrac{1}{\beta\gamma} \displaystyle\sum_{n=k+2}^{N-1} \left(\alpha^{-n} (\gamma y_n + et_{n+1}) w_n\right) & ,k = 0,1,\ldots,N-2 \\ = \dfrac{\alpha^{N+2} \mu \delta^{N+1} i_N}{\beta^2 \gamma^2} - \dfrac{\alpha^{N+2} \lambda \delta^N e_N}{\beta\gamma^2} \quad ,k = N-1 \\ = 0 \quad , \text{otherwise} \end{cases}$$

Solving the system for $\bar{z} = (z_0, z_1, \ldots, z_{N-1})'$ and applying the change of variables described in Equation 4, one obtains the optimal path for the interest rate within the number of periods considered. As all expectations used are conditional to the set of available information up to the moment of the optimization[20], the proposal is not for adoption of all the resulting path[21], but only the interest rate value determined for the current period, since conditional expectations depend on the initial moment selected.

In addition, it shall be noticed that the matrix that pre-multiplies the vector $\bar{z}$ was deliberately divided in three parts, corresponding to the weights of unemployment, inflation, and interest rate of the loss function, respectively. Then, to study cases where one does not wish to soften either unemployment or interest rates, or both of them, it is sufficient to discard the corresponding term, which is equivalent to reduce its related weight to zero, taking care only or the definition of d_k.

4. Pure Inflation Targeting (Particular Case).

A particular case of theoretical interest is the pure inflation targeting regime. The economic model remains unchanged, although the loss function shall display $\lambda = 0, \mu = 0$, that is to say, the weights attributed to the unemployment and interest rate change are zero. Then, Equation 5 in its matrix form is reduced to:

[20] The scenarios foreseen by the monetary authority regarding exogenous variables are considered as part of the information set.

[21] The adoption of the whole determined path would only make sense in case of absence of shocks and, in addition, that the model's estimated coefficients and expectations about exogenous variables did not change.

$$\begin{bmatrix} \frac{w_1}{w_1} & 1 & 1 & \cdots & 1 & 1 \\ \frac{w_2}{w_1} & \frac{w_2}{w_2} & 1 & \cdots & 1 & 1 \\ \frac{w_3}{w_1} & \frac{w_3}{w_2} & \frac{w_3}{w_3} & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \frac{w_{N-2}}{w_1} & \frac{w_{N-2}}{w_2} & \frac{w_{N-2}}{w_3} & \cdots & \frac{w_{N-2}}{w_{N-2}} & 1 \\ \frac{w_{N-1}}{w_1} & \frac{w_{N-1}}{w_2} & \frac{w_{N-1}}{w_3} & \cdots & \frac{w_{N-1}}{w_{N-2}} & \frac{w_{N-1}}{w_{N-1}} \end{bmatrix} \bullet \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_{N-3} \\ z_{N-2} \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{N-3} \\ d_{N-2} \end{bmatrix} \qquad \text{Eq. 6)}$$

Given the principle of equivalence in certainty, one may see that, in case of pure inflation targeting, the method shown to solve the problem is not simple. However, the solution is trivial, in which all terms of the summation below are clearly null.

$$L = \sum_{j=2}^{N} \delta^{j} E[(\pi_j - \pi^*)^2]$$

Therefore, $L = 0$.

This fact has an intuitive appeal, since discarding other economic variables makes it simpler to achieve the target.

Then, the restrictions in their deterministic[22] forms are used to calculate the path of interest rates that makes $L = 0$.

$$u_{k+1} = \beta . i_k + e_{k+1}$$
$$\pi_{k+2} = \alpha\pi_{k+1} + \gamma u_{k+1} + et_{k+2}$$

With this, one may solve the problem making $\pi_2 = \pi^*$, which implies having $u_1 = \frac{\pi^*}{\gamma} - \frac{\alpha\pi_1}{\gamma} - \frac{et_2}{\gamma}$ and, therefore, $i_0 = \frac{u_1}{\beta} - \frac{e_1}{\beta} = \frac{\pi^*}{\beta\gamma} - \frac{\alpha\pi_1}{\beta\gamma} - \frac{et_2}{\beta\gamma} - \frac{e_1}{\beta}$.

