Technical Note:
Combining Hodrick-Prescott Filtering with a Production Function Approach to Estimate Output Gap

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Abstract

The methodology proposed combines two of the most important techniques to estimate output gap: the production function approach and Hodrick-Prescott filtering. Three main advantages can be derived from this method: (1) it adds some economic structure to a filtering method, (2) it allows different mean levels for potential output and output and (3) it simultaneously produces estimates for potential output and its unobservable components - nairu and naicu.

1 Introduction

Estimating output gap is a problem that has been on debate for a long time. This intense discussion can be explained by the fact that output gap has become one of the most important unobserved economic time series. Its relation with inflation, known as Phillips curve, expresses a trade-off that has been very useful for central banks that implicitly or explicitly target inflation.

Many methodologies have been proposed to estimate output gap. Two of the most popular techniques are Hodrick-Prescott filtering and the production function approach. These methodologies have been continually modified to incorporate new features. Two examples of this fact are the multivariable Hodrick-Prescott filter proposed by Laxton and Tetlow (1992) and the r-filters proposed by Araújo, Areosa and Rodrigues Neto (2003).

In line with Laxton and Tetlow (1992), in this note some structure is imposed to sharpen the identification of potential output. In the present case, Hodrick-Prescott filtering is combined to a production function approach. The strategy used is to add a constraint derived from a production function to the optimization problem known as Hodrick-Prescott filter. In order to estimate the unobserved variables that appear in production function, the objective function is extended. Two methods to calibrate the model are presented.

2 Describing the problem

In line with the approach presented in Alves and Muinhos (2003), a Cobb-Douglas production function with constant returns to scale is used to assess output gap and potential output. That is:

\[ y_t = A_t(K_t c_t)^\alpha (L_t(1 - u_t))^{(1-\alpha)} \]
\[ \overline{y}_t = A_t(K_t naicu_t)^\alpha (L_t(1 - nairu_t))^{(1-\alpha)} \] (1)

where \( y_t \) is the output, \( \overline{y}_t \) is the potential output, \( A_t \) is the productivity factor, \( K_t \) is the capital stock, \( L_t \) is the labor force, \( \alpha_t \) is the income capital share, \( c_t \) is the capacity utilization, \( u_t \) is the rate of unemployment, \( naicu_t \) is the non-accelerating inflation capacity utilization and \( nairu_t \) is the non-accelerating inflation rate of unemployment.

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1 Some interesting papers that uses these methodologies are Slevin (2001), Proietti, Musso and Westermann (2002), Bolt and van Els (2000) and Cerca and Saxena (2000)
2 A discussion about this methodology can be found in Boone (2000), Brouwer (1998) and St-Amant and van Norden (1997)
As stressed in Banco Central do Brasil Inflation Report (2003), “The potential output estimated from a production function (...) depends on the behavior of variables that are difficult to measure, such as capital stock and the amount of labor”. As an strategy to eliminate unnecessary data problems and measuring errors generated on the estimation of capital stock, equations (1) and (2) are used to assess the output gap and the potential output respectively as

\[ h_t = \ln \left( \frac{y_t}{\nairu_t} \right) = \alpha \left[ \ln(c_t) - \ln(\text{naicu}_t) \right] + (1 - \alpha) \left[ \ln(1 - u_t) - \ln(1 - \nairu_t) \right] \] (3)

\[ \nairu_t = \frac{y_t}{\exp(h_t)} = y_t \left( \frac{\text{naicu}_t}{c_t} \right)^\alpha \left( \frac{1 - \nairu_t}{1 - u_t} \right)^{1-\alpha} \] (4)

As an attempt to combine Hodrick-Prescott filtering and the production function approach just presented, potential output and its unobserved components - nairu and naicu - are estimated by solving the following problem:

\[
\min_{\{\text{naicu}_t\}_{t=1}^N, \{\text{nairu}_t\}_{t=1}^N} \begin{cases} 
\beta_1 \left[ \sum_{t=1}^N (\text{nairu}_t - u_t)^2 + \lambda \sum_{t=2}^{N-1} \left( \Delta^2 \text{nairu}_t \right)^2 \right] \\
\beta_2 \left[ \sum_{t=1}^N (\text{naicu}_t - c_t)^2 + \lambda \sum_{t=2}^{N-1} \left( \Delta^2 \text{naicu}_t \right)^2 \right] \\
\beta_3 \left[ \sum_{t=1}^N (\text{naicu}_t - c_t)^2 + \lambda \sum_{t=2}^{N-1} \left( \Delta^2 \nairu_t \right)^2 \right]
\end{cases}
\]

s.t.

\[ \nairu_t = y_t \left( \frac{\text{naicu}_t}{c_t} \right)^\alpha \left( \frac{1 - \nairu_t}{1 - u_t} \right)^{1-\alpha} \] (5)

