Risk-Neutral Probability Densities\textsuperscript{1}

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1 Introduction

In a recent article, Andrade and Tabak (2001) demonstrated that implicit exchange volatility extracted from exchange options negotiated on the domestic market contains more information than models based on temporal series, such as in the case of GARCH (1,1), or on moving average. This result is consistent with financial literature. Innumerable empirical tests have been carried out for many countries and markets and, in general, the results indicate that volatility extracted from options possesses greater informational content than the econometric models that seek to forecast future volatility to be realized\textsuperscript{4}. In general, the forecast of future volatility is recovered through options pricing models based on Black and Scholes (1973).

In the 1990s, financial research went well beyond extraction of volatility through models based on Black and Scholes (1973). Methodologies were developed with the purpose of extracting market expectations implicit in options prices. Basically, the idea consisted of recovering the density functions (PDFs) of the price of the underlying asset on the maturity date of options negotiated on the market\textsuperscript{5}.

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This approach based on PDFs has been widely used in international literature in detecting qualitative changes in the expectations of market agents regarding possible events in the nature and prices that would be observed in each one of these states. The advantage of using these forecasting methods is that they are forward-looking. Because of this property, these methods generally indicate changes in expectations well before than models that utilize techniques based on time-series. Just as in the case of implicit volatilities, it is to be expected that these forecasts will have greater informational content than forecasts based on time-series.

Section 2 discusses the pricing of options and extraction of volatility forecasts through a simple model. Section 3 presents a methodology aimed at extracting risk-neutral densities and an exercise is performed for domestic exchange. Section 4 concludes the article.

2. Options and implicit volatility

The options pricing model developed by Garman-Kohlgen (1983) is one of the most commonly utilized for pricing exchange options. The call option price $C_t$ is a function of implicit volatility $\sigma_{it}$ that should be numerically calculated, resolving the equation:

$$C_t = \frac{1}{(1 + r_t)^T} \left[ F_t N(d_1) - K_t N(d_2 - \sigma_{it} \sqrt{T}) \right]$$

(1)

where

$$d_1 = \frac{\ln(F_t/K_t)}{\sigma_{i,T} \sqrt{T}} + \frac{1}{2} \sigma_{i,T} \sqrt{T}$$

(2)

$T$ denotes the number of days to maturity; 
$r_t$ is the interest daily rate; 
$F_t$ is the adjustment price of the future of R$/US$ which matures in $T$ days; 
$N(.)$ is the cumulative standard normal distribution; 
$K_t$ is the option exercise price; 
$\sigma_{it}$ is the projected volatility at instant $t$ up to maturity of the option in $T$;
The only variable not known when calculating the premium of an option is volatility, $\sigma$. This volatility is the projection of changes in the prices of the asset underlying the option that agents believe will come about up to maturity of the option. Thus, the implicit volatility extracted from the model above consists in the market forecast for exchange rate variation up to maturity of the option.

Tabak and Andrade (2001) come to the conclusion that volatility forecasts extracted from equation (1) possess informational content that is not contained in models based on time-series such as GARCH (1,1) or moving average (MA). The authors utilize the adjusted $R^2$ to compare these forecasting models and conclude empirically that the implicit volatilities for the period from February 1999 to June 2001 more effectively explain the realized volatility6. Tabak and Chang (2002) analyze the same problem for a more recent period that extends to June 2002 and conclude that volatility forecasting based on time-series, as well as that based on exponential moving average and GARCH (1,1), do not aggregate information in the construction of forecasts of future volatility to the information that already exists in implicit volatilities.

The construction of scenarios for variables like exchange rate, interest rate, equity indices among other financial variables utilize forecasts of volatilities of these same variables as their input. These forecasts can be made with models based on time-series or utilizing implicit market expectations in options prices. Given that the literature comes to results suggesting that implicit volatilities possess greater informational content, it would be important to consider these forecasts in the scenarios that are constructed.

The next section will discuss extraction of the expectations for the risk-neutral density of the underlying asset. Parallel to this, an application to domestic exchange rate will be presented.

6/ The GARCH models were originally developed by Bollerslev (1986).
3. **Risk-neutral densities**

In recent years, financial literature has advanced considerably in the extraction of the risk-neutral densities of options. A series of methods for estimating these densities was proposed in literature. Two methods have become well known in recent years.

One approach consists in specifying a parametric functional form for risk-neutral density and adjusting this distribution to observed options prices through nonlinear least squares. The most commonly used functional form in literature has been a mixture of two lognormal\(^7\). Another approach interpolates the volatility smile and utilizes the result developed by Breeden and Litzenberger (1978) to recover risk-neutral density. The latter approach, known as the “smile approach” in literature, will be used in this article.

Much of financial literature is based on the result found by Breeden and Litzenberger (1978):

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\frac{\partial^2 C[K]}{\partial K^2} = e^{-\rho t} \times f(S)
\]

Equation (3) guaranties that it is possible to recover the risk-neutral probability density of an underlying asset \(S\) from the second derivative of the premium of an option with respect to the exercise price \(K\).