[22] As previously mentioned, the values of exogenous variables in the future are expectations, so that , rigorously, the expectation operator shall be written.

Similarly, $\pi^*(1-\alpha) = \gamma . u_2 + et_3 \Rightarrow u_2 = \frac{\pi^*(1-\alpha)}{\gamma} - \frac{et_3}{\gamma}$.

Therefore one shall have: $i_1 = \frac{u_2}{\beta} - \frac{e_2}{\beta} = \frac{\pi^*(1-\alpha)}{\beta\gamma} - \frac{et_3}{\beta\gamma} - \frac{e_2}{\beta}$

Analogously, one determines $i_2, i_3, \cdots$ so that $L = \sum_{j=2}^{N} \delta^j E[(\pi_j - \pi^*)^2] = 0$

The use of this type a loss function facilitates more profound analyses of the problem, enabling even a comparison of the resulting performance with other monetary regimes[23]. However, a pure inflation target regime is too susceptible to social criticism, since it induces the idea that the authorities are not concerned with other macroeconomic variables, such as unemployment or output level.

5. The Model in the Literature[24]

This model, also consisting of an IS curve and a Phillips curve, has no exogenous variables, except for the control variable, and may be described by:

$$\begin{cases} \pi_{t+1} = \pi_t + \alpha . y_t + \varepsilon_t \\ y_{t+1} = \gamma . \pi_t + \beta . y_t - \gamma . i_t + \eta_t \end{cases} \qquad \text{(Eq.7)}$$

where:

π inflation

y product

i nominal rate of interest

ε i.i.d. white noise

η i.i.d. white noise independent of ε .

---

[23] As it may be seen in the articles by Svensson, [8], [9].

[24] This model is used in Svensson [2].

As usual, one considers a function that measures the loss of social welfare. In this case, L will be a function of inflation and output only, yet a term corresponding to the variability of interest rate, as before, may be easily introduced. One wishes to minimize:

$$L = \sum_{j=1}^{+\infty} \delta^{j} \left[E[(\pi_{j} - \pi^{*})^{2}] + \lambda . E[y_{j}^{2}] \right] = \lim_{M \to \infty} \sum_{N=1}^{M} \delta^{N} \left[E[(\pi_{N} - \pi^{*})^{2}] + \lambda . E[y_{N}^{2}] \right]$$

subject to the restrictions given by Equations 7.

It may be noticed that this is again a linear-quadratic problem, and all previous remarks still hold.

In matrix form, the model is given by:

$$\begin{bmatrix} \pi_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ \gamma & \beta \end{bmatrix} \begin{bmatrix} \pi_{t} \\ y_{t} \end{bmatrix} + \begin{bmatrix} 0 \\ -\gamma \end{bmatrix} . i_{t}$$

The solution is detailed in Appendix B. The main idea is to diagonalize matrix $A = \begin{bmatrix} 1 & \alpha \\ \gamma & \beta \end{bmatrix}$. The optimal interest rate is given by the solution of the following linear system:

$$\begin{bmatrix} a(0,0) & a(0,1) & a(0,2) & \cdots & a(0,M-1) \\ a(1,0) & a(1,1) & a(1,2) & \cdots & a(1,M-1) \\ a(2,0) & a(2,1) & a(2,2) & \cdots & a(2,M-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a(M-1,0) & a(M-1,1) & a(0,2) & \cdots & a(M-1,M-1) \end{bmatrix} \begin{bmatrix} i_{0} \\ i_{1} \\ i_{2} \\ \vdots \\ i_{M-1} \end{bmatrix} = \begin{bmatrix} b(0) \\ b(1) \\ b(2) \\ \vdots \\ b(M-1) \end{bmatrix}$$

where the coefficients are defined in Appendix B, and depend solely of $\alpha, \beta, \gamma, \delta, \lambda$

6. Conclusions.

In the model described, if the pure inflation targeting regime were adopted, that is to say, if one attributes zero weights to variation of unemployment and volatility of interest rates in the loss function, it remains always possible, except for shocks, to make inflation reach the target within two periods. Then, one obtains the value zero for the loss function.