It is worth noting that equation (4) appears as a restriction of this problem. Without this restriction, the optimization defined in (5) would give the same solution as if the following three optimizations were solved separately

\[
\min_{\{\text{nairu}_t\}_{t=1}^N} \left[ \sum_{t=1}^N (\text{nairu}_t - u_t)^2 + \lambda \sum_{t=2}^{N-1} \left( \Delta^2 \text{nairu}_t \right)^2 \right] \] (6)

\[
\min_{\{\text{naicu}_t\}_{t=1}^N} \left[ \sum_{t=1}^N (\text{naicu}_t - c_t)^2 + \lambda \sum_{t=2}^{N-1} \left( \Delta^2 \text{naicu}_t \right)^2 \right] \] (7)

\[
\min_{\{\nairu_t\}_{t=1}^N} \left[ \sum_{t=1}^N (\text{nairu}_t - y_t)^2 + \lambda \sum_{t=2}^{N-1} \left( \Delta^2 \nairu_t \right)^2 \right] \] (8)

By definition, the optimizations presented in (6), (7) and (8) correspond to using the Hodrick-Prescott filter to the series \{u_t\}_{t=1}^N, \{c_t\}_{t=1}^N and \{y_t\}_{t=1}^N in order to extract their trend, respectively assigned as \{nairu\}_{t=1}^N, \{naicu\}_{t=1}^N and \{nairu\}_{t=1}^N.

Nevertheless, the changes result considerably when (4) is added as a constraint to the problem defined in (5). For instance, it is easy to see that \{\nairu_t\}_{t=1}^N and \{y_t\}_{t=1}^N would have to be the same mean level if Hodrick-Prescott filtering were used to estimate \{\nairu_t\}_{t=1}^N as in (8). This characteristic is extremely undesirable since it does not have any economic appeal. Adding (4) corrects this problem.

\footnote{IBGE, which was in charge of measuring labor force, discontinued its series at December 2002. They created another series, based on a different methodology, that started at March 2002}


\footnote{On the following problem \(\Delta^2\) represents the second centred difference. For instance, \(\Delta^2 y_t = y_{t+1} - 2y_t + y_{t-1}\).}

\footnote{For further details about HP filtering, see Hodrick and Prescott (1997), King and Rebello (1993) and Araújo, Areosa and Rodrigues Neto (2003)}

\footnote{Although it is natural to assume that output fluctuates around potential output, it is not absurd to think that for quite long...}
3 Modifying the problem

When trying to run the optimization described in (5), a problem arises: what would be a suitable value for $\alpha$? Data from national accounts can be used to estimate an yearly series. However, this series gives evidence that $\alpha$ grew significantly during the last decade, suggesting that $\alpha$ should not be considered constant.

Since there is no reliable quarterly series for $\alpha$, Alves and Muinhos (2003) designed the following optimization problem:

$$\min \{ \alpha_t \}_{t=1}^{N} \left\{ \sum_{t=2}^{N-1} (\Delta^2 \alpha_t)^2 \right\}$$

s.t.

$$\frac{1}{4} \sum_{j=1}^{4} \alpha_{4(T-1)+j} = \overline{\alpha_T}, \forall T \in \{1, ..., N/4\}$$

$$0 \leq \alpha_t \leq 1, \forall t \in \{1, ..., N\}$$

This optimization aims to create an quarterly series, $\{\alpha_t\}_{t=1}^{N}$, using as benchmark an yearly series, $\{\overline{\alpha_T}\}_{T=1}^{N/4}$. As a constraint, it imposes that the average value of $\alpha_t$ during a given year $T$ is $\overline{\alpha_T}$.

Although estimating a quarterly series for $\alpha$ is not the main goal of this note, this step can greatly influence the results. Thereafter, a modified optimization that returns not only estimates for nairu and naicu, but also for $\alpha$, is suggested as a solution for the problem of not having a reliable quarterly series for $\alpha$.

$$\min \left\{ nairu_t, naicu_t, \alpha_t \right\}_{t=1}^{N} \left\{ \begin{array}{l}
\beta_1 \left[ \sum_{t=1}^{N} (nairu_t - u_t)^2 + \lambda \sum_{t=2}^{N-1} (\Delta^2 nairu_t)^2 \right] + \\
\beta_2 \left[ \sum_{t=1}^{N} (naicu_t - c_t)^2 + \lambda \sum_{t=2}^{N-1} (\Delta^2 naicu_t)^2 \right] + \\
\beta_3 \left[ \sum_{t=1}^{N} (\gamma_t - y_t)^2 + \lambda \sum_{t=2}^{N-1} (\Delta^2 \gamma_t)^2 \right] + \\
\beta_4 \left[ \lambda \sum_{t=2}^{N-1} (\Delta^2 \alpha_t)^2 \right]
\end{array} \right\}$$

s.t.