There are various ways of recovering density through equation (3). Shimko (1993) suggests a relatively simple way that consists in interpolating the calculated implicit volatilities for the same maturity and different exercise prices.

The idea would be to generate the estimated density in 4 stages:

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\(^7\) In this way, density is sufficiently flexible to perceive characteristics such as excess kurtosis, asymmetry and even bimodality.
1. Calculate implicit volatilities with observed prices on the market;
2. The implicit volatilities are interpolated generating a “smile” volatility curve8;
3. The prices of the options are calculated for each point on the interpolated volatility curve using Garman and Kohalgen (1983); and
4. With the aid of equation (3), the estimate risk-neutral density is generated.

This exercise was carried out for exchange rate options on the domestic market on the final business day of June 2002 (27.6). Figure 1 shows the volatility smile for these options. On that day, there were three options with the difference that there were three distinct exercise prices for the same maturity (final business day of July): R$2.7/US$, R$2.8/US$ and R$2.9/US$.

With three different exercise prices of negotiated options it was possible to calculate the implicit volatilities and interpolate these volatilities using a quadratic formula similar to that suggested by Castro (2002). It is important to emphasize that at least three exercise prices are needed to interpolate the implicit volatilities and construct the volatility smile. Therefore, one of the indispensable prerequisites for estimating risk-neutral densities is a market that has liquidity for different exercise prices.

Finally, it is possible to extract the risk-neutral probability, as presented in figure 2.

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8/ Different authors interpolate this curve in different ways, using quadratic forms, cubic splines, etc.
As seen in the previous figure, risk-neutral distribution is bimodal suggesting that agents attributed a high probability to a scenario in which exchange could come to more than R$ 3.6/US$. This expectation cannot be observed through the future of exchange (pink line in graph above)\(^9\). This performance is compatible with international literature which registers the existence of bimodality in distributions, particularly on the eve of elections.

International literature finds this type of distribution when two results are possible in the future: in the imminence of a war, on the eve of elections in which there are candidates with highly different proposals, among others.

Risk-neutral distribution reflects not only the chance that economic agents attribute to specific events, but also mirrors the value that agents attribute to this higher rate of exchange. In other words, economic agents may be ready to pay a risk premium to protect themselves against this possibility. Separating these two effects is complicated and they should be taken into consideration in the analysis of these distributions.

What is important to stress here is that the risk-neutral density above provides information that is not present in other instruments, such as the future of exchange rate or even in backward-looking models. In the example above, there are two possible...

\(^9\) In fact, futures contracts do not necessarily mirror the expectations of agents on realization of the prices of the underlying asset on the maturity date.
interpretations, as already stated. In any case, they reflect market concern at a rise in the value of the dollar to a specified level.

In figure 3, we present the evolution of the rate of exchange R$/US$ (PTAX) over the course of 2002. As one can see, expectations (via risk-neutral densities) at the end of June and early July reflected a certain degree of concern at a more accentuated high in the domestic exchange rate which, in fact, occurred in July as the PTAX reached R$3.4281/US$ on 7.31.2002.

![Figure 3. Ptax](image)

On repeating the same exercise for 5.28.2002 for exchange options with maturity on 6.28.2002, we have the following risk-neutral density.

![Figure 4. Risk Neutral Probability Density](image)

It is possible to realize that the density above has slightly positive asymmetry indicating a greater probability of an exchange rate increase. Furthermore, it is difficult to extract this type of information from exchange futures (pink line).
We can conclude that risk-neutral densities possess informational content that may, to some extent, be utilized to measure market expectations. However, one of the concerns that must be present in this type of analysis refers to the liquidity of these options and the volume of trading. It is important to have options at various exercise prices for the same maturity so that the densities can be extracted. This doesn’t always occur, making it difficult to utilize this type of instrument in constructing scenarios.

Another problem of implementation refers to the fact that, in general, options with sufficient liquidity to make this type of analysis have 1 or at the most 2 months to elapse to maturity. Thus, it is not possible to construct densities for longer terms and, consequently, the scenarios that can be constructed must always be short-term.

4. Conclusions

It is possible to extract information from options regarding the expected trajectory of important variables, such as exchange rate and interest. Literature shows that the implicit volatility extracted from these instruments possesses greater informational content than the models based on temporal series and, therefore, can generate benefits in the construction of scenarios for the future.

Moreover, it is possible to extract all the risk-neutral expected density for the underlying asset. These densities provide information (to some extent) on the expectations of economic agents regarding the distribution of the probability of the underlying asset on the date of option maturity.

Interpretation of these risk-neutral densities is no easy task. How to use these densities to aid in defining and implementing economic policy is still a question that generates considerable debate. However, the message is clear: these densities possess important information that must be taken into consideration in constructing relevant scenarios for the future.
References