The fact that the solution was obtained through a linear equations system warrants its continuity, and small disturbances in the set of information will result in small disturbances in the solution.

Although the method explained in this paper depicts a solution for a finite number of periods only, this number may be made arbitrarily large[25], limited only by the available computational capacity. Besides, assuming an infinite number of periods may not be the better solution, as discussed in the introduction.

[25] For the stationary case, the article [10] proves that the solutions for minimization in N periods converge for the solution with infinite periods as N tends to infinity.

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Appendix A

The minimization of a social loss function restricted to the model given by Equations 1 is a linear-quadratic problem, that is to say, one wishes to minimize the quadratic function subject to a system of linear equations. Therefore, the principle of equivalence in certainty, that states that its solution is equivalent to the solution of a similar deterministic problem, holds:

$$u_{k+1} = \beta . i_k + e_{k+1}$$
$$\pi_{k+2} = \alpha\pi_{k+1} + \gamma u_{k+1} + et_{k+2}$$

Therefore, there is no need to use sophisticated mathematical techniques, as dynamic programming (generally used in stochastic optimizations).

The method used consisted in calculating directly the first order conditions, as the object function is convex. Due to the presence of exogenous variables, algebraic manipulations became more extended.

This Appendix intends to provide inputs to facilitate the reader deriving the results shown in Equations 3.

$$E\pi_k = \alpha^k \pi_0 + \gamma \sum_{n=0}^{k-1} \alpha^{k-1-n} Eu_n + \sum_{n=0}^{k-1} \alpha^{k-1-n} et_{n+1}$$

$$E\pi_j = \alpha^j \pi_0 + \beta\gamma \sum_{n=0}^{j-1} \alpha^{j-1-n} i_{n-1} + \sum_{n=0}^{j-1} \alpha^{j-1-n} (\gamma e_n + et_{n+1})$$

Inflation:

With the idea of calculating $\frac{\partial E[\pi_j^2]}{\partial i_k}$, the following change of variables was used:

$P_{jk} = \pi_j - \alpha^{j-k-2} \beta\gamma i_k$, then $E[\pi_j^2] = EP_{jk}^2 + 2\alpha^{j-k-2} \beta\gamma E[P_{jk} i_k] + \alpha^{2j-2k-4} \beta^2 \gamma^2 E[i_k^2]$

Therefore, P_{jk} does not depend on i_k. Hence, EP_{jk} and $E[P_{jk}^2]$ do not depend on i_k. Hence, their derivatives with respect to i_k are zero.

$\frac{\partial E[\pi_j^2]}{\partial i_k} = 2\alpha^{j-k-2}\beta\gamma E[\pi_j]$, when $j \geq k+2$. In case $j < k+2$ this derivative is zero.

$\frac{\partial E[\pi_j]}{\partial i_k} = \alpha^{j-k-2}\beta\gamma$, when $j \geq k+2$. In case $j < k+2$ this derivative is zero.

Unemployment:

$\frac{\partial E[u_j^2]}{\partial i_k} = 2\beta^2 i_{j-1} + 2\beta e_j$, when $j = k+1$. Otherwise, this derivative is zero.

Appendix B

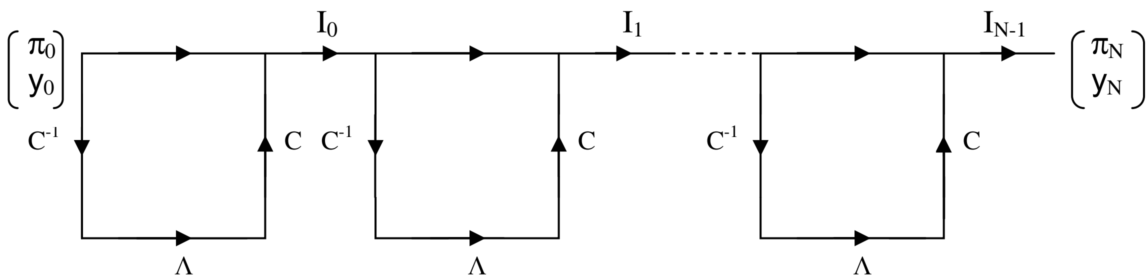
In this model, that has no exogenous variables, one used again a direct method of solution. However, before determining the first order conditions, one writes inflation (and the output gap) for an arbitrary future time, considering the initial conditions and the constants of the system. With this purpose, the matrix A shall be diagonalized. One may see that:

$$\begin{bmatrix} \pi_N \\ y_N \end{bmatrix} = (I_{N-1} \circ A \circ \cdots \circ I_0 \circ A) \begin{bmatrix} \pi_0 \\ y_0 \end{bmatrix}$$