$$\frac{1}{4} \sum_{j=1}^{4} \alpha_{4(T-1)+j} = \overline{\alpha_T}, \forall T \in \{1, ..., N/4\}$$

$$0 \leq \alpha_t \leq 1, \forall t \in \{1, ..., N\}$$

$$y_t = y_t \left( \frac{naicu_t}{c_t} \right)^{\alpha_t} \left( 1 - nairu_t \right)^{1 - \alpha_t}$$

4 Calibrating the model

There are some alternative ways that can be used to calibrate the model.

4.1 Normalizing the series

Calibrating the model proposed in (10) regards quantifying the relative importance among problems - i.e., defining $\beta_1, \beta_2, \beta_3$ and $\beta_4$.

periods, like economic expansions (or retractions), output can stay persistently above (or under) the potential output. Thereafter, one would expect to get different mean level for estimates of output and potential output, if the sample were restricted to one of those periods.

$^8 \overline{\alpha_t}$ is estimated from the empirical average labor share obtained from national accounts (available at tab04.xls, extracted from sinoticas.zip at ftp://ftp.ibge.gov.br/Contas_Nacionais/Sistema_de_Contas_Nacionais/2000_2002/)

$^9$ The problem would not change if we had normalized one of the coefficients (ex. $\beta_1 = 1$). In order to be more explicit about some issues, four coefficient were used.
As a initial adjust, one might use $\beta_1 = \frac{1}{\sigma_u^2}$, $\beta_2 = \frac{1}{\sigma_\varepsilon^2}$, $\beta_3 = \frac{1}{\sigma_\eta^2}$ and $\beta_4 = \frac{1}{\sigma_v^2}$ where $\sigma_u^2$, $\sigma_\varepsilon^2$ and $\sigma_v^2$ are respectively variances of the series $\{u_t\}_{t=1}^N$, $\{\varepsilon_t\}_{t=1}^N$ and $\{\eta_t\}_{t=1}^N$. The rationale behind this choice is to put all series at the same scale. These procedure can avoid penalizing a series just because it is naturally more volatile than the others. Furthermore, it also corrects distortions caused by the fact that different methods may have been used to measure each observable variable.

In order to fully understand this idea, imagine that $\{u_t\}_{t=1}^N$ had been normalized, creating the series $\{\tilde{u_t}\}_{t=1}^N$ that, being $\pi$ and $\sigma_{ru}$ respectively the mean and standard deviation of $\{u_t\}_{t=1}^N$, was given by:

$$\tilde{u_t} = \frac{u_t - \pi}{\sigma_u}, \forall t$$

It is easily seen that solving

$$\min_{\{\text{nairu}_t\}_{t=1}^N} \left\{ \sum_{t=1}^{N} (\text{nairu}_t - \tilde{u}_t)^2 + \lambda \sum_{t=2}^{N-1} (\Delta^2 \text{nairu}_t)^2 \right\}$$

and estimating nairu by

$$\text{nairu}_t = \sigma_u \tilde{\text{nairu}}_t + \pi$$

is equivalent\(^\text{10}\) to solving

$$\min_{\{\text{nairu}_t\}_{t=1}^N} \frac{1}{\sigma_u^2} \left\{ \sum_{t=1}^{N} (\text{nairu}_t - u_t)^2 + \lambda \sum_{t=2}^{N-1} (\Delta^2 \text{nairu}_t)^2 \right\}$$

Analogously, choosing $\beta_2 = \frac{1}{\sigma_\varepsilon^2}$ and $\beta_3 = \frac{1}{\sigma_\eta^2}$ would be respectively equivalent to normalizing $\{c_t\}_{t=1}^N$ and $\{\eta_t\}_{t=1}^N$. Following the same idea, $\beta_4$ should be made equal to $\frac{1}{\sigma_v^2}$, where $\sigma_\eta^2$ would be the variance of $\{\alpha_t\}_{t=1}^N$. However this variance is not known, since $\{\alpha_t\}_{t=1}^N$ is one of the quarterly series to be estimated. Using the variance of $\{\pi_t\}_{t=1}^{N/4}$ is not recommended because it may underestimate the “real” value, once it does not take into consideration movements that occurs during an given year. However, each value of $\alpha_t$ is between 0 and 1 and, consequently, its variance is between 0 and 0.25. Thereafter choosing $\sigma_\eta^2 = 0.25$ can be interpreted as adopting the least restrictive behavior about the variability on $\{\alpha_t\}_{t=1}^N$.

