Estimation of Exchange Rate Densities

One of the most commonly used methodologies for estimating densities for future nominal exchange rates is based on utilization of existing option market information. However, the low liquidity of exchange rate options for periods of more than one month on the Brazilian market makes it difficult to estimate these densities. This box presents a methodology based on simulation techniques in order to generate estimates for three- and six-month ahead exchange rates. The procedure involves estimation of an ARMA$(p,q)$ – GARCH$(d,e)$ model, in which $p$ and $q$ are the orders of the autoregressive terms (AR) and moving averages (MA), respectively, and $d$ and $e$ are the orders of the error and variance lags, respectively, in the conditional variance equation.

The daily exchange rate data series was used to construct the densities (Ptax ask price) for the period extending from April 5th, 1999 to February 24th, 2006. Based on this series, monthly averages are calculated (83 observations) to be used to estimate the ARMA – GARCH model. The first 59 observations are used for the estimations and the 24 remaining observations for evaluation of the performance of the methodology.

After estimating the different specifications, the one best suited to the data was an AR(1) – GARCH(1,1). Let $e_t$ be the natural log of the exchange rate at instant $t$, and $\Delta e_t$ the exchange rate variation, the following specification is drawn:

\[
\Delta e_t = \alpha + \phi \Delta e_{t-1} + \eta_t \tag{1}
\]

\[
\eta_t^2 = \omega + \beta \eta_{t-1}^2 + \gamma \eta_{t-1}^2 \tag{2}
\]
in which the residuals ($\eta_t$) are independent and identically distributed (iid) and are normally distributed with zero mean and variance $h_t$ ($\eta_t \sim N(0, h_t) )$.

The AR(1) – GARCH(1,1) specification serves as a benchmark for construction of one, three- and six-month (steps) ahead exchange rate variance trajectories. For example, for one-step ahead, 100,000 random shocks are generated through Monte Carlo simulation with normal standardized distribution according to equations (1) and (2), used below in the construction of the density for the exchange variation corresponding to the adjustment of a kernel function for the 100,000 simulated observations of the exchange rate.

In turn, the procedure for generating three-step ahead forecasts involves simulation of 100,000 random shocks for each future time period, which will be used to generate exchange rate variations recursively and, consequently, to construct the densities:

\[
\Delta e_{t+1} = \alpha + \phi \Delta e_t + \eta_{t+1} \tag{3}
\]

\[
\Delta e_{t+2} = \alpha + \phi \Delta e_{t+1} + \eta_{t+2} \tag{4}
\]

\[
\Delta e_{t+3} = \alpha + \phi \Delta e_{t+2} + \eta_{t+3} \tag{5}
\]

Figure 1 presents the simulated density (one-month ahead) for the March 24 exchange rate, constructed through non-parametric estimation of the kernel function. Note that the density is concentrated in interval $[2.7;3.2]$ and that the average rate of exchange in that period was 2.90.

In order to evaluate performance of the methodology, we resorted to the Berkowitz test (2001)\(^1\), which basically, studies the properties of the forecast error sequence. If this sequence is independent and identically (iid) with normal standard distribution, one infers that the generated densities are well calibrated. In order to confront the assumption of standard normal and independence with autoregressive models with mean $\mu$ and standard deviation $\sigma$, one uses a likelihood ratio test (LR).

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Table 1 presents the results of application of the Berkowitz test (2001) for the one, three- and six-step ahead forecasting AR(1) model. At the significance level of 5%, the critical value is presented in the last column of the table. An LR test with a value higher than the critical value implies rejection of the null hypothesis that the forecasted densities are iid with normal standardized distribution. However, as shown in the table, this did not occur in any of the three cases.

The forecasts of the simulation model were compared to the forecasts gathered by Gerin among market analysts or, in other words, the average of the Gerin sampling was compared to the expected value supplied by the simulation. Table 2 presents the mean squared errors and Theil’s U index of the forecasting errors of the expectations collected by Gerin and of the forecasts generated by the proposed model2. The simulation model shows a better performance, principally in the three-step ahead forecast (one quarter). At the same time, as expected, the errors increase as the forecasting horizon expands.

Figure 2 presents the values attained, the forecast of the simulation model and the average one-step ahead expectations collected by Gerin.

The methodology presented to generate exchange rate densities in the future produces satisfactory results, since one cannot reject the hypothesis that the densities have a normal standardized distribution and are iid. Aside from this, the simulation model generates lower forecasting errors than those obtained through utilization of the average expectations collected by Gerin. In this way, the densities constructed can be a useful tool for constructing exchange rate scenarios and for evaluating the probability that the exchange rate will surpass the predetermined values.

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2/ The Theil’s U index corresponds to the ratio of the mean squared errors of the simulation model and the expectations collected by Gerin, respectively.