Where $A = \begin{bmatrix} 1 & \alpha \\ \gamma & \beta \end{bmatrix}$, and $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ its diagonalization.

$C = \begin{bmatrix} u_1 & u_2 \\ 1 & 1 \end{bmatrix}$, where λ_1, λ_2 are the eigenvalues[26] of A and $v_1 = \begin{bmatrix} u_1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} u_2 \\ 1 \end{bmatrix}$ are the corresponding eigenvectors. Therefore, C is the change of coordinates such that

$A = C \circ \Lambda \circ C^{-1}$, or, equivalently, $\Lambda = C^{-1} \circ A \circ C$.



It can be easily proved that:

$$C^{-1} = \frac{1}{u_1 - u_2} \begin{bmatrix} 1 & -u_2 \\ -1 & u_1 \end{bmatrix}$$

---

[26] Matrix A will have a diagonal whenever $\left(\frac{1-\beta}{2\gamma} \right)^2 + \frac{\alpha}{\gamma} \neq 0 \Leftrightarrow \lambda_1 \neq \lambda_2$

$$u_j = \left(\frac{1-\beta}{2\gamma}\right) \pm \sqrt{\left(\frac{1-\beta}{2\gamma}\right)^2 + \frac{\alpha}{\gamma}} = \frac{\lambda_j - \beta}{\gamma}$$

$$\lambda_j = \left(\frac{1+\beta}{2}\right) \pm \sqrt{\left(\frac{1-\beta}{2}\right)^2 + \alpha\gamma} = \gamma . u_j + \beta$$

for $n = 0,1,2,\cdots$, let:

$I_n : \Re^2 \to \Re^2$ be a (non linear) transformation described by:

$$I_n \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y - \gamma . i_n \end{bmatrix}$$

It may be seen that $\begin{bmatrix} \pi_N \\ y_N \end{bmatrix} = (I_{N-1} \circ A \circ \cdots \circ I_0 \circ A) \begin{bmatrix} \pi_0 \\ y_0 \end{bmatrix}$

Hence, for the purpose of calculating $B = (I_{N-1} \circ A \circ \cdots \circ I_0 \circ A)$, the following formula may be used:

$B = I_{N-1} \circ C \circ \Lambda \circ T_{N-2} \circ \Lambda \circ T_{N-3} \circ \cdots \circ T_0 \circ \Lambda \circ C^{-1}$, where:

$T_n = C^{-1} \circ I_n \circ C$ represents a (non linear) transformation.

By iterating the functions, it may be proved that:

$$\begin{bmatrix}\pi_N \\ y_N\end{bmatrix} = \left(\frac{\lambda_1^N(\pi_0 - u_2 y_0) + \gamma . u_2 \sum_{j=0}^{N-2} i_j . \lambda_1^{N-1-j}}{u_1 - u_2}\right) v_1 + \left(\frac{\lambda_2^N(-\pi_0 + u_1 y_0) - \gamma . u_1 \sum_{j=0}^{N-2} i_j . \lambda_2^{N-1-j}}{u_1 - u_2}\right) v_2 +$$

$$+ \begin{bmatrix} 0 \\ -\gamma . i_{N-1}\end{bmatrix}$$

Thus, for $N \geq 2$ and $k \leq N-2$, one has:

$$\frac{\partial \pi_N}{\partial i_k} = \frac{u_1 u_2 \gamma . (\lambda_1^{N-1-k} - \lambda_2^{N-1-k})}{u_1 - u_2} \text{ , and } \frac{\partial y_N}{\partial i_k} = \frac{\gamma . (u_2 \lambda_1^{N-1-k} - u_1 \lambda_2^{N-1-k})}{u_1 - u_2}$$