### 4.2 Using a Phillips Curve

As commented in Boone, Juillard, Laxton and N’Diane (2002), nairu estimation processes that do not exploit information about inflation may result in inefficient historical measures of the nairu, biased parameter estimates, as well as inefficient forecasts of nairu. Thereafter, using a Phillips curve to calibrate this model might be recommended and the following problem is proposed:

$$\max_{\beta_1, \beta_2} \quad f \left( \{h_t\}_{t=1}^N, \{\pi_t\}_{t=1}^N \right)$$

s.t.

$$h_t = \alpha \left[ \ln(c_t) - \ln(nairu_t) \right] + (1 - \alpha) \left[ \ln(1 - u_t) - \ln(1 - \text{nairu}_t) \right]$$

$$\left( \{\text{nairu}_t\}_{t=1}^N, \{\text{nairu}_t\}_{t=1}^N \right) \in \arg\min \text{ Optim} 1$$

where $\{h_t\}_{t=1}^N$ is the output gap series, $\{\pi_t\}_{t=1}^N$ is the inflation series, $f$ is any function that quantify the adjust of $\{h_t\}_{t=1}^N$ to a Phillips curve\(^\text{11}\) and Optim1 is the problem described in (5). Simplifying the optimization proposed in (11) requires studying the first order conditions of (5).

\(^{10}\)Equivalence in this context means that, not only it gives the same solution, but also the same value for the minimum.

\(^{11}\)For instance, $f$ could represent the R\(^2\) (although this objective function should be carefully considered since it may cause overfitting).
5 Results

Although all results presented in this section are considered preliminary, they help to highlight the differences that would have occurred if all optimizations were made separately. It is important to notice that:

- The levels of the series \( \{nairu_t\}_{t=1}^N \) and \( \{u_t\}_{t=1}^N \) are clearly different. This would never have happened if \( \{nairu_t\}_{t=1}^N \) had been obtained solving problem (6) (i.e., if the Hodrick-Prescott filter had been used to estimate \( \{nairu_t\}_{t=1}^N \)). Series \( \{naicu_t\}_{t=1}^N \) presents the same characteristic.

- The series \( \{\alpha_t\}_{t=1}^N \) would be much less volatile if it were estimated running the optimization expressed in (9).
The results just presented are based on the following data:

- unemployment \(\{u_t\}_{t=1}^{N}\) - quarterly average of series n°. 1629, available at www.bcb.gov.br.
- capacity utilization \(\{c_t\}_{t=1}^{N}\) - quarterly average of series n°. 1341, available at www.bcb.gov.br.
- GDP \(\{y_t\}_{t=1}^{N}\) - series n°. 1232, available at www.bcb.gov.br.
- empirical average labor share \(\{1 - \alpha_t\}_{t=1}^{N}\) - available at tab04.xls, extracted from sinoticas.zip at ftp://ftp.ibge.gov.br/Contas_Nacionais/Sistema_de Contas_Nacionais/2000_2002/)

Additionally, \(\beta_1 = 9.347, \beta_2 = 9.159, \beta_3 = 12\) and \(\beta_4 = 0.056\).

6 Conclusions and Extensions

The most important idea in this note is that, when some structure is imposed to a econometric method, the results can change considerably. In the present case, Hodrick-Prescott filtering has been combined with a production function approach. This strategy requires estimating the non-observed components of a Cobb-Douglas production function - nairu and naicu. In this note, a “kind of Hodrick-Prescott filtering” is used. Nevertheless, many other methods could have been chosen. Most of them requires some kind of optimization whose objective function could replace the equations (6) and (7). That is, the optimization expressed in (5) could be extended to give:

\[
\min_{\{\text{nairu}_t\}_{t=1}^{N},\{\text{naicu}_t\}_{t=1}^{N}} \left\{ \sum_{t=1}^{N} (y_t - y_{t-1})^2 + \lambda \sum_{t=2}^{N} (\Delta^2 y_t)^2 \right\}
\]

s.t.

\[y_t = y_t \left( \frac{\text{naicu}_t}{c_t} \right)^\alpha \left( \frac{1 - \text{nairu}_t}{1 - u_t} \right)^{1-\alpha} \]

(12)

It is also important to emphasize that calibrating these models is not an easy task. Whenever a parameter - \(\beta_1, \beta_2, \beta_3, \) or \(\beta_4\) - is changed, the hole problem is modified, since these parameters quantify the relative importance between optimizations problems. In this note, two methods are presented. Nevertheless, any other criterion could be used. A natural extension of the method proposed in subsection 3.2 is using an IS curve or any other relation (or even combination of relations) where output gap would be important\(^{12}\).

\(^{12}\)Some of the stylized fact commented in Doménech and Gómez (2003) can be also considered.
References