$$\frac{\partial \pi_N^2}{\partial i_k} = 2\pi_N . \frac{\partial \pi_N}{\partial i_k} \text{ , e } \frac{\partial y_N^2}{\partial i_k} = 2 y_N . \frac{\partial y_N}{\partial i_k}$$

$$\frac{\partial \pi_N^2}{\partial i_k} =$$

$$= \frac{2u_1 u_2 \gamma . (\lambda_1^{N-1-k} - \lambda_2^{N-1-k})}{(u_1 - u_2)^2} . \left[\pi_0 (u_1 \lambda_1^N - u_2 \lambda_2^N) + u_1 u_2 y_0 (\lambda_2^N - \lambda_1^N) + u_1 u_2 \gamma \sum_{j=0}^{N-2} i_j (\lambda_1^{N-1-j} - \lambda_2^{N-1-j})\right]$$

$$\frac{\partial y_N^2}{\partial i_k} = \frac{2\gamma^2 i_{N-1}(u_1 \lambda_2^{N-1-k} - u_2 \lambda_1^{N-1-k})}{u_1 - u_2} +$$

$$\frac{2\gamma (u_2 \lambda_1^{N-1-k} - u_1 \lambda_2^{N-1-k})}{(u_1 - u_2)^2} \left[\pi_0 (\lambda_1^N - \lambda_2^N) + y_0 (\lambda_2^N u_1 - \lambda_1^N u_2) + \gamma \sum_{j=0}^{N-2} i_j (u_2 \lambda_1^{N-1-j} - u_1 \lambda_2^{N-1-j})\right]$$

On the other hand, the first order condition:

$$\frac{\partial L}{\partial i_k} = \sum_{N=k+2}^{+\infty} \delta^N \frac{\partial \pi_N^2}{\partial i_k} - 2\pi^* \sum_{N=k+2}^{+\infty} \delta^N \frac{\partial \pi_N}{\partial i_k} + \lambda \sum_{N=k+1}^{+\infty} \delta^N \frac{\partial y_N^2}{\partial i_k}$$ may be rewritten as:

$\frac{\partial L}{\partial i_k} = A + B$, where A represents the terms depending on the interest rate, and B the remaining terms.

$$A = \frac{2\gamma^2}{u_1 - u_2}\left[A_1 + A_2 + A_3\right]$$

$$A_1 = \frac{u_1^2 u_2^2}{u_1 - u_2} \sum_{N=k+2}^{+\infty} \sum_{j=0}^{N-2} \delta^N (\lambda_1^{N-1-k} - \lambda_2^{N-1-k})(\lambda_1^{N-1-j} - \lambda_2^{N-1-j}) i_j$$

$$A_1 = \sum_{j=0}^{k-1}\left[\frac{u_1^2 u_2^2}{u_1 - u_2} \sum_{N=k+2}^{M} \delta^N (\lambda_1^{N-1-k} - \lambda_2^{N-1-k})(\lambda_1^{N-1-j} - \lambda_2^{N-1-j})\right] i_j + \\ + \sum_{j=k}^{M-2}\left[\frac{u_1^2 u_2^2}{u_1 - u_2} \sum_{N=j+2}^{M} \delta^N (\lambda_1^{N-1-k} - \lambda_2^{N-1-k})(\lambda_1^{N-1-j} - \lambda_2^{N-1-j})\right] i_j$$

With this, A_1 may be written as a linear combination of interest rates i_j as follows:

$$A_1 = \sum_{j=0}^{M-2} a_{1,k,j} . i_j$$

Similarly, A_2 may be written as:

$$A_2 = \lambda \sum_{N=1}^{+\infty} \delta^N (u_1 \lambda_2^{N-1-k} - u_2 \lambda_1^{N-1-k}) i_{N-1} = \lambda \sum_{j=1}^{M} \delta^j (u_1 \lambda_2^{j-1-k} - u_2 \lambda_1^{j-1-k}) i_{j-1} = \sum_{j=0}^{M-1} a_{2,k,j} . i_j$$

and A_3 may be calculated as:

$$A_3 = \frac{\lambda}{u_1 - u_2} \sum_{N=k+1}^{+\infty} \sum_{j=0}^{N-2} \delta^N (u_2 \lambda_1^{N-1-k} - u_1 \lambda_2^{N-1-k})(u_2 \lambda_1^{N-1-j} - u_1 \lambda_2^{N-1-j}) i_j$$

$$A_3 = \sum_{j=0}^{k-1}\left[\frac{\lambda}{u_1 - u_2} \sum_{N=k+1}^{M} \delta^N (u_2 \lambda_1^{N-1-k} - u_1 \lambda_2^{N-1-k})(u_2 \lambda_1^{N-1-j} - u_1 \lambda_2^{N-1-j})\right] . i_j + \\ + \sum_{j=k}^{M-2}\left[\frac{\lambda}{u_1 - u_2} \sum_{N=j+2}^{M} \delta^N (u_2 \lambda_1^{N-1-k} - u_1 \lambda_2^{N-1-k})(u_2 \lambda_1^{N-1-j} - u_1 \lambda_2^{N-1-j})\right] . i_j$$

$$A_3 = \sum_{j=0}^{M-2} a_{3,k,j} . i_j$$

defining $a(k, j) = a_{1,k,j} + a_{2,k,j} + a_{3,k,j}$

This means that $a_{1,k,j}$ is the term related to $\frac{\partial \pi_N^2}{\partial i_k}$.

While $a_{2,k,j}$ and $a_{3,k,j}$ are related to $\frac{\partial y_N^2}{\partial i_k}$.

Therefore, one may define:

$$B = \frac{2\gamma}{u_1 - u_2}(B_1 + B_2 + B_3 + B_4 + B_5)$$

$$B_1(k) = \frac{u_1 u_2 \pi_0}{u_1 - u_2} \sum_{N=k+2}^{+\infty} \delta^N (\lambda_1^{N-1-k} - \lambda_2^{N-1-k})(u_1 \lambda_1^N - u_2 \lambda_2^N)$$

$$B_2(k) = \frac{u_1^2 u_2^2 y_0}{u_1 - u_2} \sum_{N=k+2}^{+\infty} \delta^N (\lambda_1^{N-1-k} - \lambda_2^{N-1-k})(\lambda_2^N - \lambda_1^N)$$

$$B_3(k) = -u_1 u_2 \pi^* \sum_{N=k+2}^{+\infty} \delta^N (\lambda_1^{N-1-k} - \lambda_2^{N-1-k})$$

$$B_4(k) = \frac{\lambda \pi_0}{u_1 - u_2} \sum_{N=k+1}^{+\infty} \delta^N (u_2 \lambda_1^{N-1-k} - u_1 \lambda_2^{N-1-k})(\lambda_1^N - \lambda_2^N)$$

$$B_5(k) = \frac{\lambda y_0}{u_1 - u_2} \sum_{N=k+1}^{+\infty} \delta^N (u_2 \lambda_1^{N-1-k} - u_1 \lambda_2^{N-1-k})(u_1 \lambda_2^N - u_2 \lambda_1^N)$$

$$b(k) = \frac{1}{\gamma}\left[B_1(k) + B_2(k) + B_3(k) + B_4(k) + B_5(k)\right]$$

and the first order condition implies:

$$A_1 + A_2 + A_3 = \frac{1}{\gamma}\left(B_1 + B_2 + B_3 + B_4 + B_5\right)$$

One has now an M equations linear system that may be represented in matrix form by:

$$\begin{bmatrix} a(0,0) & a(0,1) & a(0,2) & \cdots & a(0,M-1) \\ a(1,0) & a(1,1) & a(1,2) & \cdots & a(1,M-1) \\ a(2,0) & a(2,1) & a(2,2) & \cdots & a(2,M-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a(M-1,0) & a(M-1,1) & a(0,2) & \cdots & a(M-1,M-1) \end{bmatrix} \begin{bmatrix} i_0 \\ i_1 \\ i_2 \\ \vdots \\ i_{M-1} \end{bmatrix} = \begin{bmatrix} b(0) \\ b(1) \\ b(2) \\ \vdots \\ b(M-1) \end{bmatrix}$$

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